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On the existence and the uniqueness of solutions and on the convergence of successive approximations in the Darboux problem for certain differential equations of the type $u_{x_1 \cdots x_n} = f(x_1, \cdots, x_n, u, \cdots, u_{x_{l_1} \cdots x_{l_j}}, \cdots)$

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ON THE EXISTENCE AND THE UNIQUENESS OF SOLUTIONS AND ON THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS IN THE DARBOUX PROBLEM FOR CERTAIN DIFFERENTIAL EQUATIONS OF THE TYPE $u_{x_1 \dots x_n} = f(x_1, \dots, x_n, u, \dots, u_{x_{l_1} \dots x_{l_j}}, \dots)$

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1. Introduction. In the paper [1] W. WALTER proved the uniqueness of solutions of some initial value problems for the ordinary, parabolic and hyperbolic differential equations under certain generalized conditions of the Nagumo and Osgood type. On the basis of these uniqueness conditions, M. KWAPISZ, B. PALCZEWSKI, W. PAWELSKI [2] showed the existence and the uniqueness of solutions of the Darboux problem for the equations of the type $u_{xyz} = f(x, y, z, u, u_x, u_y, u_z, u_{xy}, u_{xz}, u_{yz})$ V. PALCZEWSKI [3] and J. S. W. WONG [4] proved, besides the existence and the uniqueness of solutions, also the uniform convergence of successive approximations in the Darboux problem for the equations of the type $u_{xy} = f(x, y, u)$ under the conditions for the uniqueness of the Krasnosielski and Krein type.

In the present paper we shall study the questions of the existence, of the uniqueness of solutions and of the convergence of successive approximations in the Darboux problem for the equations of the n-th order of the type $u_{x_1 \dots x_n} = f(x_1, \dots, x_n, u, \dots \dots, u_{x_{l_1} \dots x_{l_j}}, \dots)$ and for the systems of differential equations of the n-th order using the generalized conditions of the Krasnosielski and Krein type and of Nagumo-Perron-van Kampen type [5]. Instead of the classic method of proving the following theorems, it will be shown that these results all follow as a consequence of a certain theorem on the conctractive mappings in some generalized metric space. This theorem was first initiated in the paper by W. A. J. LUXEMBURG [6].

2. A theorem on contraction. First of all we shall define the notion of the generalized metric space.

Let Y be a non-void set; and let $d(x, y)$ be a non-negative real valued function

$0 \leqq d(x, y) \leqq +\infty$ defined on the Cartesian product $Y \times Y$ and satisfying the following conditions for arbitrary elements $x, y, z \in Y$:

a) $d(x, y) = 0$ if and only if $x = y$.

b) $d(x, y) = d(y, x)$.

c) $d(x, y) \leqq d(x, z) + d(z, y)$.

d) If the sequence $\{x_k\}_1^\infty$ of the elements $x_k \in Y$ is a d-Cauchy sequence, i.e. $\lim_{k,m\to\infty} d(x_k, x_m) = 0$, then there exists an element $x \in Y$ such that $\lim_{k\to\infty} d(x, x_k) = 0$.

An abstract set Y on which the distance is defined in this way is called the generalized complete metric space. It differs from the usual concept of the complete metric space by the fact that not every pair of elements $x, y \in Y$ necessarily has a finite distance $d(x, y)$.

Theorem 1. (Luxemburg [6]). *Let Y be a generalized complete metric space and T a mapping of Y into itself satisfying the following conditions:*

1° *There exists a constant λ, $0 < \lambda < 1$ such that*

$$d(Tx, Ty) \leqq \lambda\, d(x, y)$$

for all $x, y \in Y$ with the distance $d(x, y) < +\infty$.

2° *For every sequence of succesive aproximations $x_k = Tx_{k-1}$, $k = 1, 2, \ldots$, where x_0 is an arbitrary element of Y, there exists an index $K(x_0)$ such that $d(x_K, x_{K+l}) < +\infty$ for all $l = 1, 2, \ldots$*

3° *If x and y are two fixed points of the mapping T, i.e. $Tx = x$, $Ty = y$, then $d(x, y) < +\infty$.*

Then the equation $Tx = x$ has one and only one solution and every sequence of successive approximations $\{x_k\}_1^\infty$ converges in the distance $d(x, y)$ to this unique solution.

3. The formulation of the Darboux problem. Let us introduce the following notation and assumptions.

1. Let R^0, R be an arbitrary set of points $X = (x_1, \ldots, x_n)$, for which $0 < x_i \leqq A_i$, $0 \leqq x_i \leqq A_i$ respectively, $A_i > 0$ for all $i = 1, 2, \ldots, n$ and $n \geqq 1$. Further, let $R_{l_1 \ldots l_j}$ denote an arbitrary $(n - j)$-dimensional closed domain of points $X_{l_1 \ldots l_j} = (x_1, \ldots, x_{l_1-1}, x_{l_1+1}, \ldots, x_{l_j-1}, x_{l_j+1}, \ldots, x_n)$ such that $0 \leqq x_i \leqq A_i$ holds for $i = 1, \ldots, l_1 - 1, l_1 + 1, \ldots, l_j - 1, l_j + 1, \ldots, n$ and $j = 1, 2, \ldots, n - 1$. $(l_1, \ldots, l_j)$ denotes an arbitrary combination of j numbers from the n numbers $(1, \ldots, n)$, $l_1 < \ldots < l_j$.

2. Let us define the sets E and E^0 as Cartesian products $E = R \times \prod_{i=1}^{s} \{-\infty < z_i < +\infty\}$ and $E^0 = R^0 \times \prod_{i=1}^{s} \{-\infty < z_i < +\infty\}$ for $s = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} = (1 + 1)^n - 1 = 2^n - 1$.

3. Further, let us denote:

a) an arbitrary vector with $\binom{n}{j}$ real components by $\mathbf{U}^j = (u_{1\ldots j}, \ldots, u_{l_1\ldots l_j}, \ldots \ldots, u_{n-j+1\ldots n})$ for $j = 1, 2, \ldots, n$ and $\mathbf{U}^0 = u_0$. Let the symbol $|\mathbf{U}^j|^\gamma$ mean the vector $(|u_{1\ldots j}|^\gamma, \ldots, |u_{l_1\ldots l_j}|^\gamma, \ldots, |u_{n-j+1\ldots n}|^\gamma)$ for any real number γ and let $(\mathbf{U}, \mathbf{V})$ mean the scalar product of the vectors $\mathbf{U}$ and $\mathbf{V}$.

b) $$D_{l_1\ldots l_j} = \frac{\partial^j}{\partial x_{l_1} \ldots \partial x_{l_j}}, \quad \mathbf{D}^j = (D_{1\ldots j}, \ldots, D_{l_1\ldots l_j}, \ldots, D_{n-j+1\ldots n})$$

for $j = 1, 2, \ldots, n$ and $\mathbf{D}^0 u = u$.

4. Let us suppose that the function $f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})$ of $s + n$ variables is continuous on E.

5. Let the function $\sigma_j(X_j)$ together with its derivatives $D_{l_1\ldots l_k}\sigma_j$ of the k-th order for $k = 1, 2, \ldots, n - 1$ be continuous in the domain R_j for $j = 1, 2, \ldots, n$ and let it fulfil the conditions

$$[\sigma_i(X_i)]_{x_j=0} = [\sigma_j(X_j)]_{x_i=0}, \quad i \neq j; \quad i, j = 1, 2, \ldots, n$$

in the domain R_{st} where $s = \min(i, j)$, $t = \max(i, j)$.

6. At last, let us denote the set of all functions $z(X) \in C(R)$ with continuous derivatives $D_{l_1\ldots l_k} z$ for $k = 1, 2, \ldots, n - 1$ in the domain R satisfying the conditions

$$[z(X)]_{x_j=0} = \sigma_j(X_j), \quad j = 1, 2, \ldots, n$$

in R_j by $M(R)$.

We shall understand by the solution of the Darboux problem

$$\mathbf{D}^n u = f(X, u, \mathbf{D}^1 u, \ldots, \mathbf{D}^{n-1} u) \tag{1}$$

$$u(X) = \sigma_j(X_j) \quad \text{for} \quad x_j = 0, \tag{2}$$
$$[\sigma_i(X_i)]_{x_j=0} = [\sigma_j(X_j)]_{x_i=0} \quad \text{for} \quad i \neq j, \quad i, j = 1, 2, \ldots, n$$

any function $u(X) \in M(R)$ which has the continuous derivative $\mathbf{D}^n u$ in the domain R and satisfies equation (1) on R.

Then, the Darboux problem (1), (2) is equivalent to solving the integro-differential equation

$$u(X) = G(X) + \int_R f(\Xi, u, \mathbf{D}^1 u, \ldots, \mathbf{D}^{n-1} u)\, d\Xi \,. \tag{3}$$

The function $G(X)$ can be explicitly expressed in R by the initial functions $\sigma_i(X_i)$ for $i = 1, 2, \ldots, n$ because

$$G(X) = u(0, x_2, \ldots, x_n) + \ldots + u(x_1, \ldots, x_{n-1}, 0) -$$
$$- [u(0, 0, x_3, \ldots, x_n) + \ldots + u(x_1, \ldots, x_{n-2}, 0, 0)] + \ldots + (-1)^{n-1} u(0, \ldots, 0) \,.$$

With respect to (3), the sequence of Picard's successive approximations $\{u_k\}_1^\infty$ is defined by the equation

$$u_k(X) = G_0(X) + \int_R f(\Xi, u_{k-1}, \mathbf{D}^1 u_{k-1}, \ldots, \mathbf{D}^{n-1} u_{k-1})\, d\Xi \tag{4}$$

for $k = 1, 2, \ldots$ on R where $u_0(X)$, $G_0(X)$ are arbitrary functions of $M(R)$. The sequence of the derivatives $\{D_{l_1 \ldots l_j} u_k\}_{k=1}^\infty$ is determined by

$$\begin{aligned} D_{l_1 \ldots l_j} u_k &= D_{l_1 \ldots l_j} G_0(X) + \\ &+ \int_{R_{l_1 \ldots l_j}} f(\Xi^x_{l_1 \ldots l_j}, u_{k-1}, \mathbf{D}^1 u_{k-1}, \ldots, \mathbf{D}^{n-1} u_{k-1})\, d\Xi_{l_1 \ldots l_j} \end{aligned} \tag{4_1}$$

for $j = 1, 2, \ldots, n-1$ and $X \in R$ where $\Xi^x_{l_1 \ldots l_j}$ denotes any point of R with the components $(\xi_1, \ldots, \xi_{l_j - 1}, x_{l_j}, \xi_{l_j + 1}, \ldots, \xi_n)$; i.e. we get the point $\Xi^x_{l_1 \ldots l_j}$ so that we replace the $l_1, \ldots, l_j$-th component of the point $\Xi = (\xi_1, \ldots, \xi_n)$ by the variables $x_{l_1}, \ldots, x_{l_j}$ in this order.

4. Theorems on the existence and uniqueness. In the following theorem we shall investigate the problem (1), (2) using the generalized conditions of Krasnosielski and Krein.

Theorem 2. *Let the function $f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})$ be defined, continuous and bounded on E and let it satisfy the following conditions in E^0:*

$$\begin{aligned} &|f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}) - f(X, \mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^{n-1})| \leqq \\ &\leqq \frac{L}{x_1 \ldots x_n} \sum_{j=0}^{n-1} (\mathbf{P}^j, |\mathbf{U}^j - \mathbf{V}^j|)\,, \quad L > 0 \end{aligned} \tag{5}$$

where

$$\mathbf{P}^j = \left(\frac{p_{1\ldots j}}{\sqrt[n]{L^j}}\, x_1 \ldots x_j,\ \frac{p_{l_1\ldots l_j}}{\sqrt[n]{L^j}}\, x_{l_1} \ldots x_{l_j},\ \ldots,\ \frac{p_{n-j+1\ldots n}}{\sqrt[n]{L^j}}\, x_{n-j+1} \ldots x_n\right)$$

denotes the vector with $\binom{n}{j}$ *real components for* $j = 1, 2, \ldots, n-1$, $\mathbf{P}^0 = p_0$. *The coefficients* $p_{l_1\ldots l_j}$, p_0 *are non-negative constants at least one of which is non-vanishing. Let, further, the inequality*

$$\begin{gathered} |f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}) - f(X, \mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^{n-1})| \leqq \\ \leqq \frac{C}{x_1^\beta \ldots x_n^\beta} \sum_{j=0}^{n-1} (\mathbf{Q}^j, |\mathbf{U}^j - \mathbf{V}^j|^\alpha), \quad C > 0 \end{gathered} \tag{6}$$

where $\mathbf{Q}^j = (q_{1\ldots j} x_1^\alpha \ldots x_j^\alpha, \ldots, q_{l_1\ldots l_j} x_{l_1}^\alpha \ldots x_{l_j}^\alpha, \ldots, q_{n-j+1\ldots n} x_{n-j+1}^\alpha \ldots x_n^\alpha)$ *also denotes the vector with* $\binom{n}{j}$ *components for* $j = 1, 2, \ldots, n-1$, $\mathbf{Q}^0 = q_0$ *and* $0 < \alpha < 1$, $\beta < \alpha$ *hold. The coefficients* $q_{l_1\ldots l_j}$, q_0 *are non-negative constants at least one of which is non-vanishing and let the inequalities* $L(1-\alpha)^n < (1-\beta)^n$, $(p_0 + \sum_{j=1}^{n} \sum_{l_1,\ldots,l_j} p_{l_1\ldots l_j})^n L(1-\alpha)^n < (1-\beta)^n$ *be fulfilled. Then there exists one and only one solution* $u(X)$ *of the Darboux problem* (1), (2) *and furthermore the Picard's sequence of successive approximations* (4) *for arbitrary functions* $u_0(X)$, $G_0(X) \in M(R)$ *such that* $\mathbf{D}^n G_0(X) = 0$ *in* R, *converges uniformly to this unique solution.*

Proof. To prove this theorem we shall apply the preceding Theorem 1 on the contracted mappings. Hence we must choose a suitable complete metric space Y and an operator T mapping the space Y into itself and show that the conditions of Theorem 1 are fulfilled. We shall prove that $Y = (M(R), d)$ where the distance d is defined by the following equation

$$d(u, v) = \sup_{R^0} \frac{\sum_{j=0}^{n-1} (\mathbf{P}^j, |\mathbf{D}^j u - \mathbf{D}^j v|)}{(x_1 \ldots x_n)^{p \sqrt[n]{L}}} \tag{7}$$

for $u, v \in M(R)$ is the required metric space. The number $p > 1$ satisfies the inequalities $p^n L(1-\alpha)^n < (1-\beta)^n$, $p^n L > 1$.

The existence of the number p considered above is guaranteed by the assumption $L(1-\alpha)^n < (1-\beta)^n$. Moreover from the inequality $(p_0 + \sum_{j=1}^{n-1} \sum_{l_1,\ldots,l_j} p_{l_1\ldots l_j})^n L(1-\alpha)^n < (1-\beta)^n$ we immediately see that p can be chosen as follows

$$p_0 + \sum_{j=1}^{n-1} \sum_{l_1,\ldots,l_j} p_{l_1\ldots l_j} < p < \frac{1}{\sqrt[n]{L}} \frac{1-\beta}{1-\alpha}.$$

The function $d(u, v)$ defined by the relation (7) evidently fulfils the properties a), b), c) of the distance from Part 2. For the proof of the completeness of the metric space Y we shall use the inequality

$$(8)\qquad \max_R \sum_{j=0}^{n-1} (\mathbf{S}^j, |\mathbf{D}^j u - \mathbf{D}^j v|) \leqq d(u, v)$$

where $\mathbf{S}^j = (s_{1\ldots j}, \ldots, s_{l_1\ldots l_j}, \ldots, s_{n-j+1\ldots n})$ denotes the vector with $\binom{n}{j}$ components for $j = 1, 2, \ldots, n-1$ and $\mathbf{S}^0 = s_0$. The components $s_{l_1\ldots l_j}$ of the vector $\mathbf{S}^j$ for $j = 1, 2, \ldots, n-1$ and s_0 are non-negative constants at least one of which is non-vanishing and they can be expressed by the constants L, $p_{l_1\ldots l_j}$ and A_i for $i = 1, 2, \ldots, n$. From inequality (8) there follows that the d-convergence of the sequence $\{u_k(X)\}_1^\infty$ of functions $u_k(X) \in M(R)$ implies the convergence of the sequences $\{D_{l_1\ldots l_j} u_k\}_{k=1}^\infty$ for $j = 1, 2, \ldots, n-1$ and $\{u_k\}_1^\infty$ in the sense of the distance

$$(8_1)\qquad d(u, v) = \max_R |u - v| .$$

Consequently, there exists such a function $u(X) \in M(R)$ that

$$(8_2)\qquad \lim_{k\to\infty} u_k(X) = u(X), \quad \lim_{k\to\infty} D_{l_1\ldots l_j} u_k(X) = D_{l_1\ldots l_j} u(X)$$

for $j = 1, 2, \ldots, n-1$ in the domain R.

Let now the sequence $\{u_k(X)\}_1^\infty$ be d-Cauchy, i.e. $\lim_{k,m\to\infty} d(u_k, u_m) = 0$. Then we have

$$d(u_k, u_m) < \varepsilon$$

for any $\varepsilon > 0$ and $k, m > N(\varepsilon)$ where $N(x) > 0$ is a real valued function. From this inequality and by (8_2) we obtain that the sequence $\{(p_0/(x_1 \ldots x_n)^{p\sqrt[n]{L}})\, u_k\}_{k=1}^\infty$ uniformly converges to the function $(p_0/(x_1 \ldots x_n)^{p\sqrt[n]{L}})\, u(X)$ where $u(X) \in M(R)$ on R^0. Analogically we shall show that the sequence $\{(p_{l_1\ldots l_j} x_{l_1} \ldots x_{l_j}/\sqrt[n]{(L^j)}(x_1 \ldots x_n)^{p\sqrt[n]{L}}) \cdot D_{l_1\ldots l_j} u_k\}_{k=1}^\infty$ uniformly converges to the function $(p_{l_1\ldots l_j} x_{l_1} \ldots x_{l_j}/\sqrt[n]{(L^j)}(x_1 \ldots x_n)^{p\sqrt[n]{L}}) \cdot D_{l_1\ldots l_j} u(X)$ for $j = 1, 2, \ldots, n-1$ in the domain R^0.

From there

$$(9)\qquad \frac{p_0}{(x_1 \ldots x_n)^{p\sqrt[n]{L}}} |u_k - u| < \frac{\varepsilon}{s} \quad \text{for} \quad k > N_0(\varepsilon),$$

$$\frac{p_{l_1\ldots l_j} x_{l_1} \ldots x_{l_j}}{\sqrt[n]{(L^j)}(x_1 \ldots x_n)^{p\sqrt[n]{L}}} |D_{l_1\ldots l_j} u_k - D_{l_1\ldots l_j} u| < \frac{\varepsilon}{s} \quad \text{for} \quad k > N_{l_1\ldots l_j}(\varepsilon)$$

for $j = 1, 2, \ldots, n-1$, $s = 2^n - 1$ and suitable positive constants $N_0(\varepsilon)$, $N_{l_1\ldots l_j}(\varepsilon)$

on R^0. If we denote $\bar{N} = \max(N_0, \max_{\substack{l_1,\dots,l_j \\ j=1,2,\dots,n-1}} N_{l_1\dots l_j})$ then from inequalities (9) for $k > \bar{N}$ there follows that $d(u_k, u) \leqq \varepsilon$, i.e. $\lim_{k\to\infty} d(u_k, u) = 0$. Thereby the property d) is proved and $Y = (M(R), d)$ is a generalized complete metric space.

The operator T defined by the equation

$$(10)\qquad Tu(X) = G_0(X) + \int_R f(\Xi, u, \mathbf{D}^1 u, \dots, \mathbf{D}^{n-1} u)\, \mathrm{d}\Xi$$

for $X \in R$ maps the set $M(R)$ into itself. Hence we obtain the equation

$$(10_1)\quad D_{l_1\dots l_j} Tu(X) = D_{l_1\dots l_j} G_0(X) + \int_{R_{l_1\dots l_j}} f(\Xi^x_{l_1\dots l_j}, u, \mathbf{D}^1 u, \dots, \mathbf{D}^{n-1} u)\, \mathrm{d}\Xi_{l_1\dots l_j}$$

for $j = 1, 2, \dots, n-1$ in the domain R. Therefore, the problem to find the solution of the Darboux problem (1), (2) or of the integro-differential equation (3) is transformed to the problem of finding the fixed point of the mapping T on the set $M(R)$.

The sequence of Picard's approximations (4) is equivalent to the sequence $\{Tu_{k-1}\}_{k=1}^{\infty}$ and the sequence of the derivatives (4_1) is equivalent to the sequence $\{D_{l_1\dots l_j} Tu_{k-1}\}_{k=1}^{\infty}$ for $j = 1, 2, \dots, n-1$.

Proof of condition 1°. Let u, v be two arbitrary functions from Y with $d(u, v) < +\infty$. Then from equation (10) and hypothesis (5) we have

$$(11)\quad |Tu - Tv| \leqq \int_R |f(\Xi, u, \mathbf{D}^1 u, \dots, \mathbf{D}^{n-1} u) - f(\Xi, v, \mathbf{D}^1 v, \dots, \mathbf{D}^{n-1} v)|\, \mathrm{d}\Xi \leqq$$

$$\leqq L \int_R \frac{\sum_{j=0}^{n-1} (\mathbf{P}^j, |\mathbf{D}^j u - \mathbf{D}^j v|)}{(\xi_1 \dots \xi_n)^{p\sqrt[n]{L}}} (\xi_1 \dots \xi_n)^{p\sqrt[n]{(L)}-1}\, \mathrm{d}\Xi \leqq d(u, v) \frac{(x_1 \dots x_n)^{p\sqrt[n]{L}}}{p}$$

for $X \in R^0$. Further, by (10_1) we obtain the estimates

$$(11_1)\qquad \frac{x_{l_1} \dots x_{l_j}}{\sqrt[n]{L^j}} |D_{l_1\dots l_j} Tu - D_{l_1\dots l_j} Tv| \leqq d(u, v) \frac{(x_1 \dots x_n)^{p\sqrt[n]{L}}}{p}$$

for every $j = 1, 2, \dots, n-1$ in the domain R^0. The necessary inequality

$$d(Tu, Tv) = \lambda\, d(u, v)$$

where $\lambda = (p_0 + \sum_{j=1}^{n-1} \sum_{l_1,\dots,l_j} p_{l_1\dots l_j})/p < 1$ follows directly from relations (11) and (11_1).

For the proof of condition 2° we shall use the boundedness of the function

$f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})$ in the domain E. Let us denote $K = \sup_E |f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots$ $\ldots, \mathbf{U}^{n-1})|$. Then from equations (4) and (4_1) for any function $u_0(X) \in M(R)$ we get

$$(12) \qquad |u_2(X) - u_1(X)| \leqq 2K(x_1 \ldots x_n), (x_{l_1} \ldots x_{l_j})|D_{l_1 \ldots l_j} u_2 - D_{l_1 \ldots l_j} u_1| \leqq \leqq 2K(x_1 \ldots x_n)$$

for $j = 1, 2, \ldots, n-1$ in R. By relations (12) and assumption (6) the estimates

$$|u_3(X) - u_2(X)| \leqq \int_R |f(\Xi, u_2, \mathbf{D}^1 u_2, \ldots, \mathbf{D}^{n-1} u_2) - f(\Xi, u_1, \mathbf{D}^1 u_1, \ldots, \mathbf{D}^{n-1} u_1)| \, d\Xi \leqq$$

$$\leqq C \int_R \frac{\sum_{j=0}^{n-1} (\mathbf{Q}^j, |\mathbf{D}^j u_2 - \mathbf{D}^j u_1|^\alpha)}{\xi_1^\beta \ldots \xi_n^\beta} \, d\Xi \leqq C(q_0 + \sum_{j=1}^{n-1} \sum_{l_1, \ldots, l_j} q_{l_1 \ldots l_j})(2K)^\alpha (x_1 \ldots x_n)^{(\alpha - \beta) + 1}$$

hold for $X \in R^0$. Similarly, it is possible to show that

$$(x_{l_1} \ldots x_{l_j}) \, |D_{l_1 \ldots l_j} u_3(X) - D_{l_1 \ldots l_j} u_2(X)| \leqq$$
$$\leqq C(q_0 + \sum_{j=1}^{n-1} \sum_{l_1, \ldots, l_j} q_{l_1 \ldots l_j}) \, (2K)^\alpha \, (x_1 \ldots x_n)^{(\alpha - \beta) + 1}$$

for $j = 1, 2, \ldots, n-1$ in the domain R^0. We shall easily prove the following estimates

$$|u_{k+3}(x) - u_{k+2}(X)| \leqq$$
$$\leqq [C(q_0 + \sum_{j=1}^{n-1} \sum_{l_1, \ldots, l_j} q_{l_1 \ldots l_j})\}]^{1 + \alpha + \ldots + \alpha^k} (2K)^{\alpha^{k+1}} (x_1 \ldots x_n)^{(\alpha - \beta)(1 + \alpha + \ldots + \alpha^k) + 1}$$

$$(13) \qquad (x_{l_1} \ldots x_{l_j}) \, |D_{l_1 \ldots l_j} u_{k+3}(X) - D_{l_1 \ldots l_j} u_{k+2}(X)| \leqq$$
$$\leqq [C(q_0 + \sum_{j=1}^{n-1} \sum_{l_1, \ldots, l_j} q_{l_1 \ldots l_j})]^{1 + \alpha + \ldots + \alpha^k} (2K)^{\alpha^{k+1}} (x_1 \ldots x_n)^{(\alpha - \beta)(1 + \alpha + \ldots + \alpha^k) + 1}$$

for $k = 0, 1, \ldots$ and $j = 1, 2, \ldots, n-1$ in R^0 by the mathematical induction with respect to k. The inequality

$$(14) \qquad \sum_{j=0}^{n-1} (\mathbf{P}^j, |\mathbf{D}^j u_{k+3} - \mathbf{D}^j u_{k+2}|) \leqq [C(q_0 + \sum_{j=1}^{n-1} \sum_{l_1, \ldots, l_j} q_{l_1 \ldots l_j})]^{1 + \alpha + \ldots + \alpha^k} \cdot$$
$$\cdot \left(p_0 + \sum_{j=1}^{n-1} \sum_{l_1, \ldots, l_j} \frac{p_{l_1 \ldots l_j}}{\sqrt[n]{L^j}}\right) (2K)^{\alpha^{k+1}} (x_1 \ldots x_n)^{(\alpha - \beta)(1 + \alpha + \ldots + \alpha^k) + 1}$$

follows for $X \in R^0$ by estimate (13). The condition $p^n L(1 - \alpha)^n < (1 - \beta)^n$ guarantees the existence of such a number $N(p)$ that

$$(\alpha - \beta)(1 + \alpha + \ldots + \alpha^k) + 1 = (1 - \beta)(1 + \alpha + \ldots + \alpha^k) + \alpha^{k+1} =$$
$$= \frac{1 - \beta}{1 - \alpha}(1 - \alpha^{k+1}) + \alpha^{k+1} > p \sqrt[n]{L}$$

for all $k \geqq N(p)$. Consequently we have $d(u_{k+1}, u_k) < +\infty$ for $k \geqq N(p) + 2$. On the basis of the property c) of the distance (7) we conclude that condition 2° is proved.

Let us suppose that $u, v \in Y$ are two fixed points of the mapping T, i.e. $Tu = u$, $Tv = v$. Using the method form the proof of condition 2° we obtain for the difference of the function u, v and their partial derivatives estimates (13) and (14). Hence the third condition of Theorem 1 follows; $d(u, v) < +\infty$.

Now we easily conclude that there exists one and only one fixed point of operator (10). The sequence of successive approximations (4) due to any initial function $u_0(X) \in \in M(R)$ converges in the sense of the distance (7) to this solution. On the basis of relation (8) for any function $G_0(X) \in M(R)$ with the derivative $\mathbf{D}^n G_0(X) = 0$ in R Theorem 2 is proved.

In the following two theorems we shall generalize the Nagumo-Perron-van Kampen assumption of the paper [5] and use it to consider the convergence of successive approximations of the Darboux problem (1), (2). Before we pronounce this theorems let us define the space $(M^*(R), d_2)$.

Let the operator T be defined by the relation (10) and $T\,M(R)$ is the set of all the m ages of the set $M(R)$ under mapping T.

Let the symbol $(M^*(R), d_2)$ denote the complete metric space which we obtain by the completion of the metric space $(TM(R), d_2)$ in the sense of the distance

$$(15) \qquad d_2(u, v) = \max_R \Big[\sum_{j=0}^{n-1} (\boldsymbol{I}^j, |\mathbf{D}^j u - \mathbf{D}^j v|)\Big]$$

where $\boldsymbol{I}^j = (1, \ldots, 1)$ denotes the unit vector with $\binom{n}{j}$ components for $j = 1, 2, \ldots \ldots, n - 1$ and $\boldsymbol{I}^0 = 1$.

Then easy considerations lead to the following results:

If the sequence $\{u_k(X)\}_1^\infty$ of functions $u_k(X) \in M^*(R)$ converges in the distance (15) to a function $u(X) \in M^*(R)$, then this sequence and the sequence of the derivatives $\{D_{l_1 \ldots l_j} u_k(X)\}_{k=1}^\infty$ converge in the sense of the distance (8_1) for $j = 1, 2, \ldots, n - 1$ and there is $\lim\limits_{k\to\infty} d_1(u_k, u) = 0$, $\lim\limits_{k\to\infty} d_1(D_{l_1 \ldots l_j} u_k, D_{l_1 \ldots l_j} u) = 0$. Conversely, the convergence of the sequence $\{u_k(X)\}_{k=1}^\infty$ and of the sequences of its derivatives $\{D_{l_1 \ldots l_j} u_k(X)\}_{k=1}^\infty$ in the distance (8_1) for $j = 1, 2, \ldots, n - 1$ implies the convergence of the sequence $\{u_k(X)\}_1^\infty$ in the sense of the distance (15).

Theorem 3. *Let the function $f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})$ be defined and continuous on the domain E and let it satisfy the following assumptions:*

$$(16) \qquad |f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})| \leqq A(x_1 \ldots x_n)^p, \quad p \geqq 0, \quad A > 0$$

in E and

$$(17) \qquad |f(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}) - f(X, \mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^{n-1})| \leqq$$
$$\leqq \frac{C}{(x_1 \ldots x_n)^r} \sum_{j=0}^{n-1} (\mathbf{F}_q^j, |\mathbf{U}^j - \mathbf{V}^j|^q), \quad q \geqq 1, \quad c > 0$$

on E^0 where $\mathbf{F}_q^j = (f_{1\ldots j}(x_1 \ldots x_j)^q, \ldots, f_{l_1\ldots l_j}(x_{l_1} \ldots x_{l_j})^q, \ldots, f_{n-j+1\ldots n}(x_{n-j+1} \ldots x_n)^q)$ denotes the vector with $\binom{n}{j}$ non-negative components $f_{l_1\ldots l_j}$ for $j = 1, 2, \ldots, n-1$, $\mathbf{F}_q^0 = f_0 \geqq 0$ satisfying the condition

$$(f_0 + \sum_{j=1}^{n-1} \sum_{l_1,\ldots,l_j} f_{l_1\ldots l_j})\, C \frac{(2A)^{q-1}}{(p+1)^q} < 1$$

where $q(1+p) - r = p$. At least one of the constants $f_{l_1\ldots l_j}, f_0$ is nonvanishing. Then there exists one and only one solution $u(X)$ of the Darboux problem (1), (2) and, moreover, the Picard's sequence of successive approximations (4), for arbitrary functions $u_0(X)$, $G_0(X) \in M^*(R)$ such that $\mathbf{D}^n G_0(X) = 0$ in R converges uniformly in the domain R to this unique solution.

Proof. The proof will be given again by Theorem 1. First of all by (15) it is evident that $M^*(R) \subseteq M(R)$. Let us consider the metric space $Y = (M^*(R), d)$ with the distance

$$(18) \qquad d(u, v) = \sup_{R^\circ} \frac{\sum_{j=0}^{n-1} (\mathbf{F}_1^j, |\mathbf{D}^j u - \mathbf{D}^j v|)}{(x_1 \ldots x_n)^{p+1}}$$

and the operator T defined by relation (10). Hence there is $TY \subseteq Y$. The inequality

$$(19) \qquad \max_R \sum_{j=0}^{n-1} (\bar{\mathbf{S}}^j, |\mathbf{D}^j u - \mathbf{D}^j v|) \leqq d(u, v)$$

is obtained similarly as that of Theorem 2. $\bar{\mathbf{S}}^j = (\bar{s}_{1\ldots j}, \ldots, \bar{s}_{l_1\ldots l_j}, \ldots, \bar{s}_{n-j+1\ldots n})$ and $\bar{\mathbf{S}}^0 = \bar{s}_0$ denote the vectors with $\binom{n}{j}$ constant components $\bar{s}_{l_1\ldots l_j}$, $\bar{s}_0$ for $j = 1, 2, \ldots, n-1$ at least one of which is non-vanishing. The constants $\bar{s}_0, \bar{s}_{l_1\ldots l_j}$ depend on $f_{l_1\ldots l_j}$, f_0 and A_i, $i = 1, 2, \ldots, n$. From relations (15), (19) there follows that the d-convergence of the sequence $\{u_k(X)\}_1^\infty$ of functions $u_k(X) \in M^*(R)$ implies the d_2-convergence of this sequence.

Let now the sequence $\{u_k(X)\}_1^\infty$ be a d-Cauchy sequence, i.e. $\lim_{k,m\to\infty} d(u_k, u_m) = 0$. Then this sequence converges to a function $u(X) \in M^*(R)$ in the metrics (15) and $\lim_{k\to\infty} u_k(X) = u(X)$, $\lim_{k\to\infty} D_{l_1\ldots l_j} u_k(X) = D_{l_1\ldots l_j} u(X)$ for $j = 1, 2, \ldots, n-1$ in the domain R. Similar calculations to those of Theorem 2 lead us to the conclusion that $\lim_{k\to\infty} d(u_k, u) = 0$. Consequently $Y = (M^*(R), d)$ is a generalized complete metric space.

Proof of condition 1°. Let $u(X)$, $v(X)$ be arbitrary functions from Y with $d(u, v) < +\infty$. The completeness of the space Y and equations (10), (10_1) together with the assumption (16) guarantee that

$$
(20) \qquad |u(X) - v(X)| \leqq \frac{2A}{p+1} (x_1 \ldots x_n)^{p+1},
$$

$$
x_{l_1} \ldots x_{l_j} |D_{l_1\ldots l_j} u(X) - D_{l_1\ldots l_j} v(X)| \leqq \frac{2A}{p+1} (x_1 \ldots x_n)^{p+1}
$$

for $j = 1, 2, \ldots, n-1$ on the domain R. It follows by (17), (20) and the relation $M^*(R) \subseteq M(R)$ that

$$
|Tu(X) - Tv(X)| \leqq C \int_R \frac{\sum_{j=0}^{n-1} (\mathbf{F}_q^j, |\mathbf{D}^j u - \mathbf{D}^j v|)}{(\xi_1 \ldots \xi_n)^r} \, \mathrm{d}\Xi \leqq
$$

$$
\leqq C \left(\frac{2A}{p+1}\right)^{q-1} \int_R \frac{\sum_{j=0}^{n-1} (\mathbf{F}_1^j, |\mathbf{D}^j u - \mathbf{D}^j v|)}{(\xi_1 \ldots \xi_n)^{p+1}} (\xi_1 \ldots \xi_n)^{(p+1)(q-1)-r+p+1} \, \mathrm{d}\Xi \leqq
$$

$$
\leqq C \frac{(2A)^{q-1}}{(p+1)^q} d(u, v) (x_1 \ldots x_n)^{p+1}.
$$

Similarly, it is possible to show that

$$
x_{l_1} \ldots x_{l_j} |D_{l_1\ldots l_j} Tu - D_{l_1\ldots l_j} Tv| \leqq C \frac{(2A)^{q-1}}{(p+1)^q} d(u, v) (x_1 \ldots x_n)^{p+1}
$$

for $j = 1, 2, \ldots, n-1$ in R^0. From the last inequalities we obtain

$$
d(Tu, Tv) \leqq \left(f_0 + \sum_{j=1}^{n-1} \sum_{l_1,\ldots,l_j} f_{l_1\ldots l_j}\right) C \frac{(2A)^{q-1}}{(p+1)^q} d(u, v).
$$

This proves condition 1°.

The proofs of conditions 2° and 3° are trivial in this case, as the inequality

$$d(u_k, u_{k+1}) \leqq \left(f_0 + \sum_{j=1}^{n-1} \sum_{l_1,\dots,l_j} f_{l_1\dots l_j}\right) \frac{2A}{p+1} < +\infty\,, \quad k = 1, 2, \dots$$

is directly given for any Picard's sequence $\{u_k = Tu_{k-1}\}_{k=1}^{\infty}$ due to an arbitrary initial function $u_0(X) \in Y$ by (14).

Remark. Assumption (16) of Theorem 3 guarantees the boundedness of the function $f(X, \mathbf{U}^0, \mathbf{U}^1, \dots, \mathbf{U}^{n-1})$ in the domain E. In the following theorem we shall show that the assumption of the boundedness is not necessary.

Theorem 4. *Let the function $f(X, \mathbf{U}^0, \mathbf{U}^1, \dots, \mathbf{U}^{n-1})$ be continuous on E and let it satisfy the following conditions:*

$$(21) \qquad |f(X, \mathbf{U}^0, \mathbf{U}^1, \dots, \mathbf{U}^{n-1})| \leqq A(X)\,(x_1 \dots x_n)^p\,, \quad -1 < p < 0$$

in E^0. The function $A(X)$ is integrable on the domain R and in the $(n-j)$-dimensional domain $R_{l_1\dots l_j}$ for any $(x_{l_1}, \dots, x_{l_j})$ with $0 \leqq x_{l_k} \leqq A_k$ where $k = 1, 2, \dots, j$ and $j = 1, 2, \dots, n-1$. Moreover, the inequalities $0 \leqq A(X) \leqq A_0$, $A(X) \leqq \leqq A_0(x_{l_1} \dots x_{l_j})^{-p}$, $A_0 > 0$ are fulfilled for $j = 1, 2, \dots, n-1$ on R. Let, further, the inequality

$$(22) \qquad |f(X, \mathbf{U}^0, \mathbf{U}^1, \dots, \mathbf{U}^{n-1}) - f(X, \mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^{n-1})| \leqq$$
$$\leqq \frac{C(X)}{(x_1 \dots x_n)^r} \sum_{j=0}^{n-1} (\mathbf{H}^j_{p,q}, |\mathbf{U}^j - \mathbf{V}^j|^q)\,, \quad q \geqq 1$$

hold in E^0. The function $C(X)$ is also integrable on R and on $R_{l_1\dots l_j}$ for any $(x_{l_1}, \dots \dots, x_{l_j})$ with $0 \leqq x_{l_k} \leqq A_k$ where $k = 1, 2, \dots, j$ and $j = 1, 2, \dots, n-1$, moreover the inequalities $0 \leqq C(X) \leqq C_0$, $C(X) \leqq C_{l_1\dots l_j}(x_{l_1} \dots x_{l_j})^{-p}$ for $j = 1, 2, \dots, n-1$ hold where C_0, $C_{l_1\dots l_j}$ are positive constants. $\mathbf{H}^0_{p,q} = h_0$ and for $j = 1, 2, \dots, n-1$

$$\mathbf{H}^j_{p,q} = (h_{1\dots j}(x_1 \dots x_j)^{q(p+1)}, \dots, h_{l_1\dots l_j}(x_{l_1} \dots x_{l_j})^{q(p+1)}, \dots$$
$$\dots, h_{n-j+1}(x_{n-j+1} \dots x_n)^{q(p+1)})$$

denote the vectors with $\binom{n}{j}$ non-negative components h_0, $h_{l_1\dots l_j}$ at least one of which is non-vanishing.

If furthermore we suppose that $q(p+1) - r = p$ and

$$\left[C_0 h_0 + \sum_{j=1}^{n-1} \sum_{l_1,\dots,l_j} (p+1)^j\, C_{l_1\dots l_j} h_{l_1\dots l_j}\right] \frac{(2A_0)^{q-1}}{(p+1)^{nq}} < 1$$

then there exists one and only one solution of the Darboux problem (1), (2) *and the Picard's sequence of successive approximations* (4), *for arbitrary functions* $u_0(X)$, $G_0(X) \in M^*(R)$ *with* $\mathbf{D}^n G_0(X) = 0$ *in* R, *uniformly converges in the domain* R *to this unique solution.*

Proof. Analogously to Theorem 3 it can be shown that the metric space $Y = (M^*(R), d)$ on which the distance

$$d(u, v) = \sup_{R^\circ} \frac{\sum_{j=0}^{n-1} (\mathbf{H}^j_{p,1}, |\mathbf{D}^j u - \mathbf{D}^j v|)}{(x_1 \dots x_n)^{p+1}} \tag{23}$$

is defined, forms the complete generalized metric space. The operator T defined by relation (10) maps the space Y into itself. Then we obtain from (10), (10_1) and from assumption (21) for arbitrary $u, v \in Y$ with $d(u, v) < +\infty$ the inequalities

$$|u(X) - v(X)| \leqq 2 \int_R A(\Xi)\,(\xi_1 \dots \xi_n)^p \,\mathrm{d}\Xi \leqq \frac{2A_0}{(p+1)^n} (\xi_1 \dots \xi_n)^{p+1} \tag{24}$$

$$(x_{l_1} \dots x_{l_j})^{p+1} |D_{l_1 \dots l_j} u(X) - D_{l_1 \dots l_j} v(X)| \leqq$$

$$\leqq 2(x_{l_1} \dots x_{l_j})^{p+1} \int_{R_{l_1 \dots l_j}} A(\Xi^x_{l_1 \dots l_j})\,(\xi_1 \dots \xi_{l_j - 1} x_{l_j} \xi_{l_j + 1} \dots \xi_n)^p \,\mathrm{d}\Xi_{l_1 \dots l_j} \leqq$$

$$\leqq \frac{2A_0}{(p+1)^n} (x_1 \dots x_n)^{p+1}$$

in R^0. On the basis of assumption (22) and of inequalities (24) we get the following estimates

$$|Tu - Tv| \leqq \int_R \frac{C(\Xi) \sum_{j=0}^{n-1} (\mathbf{H}^j_{p,q}, |\mathbf{D}^j u - \mathbf{D}^j v|)}{(\xi_1 \dots \xi n)^r} \,\mathrm{d}\Xi \leqq$$

$$\leqq \left[\frac{2A_0}{(p+1)^n}\right]^{q-1} \int_R \frac{C(\Xi) \sum_{j=0}^{n-1} (\mathbf{H}^j_{p,1}, |\mathbf{D}^j u - \mathbf{D}^j v|)}{(\xi_1 \dots \xi_n)^{p+1}} (\xi_1 \dots \xi_n)^{(p+1)(q-1) - r + p + 1} \,\mathrm{d}\Xi \leqq$$

$$\leqq \frac{(2A_0)^{q-1}}{(p+1)^{nq}} C_0\, d(u, v)\,(x_1 \dots x_n)^{p+1}, \quad (x_{l_1} \dots x_{l_j})^{p+1} |D_{l_1 \dots l_j} Tu - D_{l_1 \dots l_j} Tv| \leqq$$

$$\leqq \frac{(2A_0)^{q-1}}{(p+1)^{nq}} C_{l_1 \dots l_j} (p+1)^j\, d(u, v)\,(x_1 \dots x_n)^{p+1}$$

for $X \in R^0$. Hence there follows that $d(Tu, Tv) \leqq \lambda\, d(u, v)$ where

$$\lambda = \left[C_0 h_0 + \sum_{j=1}^{n-1} \sum_{l_1,\ldots,l_j} (p+1)^j C_{l_1\ldots l_j} h_{l_1\ldots l_j}\right] \frac{(2A_0)^{q-1}}{(p+1)^{nq}}\,.$$

We shall obtain the required estimate for the proofs of conditions 2°, 3° directly by (24). Thereby Theorem 4 is proved.

5. Systems of differential equations. The results of the preceding Theorems 2, 3, 4 can be applied to certain systems of hyperbolic partial differential equations.

First of all we introduce some new notation and assumptions again.

1. We shall consider the sets $E_1 = R \times \prod_{i=1}^{s_1} \{-\infty < z_i < +\infty\}$, $E_1^0 = R^0 \times \prod_{i=1}^{s_1} \{-\infty < z_i < +\infty\}$ where $s_1 = m + m\binom{n}{1} + \ldots + m\binom{n}{n-1} = m(2^n - 1)$ and $m \geqq 1$ denotes an integer. Further, let us denote $\Delta = \bigcup_{i=1}^{n} \delta_i$ where $\delta_i = \{X : X \in R, x_i = 0\}$ for $i = 1, 2, \ldots, n$.

2. Let the norm of the vector $\boldsymbol{B} = (b_1, \ldots, b_t)$ be defined by equation

$$\|\boldsymbol{B}\| = \sum_{j=1}^{t} |b_j|\,.$$

3. Let a)

$$\mathbf{U}^j = \begin{pmatrix} u^1_{1\ldots j} \ldots u^m_{1\ldots j} \\ \cdots\cdots\cdots\cdots \\ u^1_{l_1\ldots l_j} \ldots u^m_{l_1\ldots l_j} \\ \cdots\cdots\cdots\cdots \\ u^1_{n-j+1\ldots n} \ldots u^m_{n-j+1\ldots n} \end{pmatrix} = (\boldsymbol{U}_{1\ldots j}, \ldots, \boldsymbol{U}_{l_1\ldots l_j}, \ldots, \boldsymbol{U}_{n-j+1\ldots n})$$

denote an arbitrary matrix of the type $\frac{1}{m}\binom{n}{j}$ for $j = 1, 2, \ldots, n-1$ and $\mathbf{U}^0 = (u^1_0, \ldots, u^m_0)$. The symbol $\boldsymbol{U}_{l_1\ldots l_j}$ denotes the vector $(u^1_{l_1\ldots l_j}, \ldots, u^m_{l_1\ldots l_j})$ and $\boldsymbol{U}_0 = \mathbf{U}^0$.

Let us denote the vector $(\|\boldsymbol{U}_{1\ldots j}\|^\gamma, \ldots, \|\boldsymbol{U}_{l_1\ldots l_j}\|^\gamma, \ldots, \|\boldsymbol{U}_{n-j+1\ldots n}\|^\gamma)$ for $j = 0, 1, \ldots, n-1$ and a real number γ by $\|\mathbf{U}^j\|^\gamma$.

b) $\boldsymbol{U}(X) = (u^1(X), \ldots, u^m(X))$ let be a sufficiently regular vector function in the domain R. Then, let us denote

$$\mathbf{D}^j\boldsymbol{U} = \begin{pmatrix} D_{1\ldots j}u^1 \ldots D_{1\ldots j}u^m \\ \cdots\cdots\cdots\cdots \\ D_{l_1\ldots l_j}u^1 \ldots D_{l_1\ldots l_j}u^m \\ \cdots\cdots\cdots\cdots \\ D_{n-j+1\ldots n}u^1 \ldots D_{n-j+1\ldots n}u^m \end{pmatrix} = (D_{1\ldots j}\boldsymbol{U}, \ldots, D_{l_1\ldots l_j}\boldsymbol{U}, \ldots, D_{n-j+1\ldots n}\boldsymbol{U})\,.$$

4. We shall suppose that the vector function

$$\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}) = (f_1(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}), \ldots, f_m(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}))$$

of $n + s_1$ variables is continuous in E_1.

5. Let us assume that the vector function $\boldsymbol{\Phi}(X) = (\Phi^1(X), \ldots, \Phi^m(X))$ is defined and continuous in the domain Δ and it has continuous derivatives $D_{i_1 \ldots i_j} \boldsymbol{\Phi}(X)$ of the j-th order in any domain δ_i, $i = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, n - 1$ and that $\mathbf{D}^n \boldsymbol{\Phi}(X) = \mathbf{O} = (0, \ldots, 0)$ on R.

6. Further, let $M_1(R)$ denote the set of the vector functions $\mathbf{Z}(X) = (Z_1(X), \ldots \ldots, Z_m(X)) \in C(R)$ with the following properties:

a) The derivatives $D_{i_1 \ldots i_j} \mathbf{Z}$ are continuous in the domain R for $j = 1, 2, \ldots, n - 1$.

b) $\mathbf{Z}(X) = \boldsymbol{\Phi}(X)$ for $X \in \Delta$.

We are now able to formulate the Darboux problem and the concept of its solution.

We shall understand by the solution of the Darboux problem

$$(1') \qquad \mathbf{D}^n \mathbf{U} = \mathbf{F}(X, \mathbf{U}, \mathbf{D}^1 \mathbf{U}, \ldots, \mathbf{D}^{n-1} \mathbf{U})$$

$$(2') \qquad \mathbf{U}(X) = \boldsymbol{\Phi}(X) \quad \text{for} \quad X \in \Delta$$

any function $\mathbf{U}(X) \in M_1(R)$ which has the continuous derivative $\mathbf{D}^n \mathbf{U}$ on R and satisfies equation (1′) in R.

The Darboux problem (1′), (2′) is equivalent to solving the system of integro-differential equations

$$(3') \qquad \mathbf{U}(X) = \boldsymbol{\Phi}_0(X) + \int_R \mathbf{F}(\Xi, \mathbf{U}, \mathbf{D}^1 \mathbf{U}, \ldots, \mathbf{D}^{n-1} \mathbf{U}) \, \mathrm{d}\Xi$$

where $\boldsymbol{\Phi}_0(X) = \boldsymbol{\Phi}(0, x_2, \ldots, x_n) + \boldsymbol{\Phi}(x_1, \ldots, x_{n-1}, 0) - [\boldsymbol{\Phi}(0, 0, x_3, \ldots, x_n) + \ldots \ldots + \boldsymbol{\Phi}(x_1, \ldots, x_{n-2}, 0, 0)] + \ldots + (-1)^{n-1} \boldsymbol{\Phi}(0, \ldots, 0)$.

Then, the Picard's sequence of successive approximations $\{\mathbf{U}_k\}_1^\infty$ shall be defined by the equation

$$(4') \qquad \mathbf{U}_k(X) = \boldsymbol{\Phi}_0(X) + \int_R \mathbf{F}(\Xi, \mathbf{U}_{k-1}, \mathbf{D}^1 \mathbf{U}_{k-1}, \ldots, \mathbf{D}^{n-1} \mathbf{U}_{k-1}) \, \mathrm{d}\Xi$$

for any function $\mathbf{U}_0 \in M_1(R)$ and $k = 1, 2, \ldots$

Now let us state the following theorems:

Theorem 5. *Let the vector function* $\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})$ *be defined, continuous and bounded in the domain* E_1 *and let it satisfy the conditions:*

$$(5') \qquad \|\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}) - \mathbf{F}(X, \mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^{n-1})\| \leqq \leqq \frac{L}{x_1 \ldots x_n} \sum_{j=0}^{n-1} (\mathbf{P}^j, \|\mathbf{U}^j - \mathbf{V}^j\|), \quad L > 0,$$

$$(6') \qquad \|\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}) - \mathbf{F}(X, \mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^{n-1})\| \leqq \leqq \frac{C}{x_1^\beta \ldots x_n^\beta} \sum_{j=0}^{n-1} (\mathbf{Q}^j, \|\mathbf{U}^j - \mathbf{V}^j\|^\alpha), \quad C > 0$$

where $\mathbf{P}^j$, $\mathbf{Q}^j$ *denote the vectors from Theorem 2, in* E_1^0. *If the inequalities* $0 < \alpha < 1$, $L(1 - \alpha)^n < (1 - \beta)^n$, $(p_0 + \sum_{j=1}^{n-1} \sum_{l_1, \ldots, l_j} p_{l_1 \ldots l_j})^n L(1 - \alpha)^n < (1 - \beta)^n$ *hold, then* $\beta < \alpha$, *there exists one and only one solution of the Darboux problem* (1'), (2') *and the Picard's sequence of successive approximations* (4') *converges in the sense of the norm* $\| \;\|$ *defined above to this unique solution for any initial function* $\mathbf{U}_0(X) \in \in M_1(R)$ *on* R.

If we choose the generalized metric space $Y = (M_1(R), d)$ with the metrics

$$d(\mathbf{U}, \mathbf{V}) = \sup_{R^*} \frac{\sum_{j=0}^{n-1} (\mathbf{P}^j, \|\mathbf{D}^j\mathbf{U} - \mathbf{D}^j\mathbf{V}\|)}{(x_1 \ldots x_n)^{p \sqrt[n]{L}}}$$

where p fulfils the same conditions as in Theorem 1, then the proof of this theorem should proceed similarly with the proof of Theorem 2.

Let $T\; M_1(R)$ denote the set of all the images of the set $M_1(R)$ in the mapping

$$(7') \qquad T\,\mathbf{U}(X) = \Phi_0(X) + \int_R \mathbf{F}(\Xi, \mathbf{U}, \mathbf{D}^1\mathbf{U}, \ldots, \mathbf{D}^{n-1}\mathbf{U})\, \mathrm{d}\Xi\,.$$

If we denote the complete metric space which was obtained by the completion of the metric space $(T\; M_1(R), d_3)$ in the sense of the distance

$$(8') \qquad d_3(\mathbf{U}, \mathbf{V}) = \max_R \sum_{j=0}^{n-1} (\mathbf{I}^j, \|\mathbf{D}^j\mathbf{U} - \mathbf{D}^j\mathbf{V}\|)$$

by $(M_1^*(R), d_3)$, then the following theorems hold:

Theorem 6. *Let the vector function* $\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})$ *be defined and continuous in the domain* E_1 *and let it satisfy the assumptions*

$$(9') \qquad \|\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})\| \leqq A(x_1 \ldots x_n)^p, \quad p \geqq 0, \quad A > 0$$

in E_1 and

(10′) $$\|\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}) - \mathbf{F}(X, \mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^{n-1})\| \leqq$$
$$\leqq \frac{C}{(x_1 \ldots x_n)^r} \sum_{j=0}^{n-1} (\mathbf{F}_q^j, \|\mathbf{U}^j - \mathbf{V}^j\|^q), \quad q \geqq 1, \quad C > 0$$

in E_1^0 where F_q^j denotes the vector from Theorem 3. If the conditions $q(1 + p) - r = p$

$$\left(f_0 + \sum_{j=1}^{n-1} \sum_{l_1,\ldots,l_j} f_{l_1 \ldots l_j}\right) C \frac{(2A)^{q-1}}{(p + 1)^q} < 1$$

are fulfilled, then there exists one and only one solution $\mathbf{U}(X)$ of the Darboux problem (1′), (2′) and furthermore the Picard's sequence of successive approximations (4′) for any initial function $\mathbf{U}_0(X) \in M_1^*(R)$ converges in the sense of the norm $\| \ \|$ to this unique solution on R.

Theorem 7. *Let the vector function $\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})$ be continuous in E_1 and in the domain E_1^0 let is satisfy the conditions*

(11′) $$\|\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1})\| \leqq A(X) (x_1 \ldots x_n)^p, \quad -1 < p < 0$$

where the scalar function $A(X)$ is integrable in the domains R, $R_{l_1 \ldots l_j}$ and, moreover, it fulfils the inequalities $0 \leqq A(X) \leqq A_0$, $A(X) \leqq A_0(x_{l_1} \ldots x_{l_j})^{-p}$, $A_0 > 0$ for $j = 1, 2, \ldots, n - 1$ in E_1.

(12′) $$\|\mathbf{F}(X, \mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^{n-1}) - \mathbf{F}(X, \mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^{n-1})\| \leqq$$
$$\leqq \frac{C(X)}{(x_1 \ldots x_n)^r} \sum_{j=0}^{n-1} (\mathbf{H}_{p,q}^j, \|\mathbf{U}^j - \mathbf{V}^j\|^q), \quad q \geqq 1$$

where $\mathbf{H}_{p,q}^j$ denotes as defined the vector as in Theorem 4. The scalar function $C(X)$ is integrable in R and $R_{l_1 \ldots l_j}$ for $j = 1, 2, \ldots, n - 1$. Moreover, let it fulfil the inequalities $0 \leqq C(X) \leqq C_0$, $C(X) \leqq C_{l_1 \ldots l_j}(x_{l_1} \ldots x_{l_j})^{-p}$ where $C_0, C_{l_1 \ldots l_j}$ are positive constants for $j = 1, 2, \ldots, n - 1$ in R. Further, if

$$\left[C_0 h_0 + \sum_{j=1}^{n-1} \sum_{l_1,\ldots,l_j} (p + 1)^j C_{l_1 \ldots l_j} h_{l_1 \ldots l_j}\right] \frac{(2A_0)^{q-1}}{(p + 1)^{nq}} < 1$$

and $q(p + 1) - r = p$, then there exists one and only one solution of the Darboux problem (1′), (2′) and the Picard's sequence of successive approximations by (4′) for any initial function $\mathbf{U}_0(X) \in M_1^*(R)$ converges in the sense of the norm $\| \ \|$ to this unique solution on R.

We omit the proofs of Theorems 6, 7 because if we choose a suitable metrics on $M_1^*(R)$ they would proceed similarly to the proofs of Theorems 3 and 4.

Remark. In Theorems 5, 6, 7 an arbitrary norm $\|\boldsymbol{B}\|_1$ which is equivalent to the norm $\|\boldsymbol{B}\| = \sum_{j=1}^{t} |b_j|$ (in the sense of convergence) can be taken instead of the norm $\|\boldsymbol{B}\|$.

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