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REGULAR FACTORS IN POWERS OF CONNECTED GRAPHS

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Let G be a graph (in the sense of [1] or [3]). We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. The number $|V(G)|$ is called *the order of* G. If $W \subseteq V(G)$, then we denote by $\langle W \rangle_G$ the subgraph of G induced by W. If a spanning subgraph F of G is a regular graph of a degree $m \geqq 0$, then we say that F is an m-factor of G. For every integer $n \geqq 1$, by the n-th power G^n of G we mean the graph with $V(G^n) = V(G)$ and

$$E(G^n) = \{uv;\, u, v \in V(G) \text{ with the property that } 1 \leqq d_G(u, v) \leqq n\}\,,$$

where d_G denotes the distance between vertices in G.

If $n \geqq 1$ is an odd integer and G has an n-factor, then the order of G is even. Chartrand, Polimeni and Stewart [2] and Sumner [6] proved that if G is a connected graph of an even order, then G^2 has a 1-factor. Nebeský [4] proved that if G is a connected graph of an even order $\geqq 4$, then G^4 has a 3-factor. In the present paper these results will be generalized for every odd integer $n \geqq 1$. We shall prove the following theorem:

Theorem 1. *Let $n \geqq 1$ be an odd integer, and let G be a connected graph of an even order $p \geqq n + 1$. Then G^{n+1} has an n-factor.*

In the present paper we shall prove one more theorem, which complements Theorem 1.

Theorem 2. *Let $n \geqq 2$ be an even integer, and let G be a connected graph of an order $p \geqq n + 1$. Then G^{n+1} has an n-factor.*

Let G be the tree (homeomorphic with the star $K(1, n + 2)$) of an order $p > n(n + 1)$ which is given in Fig. 1. Then G^n has no n-factor. This means that the value $n + 1$ of the power in Theorems 1 and 2 is the best possible.

Note that for $n = 2$ a stronger result is known. Sekanina [5] proved that if G is a connected graph, then G^3 is hamiltonian connected.

To prove Theorem 1 and 2 we use two lemmas and three remarks.

Let T be a nontrivial tree, and let u and v be adjacent vertices of T. Then $T - uv$ is a forest with exactly two components. We denote by $T(u, v)$ or $T(v, u)$ the component of $T - uv$ which contains u or v, respectively.

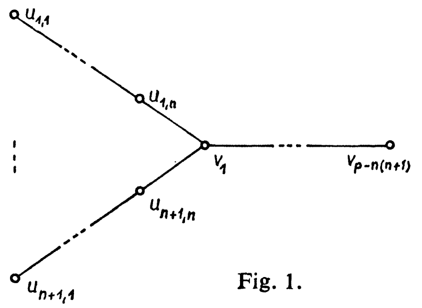

Fig. 1.

Let T be a tree, and let $u \in V(T)$. We shall say that $W \subseteq V(T)$ is a u-set in T, if either $W = \{u\}$ or there exist distinct components $T_1, \ldots, T_i$ $(i \geqq 1)$ of $T - u$ such that either $W = V(T_1) \cup \ldots \cup V(T_i)$ or $W = \{u\} \cup V(T_1) \cup \ldots \cup V(T_i)$.

Lemma 1. *Let T be a tree of an order $p > n + 1$, where $n \geqq 1$. Then there exists $u \in V(T)$ and disjoint u-sets W' and W'' in T such that*

(1) $W' \cup W'' \neq V(T)$ *and* $T - (W' \cup W'')$ *is a tree;*

(2) $|W'| \leqq n$ *and* $|W''| \leqq n$;

(3) $n < |W' \cup W''|$;

(4) *if* $|W' \cup W''| \neq n + 1$, *then* $|W' \cup W''|$ *is even.*

Proof. Since $p > n + 1$, there exist adjacent vertices u and v such that $|V(T(u, v))| > n$ and

$$|V(T(w, u))| \leqq n \text{ for every vertex } w \neq v \text{ such that } uw \in E(T).$$

(Otherwise, in T we can construct an infinite sequence of distinct vertices beginning in an arbitrary vertex of degree one, which contradicts the finiteness of $V(T)$).

Let $T_1, \ldots, T_i$ $(i \geqq 1)$ be all the components of $T - u$ which are different from $T(v, u)$. Denote $M_1 = V(T_1), \ldots, M_i = V(T_i)$ and $m = |M_1| + \ldots + |M_i|$. Clearly, $m = |V(T(u, v))| - 1$. Without loss of generality we assume that

$$n \geqq |M_1| \geqq \ldots \geqq |M_i| > 0 .$$

Since $|V(T(u, v))| > n$, we have $m \geqq n$. We shall construct disjoint u-sets W' and W'' with the properties (1)–(4). We distinguish the following cases and subcases:

1. $m = n$. We put $W' = M_1 \cup \ldots \cup M_i$ and $W'' = \{u\}$.

2. $m > n$. It is obvious that there exists an integer f, $1 \leqq f < i$, such that

$$(n + 1)/2 \leqq |M_1| + \ldots + |M_f| \leqq n .$$

Denote $m_1 = |M_1| + \ldots + |M_f|$.

2.1. $m - m_1 \leqq n$. If m is even, then we put $W' = M_1 \cup \ldots \cup M_f$ and $W'' = M_{f+1} \cup \ldots \cup M_i$. Assume that m is odd. If $m_1 < m - m_1$, then we put $W' = \{u\} \cup M_1 \cup \ldots \cup M_f$ and $W'' = M_{f+1} \cup \ldots \cup M_i$. If $m - m_1 < m_1$, then we put $W' = M_1 \cup \ldots \cup M_f$ and $W'' = \{u\} \cup M_{f+1} \cup \ldots \cup M_i$.

2.2. $m - m_1 > n$. Then there exists $g, f < g < i$, such that

$$(n + 1)/2 \leqq |M_{f+1}| + \ldots + |M_g| \leqq n .$$

Denote $m_2 = |M_{f+1}| + \ldots + |M_g|$.

2.2.1. $m_1 + m_2$ is even. Then we put $W' = M_1 \cup \ldots \cup M_f$ and $W'' = M_{f+1} \cup \ldots \cup M_g$.

2.2.2. $m_1 + m_2$ is odd.

2.2.2.1. $m - (m_1 + m_2) \leqq n$.

2.2.2.1.1. $m - m_1$ is even. Then we put $W' = M_{f+1} \cup \ldots \cup M_g$ and $W'' = M_{g+1} \cup \ldots \cup M_i$.

2.2.2.1.2. $m - m_1$ is odd. Then $m - m_2$ is even.

2.2.2.1.2.1. $m - m_2 > n$. Then $m_1 + (m - (m_1 + m_2)) > n$. We put $W' = M_1 \cup \ldots \cup M_f$ and $W'' = M_{g+1} \cup \ldots \cup M_i$.

2.2.2.1.2.2. $m - m_2 \leqq n$. If m is even, then we put $W' = M_1 \cup \ldots \cup M_f \cup M_{g+1} \cup \ldots \cup M_i$ and $W'' = M_{f+1} \cup \ldots \cup M_g$. Assume that m is odd. If $m_2 < m - m_2$, then we put $W' = M_1 \cup \ldots \cup M_f \cup M_{g+1} \cup \ldots \cup M_i$ and $W'' = \{u\} \cup M_{f+1} \cup \ldots \cup M_g$. If $m_2 > m - m_2$, then we put $W' = \{u\} \cup M_1 \cup \ldots \cup M_f \cup M_{g+1} \cup \ldots \cup M_i$ and $W'' = M_{f+1} \cup \ldots \cup M_g$.

2.2.2.2. $m - (m_1 + m_2) > n$. Then there exists an integer h, $g < h < i$, such that

$$(n + 1)/2 \leqq |M_{g+1}| + \ldots + |M_h| \leqq n .$$

Denote $m_3 = |M_{g+1}| + \ldots + |M_h|$.

2.2.2.2.1. $m_3 + m_1$ is even. Then we put $W' = M_1 \cup \ldots \cup M_f$ and $W'' = M_{g+1} \cup \ldots \cup M_h$.

2.2.2.2.2. $m_3 + m_1$ is odd. Then $m_3 + m_2$ is even. We put $W' = M_{f+1} \cup \ldots \cup M_g$ and $W'' = M_{g+1} \cup \ldots \cup M_h$.

The proof of the lemma is complete.

Remark 1. Let T be a tree, $u \in V(T)$, $n \geqq 1$, and let $W_1, \ldots, W_k$ $(k \geqq 2)$ be disjoint u-sets such that $|W_1| \leqq n, \ldots, |W_k| \leqq n$. Then every set W_h, $1 \leqq h \leqq k$, can be arranged into a sequence $w_{h,1}, \ldots, w_{h,|W_h|}$ such that, for every g, $1 \leqq g \leqq |W_h|$,

$$\text{if } u \in W_h \text{, then } d_T(w_{h,g}, u) < g \text{, and if } u \notin W_h \text{, then } d_T(w_{h,g}, u) \leqq g .$$

This means that if $u \in W_h$, then $w_{h,1} = u$.

Let h' and h'' be arbitrary integers such that $1 \leqq h' < h'' \leqq k$. Assume that g' and g'' are integers such that $1 \leqq g' \leqq |W_{h'}|$ and $1 \leqq g'' \leqq |W_{h''}|$ and that $u \in W_{h'} \cup$

$\cup W_{h''}$ implies $g' + g'' \leqq n + 2$, and $u \notin W_{h'} \cup W_{h''}$ implies $g' + g'' \leqq n + 1$. Then $d_T(w_{h',g'}, w_{h'',g''}) = d_T(w_{h',g'}, u) + d_T(w_{h'',g''}, u) \leqq n + 1$.

Denote $w_1 = w_{h',m'}$, $w_2 = w_{h',m'-1}, \ldots, w_{m'} = w_{h',1}$, $w_{m'+1} = w_{h'',1}$, $w_{m'+2} = w_{h'',2}, \ldots, w_m = w_{h'',m''}$, where $m' = |W_{h'}|$, $m'' = |W_{h''}|$, and $m = m' + m''$. Thus the set $W_{h'} \cup W_{h''}$ has been arranged into the sequence

$$w_1, \ldots, w_m .$$

Let $1 \leqq i \leqq j \leqq m$, and let $j - i \leqq n$. If $j \leqq m'$ or $i > m'$, then $d_T(w_i, w_j) \leqq n$. If $i \leqq m'$ and $m' < j$, then $d_T(w_i, w_j) = d_T(w_{h',m'-i+1}, w_{h'',j-m'}) = d_T(w_{h',m'-i+1}, u) + d_T(w_{h'',j-m'}, u) \leqq (m' - i + 1) + (j - m') = j - i + 1 \leqq n + 1$. Thus we have that if $1 \leqq i < j \leqq m$ and $j - i \leqq n$, then $d_T(w_i, w_j) \leqq n + 1$.

Let W be a finite nonempty set. Then we denote by $K(W)$ the complete graph whose vertex set is W.

Remark 2. Let T be a tree, $n \geqq 1$, and let $w_1, \ldots, w_m$ be a sequence of distinct vertices in T which has been obtained in the way described in Remark 1. Let m be even and $n + 1 \leqq m \leqq 2n$. Denote

$$\begin{aligned} E_0 = \{ & w_1 w_{(m/2)+1}, w_1 w_{(m/2)+2}, \ldots, w_1 w_{n+1}, \\ & w_2 w_{(m/2)+2}, w_2 w_{(m/2)+3}, \ldots, w_2 w_{n+2}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & w_{m/2} w_m, w_{m/2} w_{m+1}, \ldots, w_{m/2} w_{n+(m/2)} \}, \end{aligned}$$

where every index $i > m$ is to be replaced by the index $i - (m/2)$. We denote by F the graph with $V(F) = \{w_1, \ldots, w_m\}$ and

$$E(F) = E(K(\{w_1, \ldots, w_{m/2}\})) \cup E(K(\{w_{(m/2)+1}, \ldots, w_m\})) \cup E_0 .$$

Then F is an n-factor of the graph $\langle\{w_1, \ldots, w_m\}\rangle_{T^{n+1}}$.

Remark 3. Let m and n be integers such that $0 < m < n$. It follows from Theorems 9.1 and 9.6 in [3] that K_n has an m-factor if and only if at least one of the integers m and n is even.

Lemma 2. *Let T be a tree of an order $p \geqq n + 1$, where $n \geqq 1$. Assume that if n is odd, then p is even. Then T^{n+1} has an n-factor.*

Proof. If $p = n + 1$, then $T^{n+1} = K(V(T))$ and thus T^{n+1} is a regular graph of the degree n. Assume that $p > n + 1$, and that for every tree T^* of an order p^*, where (i) $n + 1 \leqq p^* < p$, and (ii) if n is odd, then p^* is even, it is proved that $(T^*)^{n+1}$ has an n-factor. Since $p > n + 1$, it follows from Lemma 1 that there exists $u \in V(T)$ and disjoint u-sets W' and W'' which fulfil (1)–(4). Clearly, if n is odd, then $|V(T)|$ and $|W' \cup W''|$ are even, and therefore $|V(T) - (W' \cup W'')|$ is also even.

First, assume that $|V(T) - (W' \cup W'')| \geqq n + 1$. The induction assumption yields that $(T - (W' \cup W''))^{n+1}$ has an n-factor. If $|W' \cup W''| = n + 1$, then $\langle W' \cup W'' \rangle_{T^{n+1}} = K(W' \cup W'')$ and thus T^{n+1} has an n-factor. Let $|W' \cup W''| > n + 1$. Then $W' \cup W''$ is even. The set $W' \cup W''$ can be arranged into a sequence $w_1, \ldots, w_m$ described in Remark 1. From this fact and from Remark 2 it follows that there exists an n-factor of the graph $\langle W' \cup W'' \rangle_{T^{n+1}}$. Hence, T^{n+1} has an n-factor.

We now assume that $|V(T) - (W' \cup W'')| \leqq n$. We distinguish the following cases and subcases:

1. There exist disjoint u-sets W_1 and W_2 such that $|W_1| \leqq |W_2| \leqq n$ and that $W_1 \cup W_2 = V(T) - \{u\}$.

1.1. p is even. Then $|W_1| < |W_2|$ and $|W_1 \cup \{u\}| \leqq n$. The set $\{u\} \cup W_1 \cup W_2$ can be arranged into a sequence $w_1, \ldots, w_m$ (where $m = p$) described in Remark 1. Since $n + 2 \leqq m \leqq 2n$ and m is even, it follows from Remark 2 that there exists an n-factor T^{n+1}.

1.2. p is odd. Then n is even. The set $W_1 \cup W_2$ can be arranged into a sequence $w_1, \ldots, w_m$ (where $m = p - 1$) described in Remark 1. Since n is even, we have that $n + 2 \leqq m \leqq 2n$. Consider the graph F defined in Remark 2. Since $m \geqq n + 2$, there exist positive even integers $i \leqq m/2$ and $j \leqq m/2$ such that $i + j = n$. Let F' be the graph obtained from the graph

$$F - \{w_1w_2, w_3w_4, \ldots, w_{i-1}w_i, w_{(m/2)+1}w_{(m/2)+2}, w_{(m/2)+3}w_{(m/2)+4}, \ldots, w_{(m/2)+j-1}w_{(m/2)+j}\}$$

by adding the vertex u and the edges $uw_1, uw_2, \ldots, uw_i, uw_{(m/2)+1}, uw_{(m/2)+2}, \ldots, uw_{(m/2)+j}$. Then F' is an n-factor of T^{n+1}.

2. For arbitrary disjoint u-sets W_1 and W_2 such that $|W_1| \leqq n$ and $|W_2| \leqq n$ it holds that $W_1 \cup W_2 \neq V(T) - \{u\}$. Since $|W'| \leqq n$, $|W''| \leqq n$, and $|V(T) - (W' \cup W'')| \leqq n$, we conclude that there exist disjoint u-sets A, B and C such that $|A| \leqq n$, $|B| \leqq n$, $|C| \leqq n$, $|A \cup B| > n$, $|B \cup C| > n$, $|A \cup C| > n$, and $A \cup B \cup C = V(T) - \{u\}$. Denote $a = |A|$, $b = |B|$, and $c = |C|$. Without loss of generality we assume that $a \geqq b \geqq c$.

2.1. Either $a + b$ is odd or $c < b$. If $a + b$ is odd, then $n \geqq a > b$, and we put $\bar{A} = A$, $\bar{B} = B \cup \{u\}$ and $\bar{C} = C$; if $a + b$ is even, then $c < b$, and we put $\bar{A} = A$, $\bar{B} = B$ and $\bar{C} = C \cup \{u\}$. Denote $\bar{a} = |\bar{A}|$, $\bar{b} = |\bar{B}|$ and $\bar{c} = |\bar{C}|$. Thus $n \geqq \bar{a} \geqq \bar{b} \geqq \bar{c}$, $\bar{b} + \bar{c} > n$, and $\bar{a} + \bar{b}$ is even. In accordance with Remark 1 the set $\bar{C}$ can be arranged into a sequence $z_1, \ldots z_{\bar{c}}$ such that for every g, $1 \leqq g \leqq \bar{c}$, $u \in \bar{C}$ implies $d_T(z_g, u) < g$ and $u \notin C$ implies $d_T(z_g, u) \leqq g$ (hence ,if $u \in \bar{C}$, then $z_1 = u$). Analogously we can arrange the sets $\bar{A}$ and $\bar{B}$. Moreover, in accordance with Remark 1, the set $\bar{A} \cup \bar{B}$ can be arranged into a sequence $w_1, \ldots, w_m$ (where $m = \bar{a} + \bar{b}$) with the properties described in Remark 1 and such that $w_1, \ldots, w_{\bar{a}} \in \bar{A}$ and $w_{\bar{a}+1}, \ldots, w_m \in \bar{B}$ (if $u \in \bar{B}$, then $w_{\bar{a}+1} = u$). According to Remark 1, for $1 \leqq i \leqq \bar{c}$ and $1 \leqq j \leqq \bar{b}$, the inequality $i + j \leqq n + 2$ implies $d_T(z_i, w_{\bar{a}+j}) \leqq n + 1$. Let F be the regular graph constructed in Remark 2. Thus $V(F) = \{w_1, \ldots, w_m\}$.

Let $\bar{c}$ be odd; since $p = \bar{a} + \bar{b} + \bar{c}$ and $\bar{a} + \bar{b}$ is even, we have that p is odd and therefore n is even. This means that at least one of the integers $\bar{c}$ and n is even. Thus at least one of the integers $\bar{c}$ and $n - \bar{c} + 1$ is even.

2.1.1. $\bar{c} < (n + 1)/2$. Since $\bar{b} + \bar{c} \geqq n + 1$, we have $m - \bar{a} = \bar{b} \geqq n - \bar{c} + 1 > \bar{c}$. It follows from Remark 3 that $K(\{w_{\bar{a}+1}, \ldots, w_{\bar{a}+1+n-\bar{c}}\})$ has a $\bar{c}$-factor, say H_1. This means that the graph obtained from the graphs $F - E(H_1)$ and $K(\bar{C})$ by adding the edges

$$z_{\bar{c}}w_{\bar{a}+1},\ z_{\bar{c}}w_{\bar{a}+2},\ \ldots,\ z_{\bar{c}}w_{\bar{a}+1+n-\bar{c}}\,,$$

$$z_{\bar{c}-1}w_{\bar{a}+1},\ z_{\bar{c}-1}w_{\bar{a}+2},\ \ldots,\ z_{\bar{c}-1}w_{\bar{a}+1+n-\bar{c}}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$z_1w_{\bar{a}+1},\ z_1w_{\bar{a}+2},\ \ldots,\ z_1w_{\bar{a}+1+n-\bar{c}}$$

is an n-factor of T^{n+1}.

2.1.2. $c > (n + 1)/2$. Then $n - \bar{c} + 1 < \bar{c} \leqq \bar{b}$. According to Remark 3, $K(\{w_{\bar{a}+1}, \ldots, w_{\bar{a}+\bar{c}}\})$ has an $(n - \bar{c} + 1)$-factor, say H_2. The graph obtained from the graphs $F - E(H_2)$ and $K(\bar{C})$ by adding the edges

$$z_{\bar{c}}w_{\bar{a}+1},\ \ldots,\ z_{\bar{c}}w_{\bar{a}+1+n-\bar{c}}\,,$$

$$z_{\bar{c}-1}w_{\bar{a}+2},\ \ldots,\ z_{\bar{c}-1}w_{\bar{a}+2+n-\bar{c}}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$z_1w_{\bar{a}+\bar{c}},\ \ldots,\ z_1w_{\bar{a}+n}\,,$$

where every index $i > \bar{a} + \bar{c}$ is to be replaced by the index $i - \bar{c}$, is an n-factor of T^{n+1}.

2.1.3. $\bar{c} = (n + 1)/2$. Then n is odd, and thus $\bar{c}$ is even. Obviously, $\bar{c} = n - \bar{c} + 1$. We denote by d the integer $\bar{a}$ if $u \notin \bar{B}$, or the integer $\bar{a} + 1$ if $u \in \bar{B}$. Obviously, $m - d \geqq \bar{c}$. We denote by d' that of the integers $d - 1$ and d which has the same parity as $m/2$. It is not difficult to see that $d' \geqq \bar{c}$. For every i, $1 \leqq i \leqq \bar{c}$, we have $d_T(z_i, w_{d'-\bar{c}+1}) \leqq d_T(z_i, w_{d-1-\bar{c}+1}) \leqq n + 1$. The graph obtained from the graphs $K(\bar{C})$ and

$$F - E(K(\{w_{d+1}, \ldots, w_{d+\bar{c}}\}) - \{w_{d'}w_{d'-1},\ w_{d'-2}w_{d'-3},\ \ldots,\ w_{d'-\bar{c}+2}w_{d'-\bar{c}+1}\}$$

by adding the edges

$$z_{\bar{c}}w_{d+1},\ \ldots,\ z_{\bar{c}}w_{d+\bar{c}-1}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$z_1w_{d+\bar{c}},\ \ldots,\ z_1w_{d+2\bar{c}-2}\,,$$

where every index $i > d + \bar{c}$ means $i - \bar{c}$, and the edges

$$z_{\bar{c}}w_{d'},\ z_{\bar{c}-1}w_{d'-1},\ \ldots,\ z_1w_{d'-\bar{c}+1}\,,$$

is an n-factor of T^{n+1}.

2.2. $a + b$ is even and $c = b$. Thus $c \geqq (n + 1)/2$. Since $p = a + 2c + 1$, $p + c$ is odd. This means that if c is even, then n is even. The set $A \cup B$ can be arranged into a sequence $w_1, \ldots, w_m$ (where $m = a + c$) with the properties described in Remark 1 and such that $w_1, \ldots, w_a \in A$ and $w_{a+1}, \ldots, w_m \in B$. The set C can be arranged into a sequence $z_1, \ldots, z_c$ such that $d_T(z_i, u) \leqq i$ for every i, $1 \leqq i \leqq c$. Let F be the graph defined in Remark 2. Hence $V(F) = \{w_1, \ldots, w_m\}$.

2.2.1. n is even. Then $c \neq (n + 1)/2$. This means that $c > (n + 1)/2$ and therefore $n - c + 1 < c$. This means that either c or $n - c + 1$ is even. It follows from Remark 3 that $K(\{w_{a+1}, \ldots, w_{a+c}\})$ has an $(n - c + 1)$-factor, say H_1'. Let F_1 be the graph obtained from the graphs $K(C)$ and $F - E(H_1')$ by adding the edges

$$z_c w_{a+1}, \ldots, z_c w_{a+n-c+1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$z_1 w_{a+c}, \ldots, z_1 w_{a+n},$$

where every index $i > a + c$ means $i - c$. It is easy to see that F_1 is an n-factor of $\langle V(T - u)\rangle_{T^{n+1}}$. Since $m/2 \geqq c > (n + 1)/2$, there exist positive even integers $j \leqq m/2$ and $k \leqq c$ such that $j + k = n$. The graph obtained from the graph

$$F_1 - \{w_1w_2, w_3w_4, \ldots, w_{j-1}w_j, z_1z_2, z_3z_4, \ldots, z_{k-1}z_k\}$$

by adding the vertex u and the edges

$$uw_1, uw_2, \ldots, uw_j, uz_1, uz_2, \ldots, uz_k$$

is an n-factor of T^{n+1}.

2.2.2. n is odd. Then c is odd and therefore $n - c$ is even. Since $c \geqq (n + 1)/2$, we have $n - c < c$. Since $n - c$ is even, we have that $K(\{w_{a+1}, \ldots, w_{a+c}\})$ has an $(n - c)$-factor, say H_2'. Let F_2 be the graph obtained from the graphs $F - E(H_2)$ and $K(C)$ by adding the edges

$$z_c w_{a+1}, \ldots, z_c w_{a+n-c},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$z_1 w_{a+c}, \ldots, z_1 w_{a+n-1},$$

where every index $i > a + c$ means $i - c$. Therefore, every vertex w_j, $1 \leqq j \leqq m$, has the degree n in F_2, and every vertex z_k, $1 \leqq k \leqq c$, has the degree $n - 1$ in F_2. Obviously, $n - c < m/2$. The graph obtained from the graph

$$F_2 - \{w_1w_2, w_3w_4, \ldots, w_{n-c-1}w_{n-c}\}$$

by adding the edges

$$uw_1, \ldots, uw_{n-c}, uz_1, \ldots, uz_c$$

is an n-factor of T^{n+1}.

Thus the lemma is proved.

Proof of Theorems 1 and 2. Let G be a graph satisfying the conditions of Theorems 1 or 2. Then G is connected, and thus there exists a spanning tree of G, say T. According to Lemma 2, T^{n+1} has an n-factor. Thus G^{n+1} has an n-factor, which completes the proof.

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