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Časopis pro pěstování matematiky, roč. 107 (1982), Praha

# A NOTE ON HIGHER MONOTONICITY PROPERTIES OF CERTAIN STURM-LIOUVILLE FUNCTIONS II

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## 1. DEFINITIONS AND NOTATION

A function $\varphi(x)$ is said to be $n$-times monotonic (or monotonic of order $n$) on an interval $I$ if

$$(1.1)\qquad (-1)^i \varphi^{(i)}(x) \geqq 0\,, \quad i = 0, 1, \ldots, n\,; \quad x \in I\,.$$

For such a function we write $\varphi(x) \in M_n(I)$ or $\varphi(x) \in M_n(a, b)$ provided $I$ is an open interval $(a, b)$. If the strict inequality holds throughout (1.1) we write $\varphi(x) \in M_n^*(I)$ or $\varphi(x) \in M_n^*(a, b)$. We say that $\varphi(x)$ is completely monotonic on $I$ if (1.1) holds for $n = \infty$.

A sequence $\{\mu_k\}_{k=1}^{\infty}$ denoted simply by $\{\mu_k\}$, is said to be $n$-times monotonic if

$$(1.2)\qquad (-1)^i \Delta^i \mu_k \geqq 0\,, \quad i = 0, 1, \ldots, n\,; \quad k = 1, 2, \ldots\,.$$

Here $\Delta\mu_k = \mu_{k+1} - \mu_k$, $\Delta^2\mu_k = \Delta(\Delta\mu_k)$ etc. For such a sequence we write $\{\mu_k\} \in M_n$. If the strict inequality holds throughout (1.2) we write $\{\mu_k\} \in M_n^*$. $\{\mu_k\}$ is called completely monotonic if (1.2) holds for $n = \infty$.

As usual, we write $[a, b)$ to denote the interval $\{x \mid a \leqq x < b\}$. $\varphi(x) \in C_n(I)$ means that $\varphi(x)$ has continuous derivatives up to and including the $n$-th order.

$$D_x(\varphi(x)) \text{ denotes the first derivative } \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\,,$$

$$D_x^n(\varphi(x) \text{ denotes the } n\text{-th derivative } \frac{\mathrm{d}^n\varphi(x)}{\mathrm{d}x^n}\,.$$

## 2. PRELIMINARY REMARKS

Consider a differential equation

$$(2.1)\qquad [g(x)\, y']' + f(x)\, y = 0$$

with $f(x)$ and $g(x)$ continuous ($g(x) > 0$) for $a < x < b$. Let $y_1(x)$, $y_2(x)$ be linearly independent solutions of (2.1) on an open interval $(a, b)$, not necessarily $(a, \infty)$.

Let

(2.2) $$p(x) = y_1^2(x) + y_2^2(x),$$

and let $p^{(N)}(x)$ exist for a positive integer $N$ (of course $f(x), g(x) \in C_{N-2}(a, b)$). The function $y(x)$ is an arbitrary non-trivial solution of (2.1) on $(a, b)$ and $\{x_1, x_2, \ldots\}$ denotes any finite or infinite increasing sequence of consecutive zeros on $(a, b)$ of a non-trivial solution $z(x)$ of (2.1). We put

(2.3) $$w(x) = y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x),$$

where the solutions $y_1(x), y_2(x)$ are normalized so that $w(x) > 0$. In [3]([5] §5) and [6] the results concerning higher monotonicity properties of certain sequences depending on $\{x_1, x_2, \ldots\}$ were inferred from certain hypotheses on the function $p(x)$. In this paper these results will be extended by means of Kummer transformation.

## 3. PRINCIPAL RESULTS

**Theorem 3.1.** *Suppose that $y_1(x), y_2(x)$ are linearly independent solutions of (2.1) on $(a, b)$ and that $p(x)$ is defined by (2.2) and $w(x)$ by (2.3). Suppose that also $W(x)$ and $\psi(x)$ are functions chosen in such a way that the integrals involved in (3.1), (3.2) and (3.3) exist and that, for $n = 0, 1, \ldots, N$,*

(3.1) $$\left(p(x)\, w^{-1}(x) \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left[W(x) \left(\frac{p(x)}{\psi^2(x)}\right)^{1+1/2\lambda-\alpha}\right]$$

*has constant* sign $\varepsilon_n = \pm 1$ *on $(a, b)$, where $\lambda > -1$ and $\alpha < 1 + \frac{1}{2}\lambda$. Then*

(3.2) $$\operatorname{sgn}\left\{\Delta^n \int_{x_k}^{x_{k+1}} W(x) \left[\frac{w(x)}{\psi^2(x)}\right] \left[\frac{p(x)}{\psi^2(x)}\right]^{-\alpha} \left|\frac{y(x)}{\psi(x)}\right|^{\lambda} \mathrm{d}x\right\} = \varepsilon_n,$$

$n = 0, 1, \ldots, N,\ k = 1, 2, \ldots$.

If for a given $n$ the expression in (3.1) is strictly positive (negative), the same is true of the corresponding differences in the conclusion of the Theorem 3.1.

Proof. Abel's formula for the Wronskian shows that $w(x) = c/g(x)$, where $c$ is a positive constant. We make the change of variable

(3.3) $$\xi = \int_a^x \frac{\mathrm{d}u}{g(u)\, \psi^2(u)}, \quad \psi > 0,$$

so that the equation (2.1) becomes

(3.4) $$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} + \varphi(\xi)\, \eta = 0,$$

where $\eta(\xi) = y(x)/\psi(x)$ and $\varphi(\xi) = [(g\psi')' + f\psi]\,\psi^3 g$ (see e.g. [1]); hence we have

$$\frac{y_1^2(x) + y_2^2(x)}{\psi^2(x)} = \frac{p(x)}{\psi^2(x)} = \eta_1^2(\xi) + \eta_2^2(\xi) = \pi(\xi)\,.$$

Another change of variable $\eta(\xi) = \sqrt{(\pi(\xi))}\,u(t)$, $\pi(\xi) = \xi'(t)$ transforms the equation (3.4) into $u''(t) + u(t) = 0$ (see [3] p. 59). Since $\pi(\xi) = \xi'(t) > 0$ on $I$, there is a one-to-one correspondence between the zeros of $\zeta(\xi) = z(x)/\psi(x)$ and those of $v(t) = (\sqrt{(\pi(\xi))})^{-1}\,\zeta(\xi)$ ($v(t)$ is the function corresponding to $z(x)$ after these two changes of variables).

But $v(t) = A\cos(t - b)$ where $A$ and $b$ are constants, so that the consecutive zeros $t_k$ of $v(t)$ are equidistant with $\Delta t_k = \pi$, $k = 1, 2, \ldots$, where $t_k$ is the zero of $v(t)$ corresponding to $\xi_k$. If $\eta(\xi) = \zeta(\xi)$, then $u(t) = v(t)$. Thus

$$M_k(W, \lambda) = \int_{x_k}^{x_{k+1}} W(x)\,\frac{w(x)}{\psi^2(x)}\left[\frac{p(x)}{\psi^2(x)}\right]^{-\alpha}\left|\frac{y(x)}{\psi(x)}\right|^{\lambda}\mathrm{d}x =$$

$$= c\int_{\xi_k}^{\xi_{k+1}} W[x(\xi)]\,[\pi(\xi)]^{-\alpha}\,|\eta(\xi)|^{\lambda}\,\mathrm{d}\xi =$$

$$= c\int_{t_k}^{t_{k+1}} W[x(\xi(t))]\,[\xi'(t)]^{1+\frac{1}{2}\lambda-\alpha}\,|u(t)|^{\lambda}\,\mathrm{d}t$$

and as in [4, p. 1245], in virtue of $(\mathrm{d}x/\mathrm{d}\xi)\,.\,(\mathrm{d}\xi/\mathrm{d}t) = g(x)\,p(x) = p(x)\,w^{-1}(x)$ we have

$$\Delta^n M_k = c\pi^n\int_{t_k}^{t_{k+1}} D_t^n\{W[x(\xi(t + \theta n\pi))]\,[\xi'(t + \theta n\pi)]^{1+\frac{1}{2}\lambda-\alpha}\}\,|u(t)|^{\lambda}\,\mathrm{d}t\,,$$

$$0 < \theta(t) < 1\,.$$

Hence we have

$$\operatorname{sgn}\Delta^n M_k = \operatorname{sgn} D_t^n\left\{W(x)\left[\frac{p(x)}{\psi^2(x)}\right]^{1+\frac{1}{2}\lambda-\alpha}\right\}.$$

Since

$$\operatorname{sgn}\Delta M_k = \operatorname{sgn}\frac{\mathrm{d}}{\mathrm{d}x}\left\{W(x)\left[\frac{p(x)}{\psi^2(x)}\right]^{1+\frac{1}{2}\lambda-\alpha}\right\}\frac{\mathrm{d}x}{\mathrm{d}\xi}\cdot\frac{\mathrm{d}\xi}{\mathrm{d}t}\,,$$

we obtain

$$\operatorname{sgn}\Delta^n M_k = \operatorname{sgn}\left(p(x)\,w^{-1}(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left[W(x)\left[\frac{p(x)}{\psi^2(x)}\right]^{1+\frac{1}{2}\lambda-\alpha}\right]$$

and the proof is complete.

**Corollary 3.1.** *Suppose that for a positive integer $N$,*

$$(3.5)\qquad (-1)^n \left(\frac{p(x)}{\psi^2(x)}\right)^{(n)} > 0\,, \quad n = 0, 1\,,$$

$$(-1)^n \left(\frac{p(x)}{\psi^2(x)}\right)^{(n)} \geqq 0\,, \quad n = 2, 3, \ldots, N\,,$$

$$(3.6)\qquad W(x) > 0\,, \quad (-1)^n\, W^{(n)}(x) \geqq 0\,, \quad n = 1, 2, \ldots, N$$

*and*

$$(3.7)\qquad (-1)^n\, D_x^n \left\{\frac{\psi^2(x)}{w(x)}\right\} \geqq 0\,, \quad n = 0, 1, \ldots, N\,.$$

*Then for a fixed $\lambda > -1$ and $\alpha < 1 + \frac{1}{2}\lambda$, we have*

$$(3.8)\qquad (-1)^n\, \Delta^n \int_{x_k}^{x_{k+1}} W(x) \frac{w(x)}{\psi^2(x)} \left[\frac{p(x)}{\psi^2(x)}\right]^{-\alpha} \left|\frac{y(x)}{\psi(x)}\right|^{\lambda} \mathrm{d}x > 0\,,$$

$n = 0, 1, \ldots, N$, $k = 1, 2, \ldots$ .

All of the above remains true if the factor $(-1)^n$ is deleted simultaneously from (3.5), (3.6), (3.7), (3.8).

Proof. It is a question of checking that the present hypotheses imply those in Theorem 3.1. This follows easily from [4, Lemmas 2.1 and 2.2].

Remark 1. If we choose $\psi(x) \equiv 1$ in Theorem 3.1, then we obtain [6, Theorem 3.1] and the same is true for Corollary 3.1.

Example 1. Consider a differential equation

$$(3.9)\qquad y'' + \left[v^2 x^{-2v-2} - \frac{(v^2 - 1)}{4x^2}\right] y = 0$$

on $(0. \infty)$, where $v \neq 0$. It has linearly independent solutions

$$y_1(x) = x^{(v+1)/2} \cos x^{-v}\,, \quad y_2(x) = x^{(v+1)/2} \sin x^{-v}\,,$$

so we may choose $p(x) = x^{v+1}$. If we consider the quantities $M_k$ given by (3.2) for this equation, choosing $W(x) \equiv 1$, $\lambda = 0$, $\alpha = 0$ and $\psi(x) = \sqrt{x}$ for simplicity, we find that

$$\operatorname{sgn} \Delta^n M_k = \operatorname{sgn} \left(x^{v+1} \frac{\mathrm{d}}{\mathrm{d}x}\right)(x^v) = n!\; v^n x^{(n+1)v}\,, \quad n = 0, 1, 2, \ldots\,,$$

where

$$M_k = \int_{x_k}^{x_{k+1}} \frac{1}{x}\, \mathrm{d}x = \ln \frac{x_{k+1}}{x_k}\,.$$

Thus, for $v > 0$ we get

$$\Delta^n \left\{ \ln \frac{x_{k+1}}{x_k} \right\} > 0\,, \quad n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots$$

and for $v < 0$ we obtain

$$(-1)^n \Delta^n \left\{ \ln \frac{x_{k+1}}{x_k} \right\} > 0\,, \quad n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots.$$

Chossing $\psi(x) \equiv 1$ we obtain [6, 2.8].

**Theorem 3.2.** Suppose that solutions $y(x)$ and $z(x)$ of (2.1) are linearly independent and that, for a positive integer $N$, there exists a pair of linearly independent solutions $y_1(x)$, $y_2(x)$ and a function $\psi(x)$ for which

$$(3.10) \qquad (-1)^n \left( \frac{p(x)}{\psi^2(x)} \right)^{(n)} > 0\,, \quad n = 0, 1\,,$$

$$(-1)^n \left( \frac{p(x)}{\psi^2(x)} \right)^{(n)} \geqq 0\,, \quad n = 2, 3, \ldots, N\,,$$

$$(3.11) \qquad (-1)^n D_x^n \left\{ \frac{\psi^2(x)}{w(x)} \right\} \geqq 0\,, \quad n = 0, 1, 2, \ldots, N$$

and for $a < x < b$

$$(3.12) \qquad W(x) > 0\,, \quad (-1)^n W^{(n)}(x) \geqq 0\,, \quad n = 1, 2, \ldots, N\,.$$

Then for any $\alpha > 0$

$$(3.13) \qquad (-1)^n \Delta^n W(x_k) \left| \frac{y(x_k)}{\psi(x_k)} \right|^\alpha > 0\,, \quad \begin{array}{l} n = 0, 1, 2, \ldots, N\,, \\ k = 1, 2, \ldots \end{array}$$

and

$$(3.14) \qquad (-1)^n \Delta^n W(x_k) \left| \frac{w(x_k)}{\psi(x_k)\, z'(x_k)} \right|^\alpha > 0\,, \quad \begin{array}{l} n = 0, 1, 2, \ldots, N\,, \\ k = 1, 2, \ldots. \end{array}$$

All of the above remains true if the factor $(-1)^n$ is deleted simultaneously from (3.10), (3.11), (3.12) and (3.13).

Remark 2. [5, Theorem 5.2] is obtained from Theorem 3.2 by choosing $\psi(x) \equiv 1$.

Proof. Making the change of variable (3.3) we have

$$W(x) \left| \frac{y(x)}{\psi(x)} \right|^\alpha = V(\xi)\, |\eta(\xi)|^\alpha\,,$$

where $V(\xi) = W[x(\xi)]$ is an $N$-times monotonic function of $\xi$. The differential equation now has the form (3.4). Another change of variable $\xi'(t) = \pi(\xi)$, $\eta(\xi) =$

$= \sqrt{(\pi(\xi))}\, u(t)$ reduces it to the form $u''(t) + u(t) = 0$. Thus we have

$$W(x) \left|\frac{y(x)}{\psi(x)}\right|^{\alpha} = V(\xi)\,|\eta(\xi)|^{\alpha} = V(\xi)\,[\pi(\xi)]^{\alpha/2}\,|u(t+B)|^{\alpha},$$

$$W(x_k) \left|\frac{y(x_k)}{\psi(x_k)}\right|^{\alpha} = C\, V[\xi(t_k)]\,\{\pi[\xi(t_k)]\}^{\alpha/2},$$

where $C \in \langle 0, 1\rangle$ is a constant independent of $k$. As in the proof of Theorem 3.1 we have $\pi(\xi) > 0$, $\pi'(\xi) < 0$, $\pi(\xi) \in M_N(\xi(a), \xi(b))$, $V(\xi) > 0$, $V(\xi) \in M_N(\xi(a), \xi(b))$. Hence if we write $q(\xi) = V(\xi)\,[\pi(\xi)]^{\alpha/2}$ we get using [5, Lemma 2.2]

$$(-1)^n D_t^n[q(\xi)] > 0\,, \quad n = 0, 1, 2, \ldots, N\,. \tag{3.15}$$

We have for $n = 0, 1, 2, \ldots, N$

$$(-1)^n \Delta^n W(x_k) \left|\frac{y(x_k)}{\psi(x_k)}\right|^{\alpha} = (-1)^n C\, D_t^n\{q[\xi(t_k + \theta n\pi)]\}\,, \quad 0 < \theta(t) < 1$$

by using the mean value theorem for higher order derivatives and differences, see [3, p. 60]. Hence in view of (3.15) we get the desired (3.13).

The result (3.14) easily follows from the Wronskian of $y(x)$ and $z(x)$ ($w(x) = y'(x)\,z(x) - y(x)\,z'(x)$) by dividing it by $\psi(x)$ and substituting $x = x_k$, so that $z(x_k) = 0$ and $y(x_k)/\psi(x_k) = w(x_k)/(\psi(x_k)\,z'(x_k))$. The Wronskian of $y(x), z(x)$ is a constant non-zero multiple of $w(x)$.

The last sentence in the statement of Theorem 3.2 follows by making obvious changes in the above proof.

Example 2. Consider a differential equation

$$\left(\frac{1}{x^{2a-1}}\, y'\right)' + \frac{a^2+1}{x^{2a+1}}\, y = 0\,, \tag{3.16}$$

which has linearly independent solutions

$$y_1 = x^a \cos \ln x\,, \quad y_2 = x^a \sin \ln x\,.$$

Now $p(x) = x^{2a}$. If we choose $\psi(x) = x^{a-1/2}$, then $g\psi^2 = 1$ and $p(x)/\psi^2(x) = x$. Thus, the hypotheses of Theorem 3.2 are fulfilled with $(-1)^n$ deleted for any $a \in (-\infty, \infty)$.

**Corollary 3.2.** *Under the hypotheses of Theorem* 3.2 *we have*

$$(-1)^n \Delta^n \log \left|\frac{y(x_k)}{\psi(x_k)}\right| \geqq 0\,, \qquad n = 1, 2, \ldots, N\,, \quad k = 1, 2, \ldots \tag{3.17}$$

*and*

$$(-1)^n \Delta^n \log \left|\frac{w(x_k)}{\psi(x_k)\, z'(x_k)}\right| \geqq 0\,, \qquad n = 1, 2, \ldots, N\,, \quad k = 1, 2, \ldots\,. \tag{3.18}$$

These results remain true if the factor $(-1)^n$ is deleted simultaneously from (3.10), (3.11), (3.12), (3.17) and (3.18).

Proof. From Theorem 3.2 we have for each $\alpha > 0$

$$(-1)^n \Delta^n \frac{[y(x_k)/\psi(x_k)] - 1}{\alpha} > 0, \qquad \begin{matrix} n = 1, 2, \ldots, N, \\ k = 1, 2, \ldots. \end{matrix}$$

Taking the limit as $\alpha \to 0^+$ and using the L'Hospital rule, we get (3.17). Similarly, (3.18) follows from (3.14).

**Corollary 3.3.** *If the hypotheses of Theorem 3.2 hold and if, in addition,*

(3.19) $$(-1)^n D_x^n\{[W(x)]^\alpha \geqq 0, \quad a < x < b, \quad n = 1, 2, \ldots, N$$

*for each* $\alpha > 0$, *then*

(3.20) $$(-1)^n \Delta^n \log\left\{W(x_k)\left|\frac{y(x_k)}{\psi(x_k)}\right|\right\} \geqq 0, \qquad \begin{matrix} n = 1, 2, \ldots, N, \\ k = 1, 2, \ldots \end{matrix}$$

*and*

(3.21) $$(-1)^n \Delta^n \log\left\{\frac{W(x_k)\, w(x_k)}{\psi(x_k)\, z'(x_k)}\right\} \geqq 0, \qquad \begin{matrix} n = 1, 2, \ldots, N, \\ k = 1, 2, \ldots. \end{matrix}$$

The results remain true if the factor $(-1)^n$ is deleted simultaneously from (3.10), (3.11), (3.12), (3.19), (3.20) and (3.21).

This corollary may be proved in the same way as Corollary 3.2. As an example we introduce the solutions of the differential equations (3.9) and (3.16) from Examples 1 and 2, which fulfil the hypotheses of Theorem 3.2.

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