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ON THE COMPLETENESS-NUMBER OF A FINITE GRAPH

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In this note the method of adjoining a graph G' to a given finite non-directed graph G without isolated vertices is described and the equality $\omega(G) = \chi(G')$ is proved.

We shall deal with finite non-directed simple graphs without loops. The completeness-number $\omega(G)$ of the graph G is defined as follows:

Definition 1. We shall say that the system $\mathfrak{S}$ of the complete subgraphs of the graph $G = \langle U, H \rangle$ covers G, if every vertex $u \in U$ and every edge $h \in H$ belongs to some subgraph $F \in \mathfrak{S}$. The smallest cardinality of the system $\mathfrak{S}$ covering G is called the completeness-number of the graph G and denoted by $\omega(G)$.

Note. Let $G = \langle U, H \rangle$, $a \notin U$, $G_1 = \langle U \cup \{a\}, H \rangle$; let a be an isolated vertex of the graph G_1. Then obviously, $\omega(G_1) = \omega(G) + 1$.

On investigating the completeness-number of the graph G we are enabled, according to the note above, to pass over to the graph which is obtained by removing the isolated vertices of the graph G.

Definition 2. We say that the edges h_1, h_2 of the graph G are *quasineighbours*, if $h_1 \neq h_2$ and h_1, h_2 both belong to a certain complete subgraph of G.

Let $G = \langle U, H \rangle$ be the graph without isolated vertices. We shall denote by $G' = \langle U', H' \rangle$ the graph, which satisfies the following conditions:

Condition 1: the edges of G correspond uniquely to the vertices of G' (let us denote the one-to-one mapping of the set H onto the set U' by φ).

Condition 2: the vertices u'_1, u'_2 of the graph G' are connected by a certain edge in G' if and only if the corresponding edges, i.e. $\varphi^{-1}(u'_1)$ and $\varphi^{-1}(u'_2)$ are not quasineighbours in G.

If G is an arbitrary graph without isolated vertices, then obviously there exists just one graph G' with the required properties (with the exception of isomorphism).

Theorem. *Let G be the graph without isolated vertices. Then*

$$\omega(G) = \chi(G'),$$

where G' is the graph satisfying the conditions 1 *and* 2 *and* $\chi(G')$ *is its chromatic number.*

Proof: Let $G = \langle U, H \rangle$ be an arbitrary graph satisfying the condition of the theorem. Let us construct the graph $G' = \langle U', H' \rangle$ satisfying the conditions 1 and 2. Let $\mathfrak{A}$ be its chromatic decomposition (i.e. $\mathfrak{A}$ is the decomposition of the set U' and if $u'_1, u'_2 \in A$, $A \in \mathfrak{A}$, then u'_1 and u'_2 are not connected by any edge in G'). We shall construct the system $\mathfrak{G}$ of complete subgraphs G covering G, which has the same cardinality as $\mathfrak{A}$ has. For each $A \in \mathfrak{A}$ let us define

$$U_A = \{u \in U;\ \text{there exists}\ \ h \in \varphi^{-1}(A)\ \text{so that } u \text{ is its end vertex}\}\,.$$

The subgraph $\bar{U}_A$ of the graph G, which is induced by the set of vertices U_A (i.e. the graph $\langle U_A, (U_A \times U_A) \cap H \rangle$) is obviously complete. Let $\mathfrak{G} = \{\bar{U}_A;\ A \in \mathfrak{A}\}$. $\mathfrak{G}$ is the required system covering G. We have proved the inequality $\omega(G) \leqq \chi(G')$.

Let now $\mathfrak{G}$ be an arbitrary system of complete subgraphs of the graph G covering G. We shall construct the chromatic decomposition $\mathfrak{A}$ of the graph G' satisfying the following inequality: card $\mathfrak{A} \leqq$ card $\mathfrak{G}$.

For each $F \in \mathfrak{G}$, if $F = \langle V, K \rangle$, we put $\tilde{U}'_F = \varphi(K)$. Let us denote the system $\{\tilde{U}'_F;\ F \in \mathfrak{G}\}$ by $\mathfrak{A}_0$. For each $F \in \mathfrak{G}$ $\tilde{U}'_F$ is the set of internal stability of the graph G', because two arbitrary edges of the complete subgraph F are quasineighbours. The system $\mathfrak{G}$ covers G, hence it covers each edge of the graph G too. Of necessity, each vertex $u' \in U'$ belongs to some $\tilde{U}'_F \in \mathfrak{A}_0$. Therefore, $\mathfrak{A}_0$ is the covering of U'. Starting from the system $\mathfrak{A}_0$, we shall construct the systems $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_m$ as follows: if A, $B \in \mathfrak{A}_i$, $A \cap B \neq \emptyset$ $(i \geqq 0)$, we replace the set B by the set $B - A$ if $B - A \neq \emptyset$. If $B - A = \emptyset$, then the set B will be left out. After a finite number of steps we get the system $\mathfrak{A}_m$, which is the chromatic decomposition of the set U'. We shall put $\mathfrak{A} = \mathfrak{A}_m$; thus card $\mathfrak{A} \leqq$ card $\mathfrak{A}_0$ = card $\mathfrak{G}$ holds. Hence $\chi(G') \leqq \omega(G)$ q.e.d.

Note. Let $G_1 = \langle U, H \rangle$ be an arbitrary graph. There need not always exist such a graph G that $G' = G_1$ (G' denotes the graph adjoined to the graph G satisfying the conditions 1 and 2). The necessary condition for the existence of such a graph is the following

condition 3: the intersection of an arbitrary system of maximal (in the sense of inclusion) complete subgraphs of the graph G_1^* has a number of vertices equal to some of the numbers $0, 1, 3, \ldots, k(k-1)/2, \ldots$ Here G_1^* is the complementary graph to G_1 taken without the loops, i.e.

$$G_1^* = \langle U, \{(u, v);\ u \neq v,\ (u, v) \notin H\} \rangle\,,$$

by the intersection of the system consisting of one maximal complete subgraph we understand the subgraph itself and the numbers $0, 1, 3, \ldots, k(k-1)/2, \ldots$ determine the number of edges in a complete subgraph of the order $1, 2, \ldots, k, \ldots$

The condition 3, however, is not sufficient for the existence of graph G. This can be seen from the example of graph G_1 such that $G_1^* = G_{11} \cup G_{12} \cup G_{13}$ consists of three complete hexagons:

$$G_{11} = \langle\{u_i;\ 1 \leqq i \leqq\ 6\},\ \{(u_i, u_j);\ i \neq j, 1 \leqq i, j \leqq\ 6\}\rangle,$$
$$G_{12} = \langle\{u_i;\ 4 \leqq i \leqq\ 9\},\ \{(u_i, u_j);\ i \neq j, 4 \leqq i, j \leqq\ 9\}\rangle,$$
$$G_{13} = \langle\{u_i;\ 7 \leqq i \leqq 12\},\ \{(u_i, u_j);\ i \neq j, 7 \leqq i, j \leqq 12\}\rangle.$$

References

[1] *K. Čulik:* Applications of graph theory to mathematical logic and linguistics. Theory of graphs and its applications. Proceedings of the Symposium held in Smolenice (Czechoslovakia), June 17—20, 1963. Nakladatelství ČSAV, Praha 1964, 13—20.

Výtah

O ČÍSLE ÚPLNOSTI KONEČNÉHO GRAFU

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V práci je posán jistý způsob, jak přiřadit danému konečnému neorinetovanému grafu G bez izolovaných uzlů graf G'. Je dokázána rovnost $\omega(G) = \chi(G')$, kde $\omega(G)$ je číslo úplnosti grafu G a $\chi(G')$ je chromatické číslo grafu G'. $\omega(G)$ se definuje jako nejmenší mohutnost soustavy $\mathfrak{S}$ úplných podgrafů grafu G, pokrývající G (tj. všechny uzly a hrany G).

Резюме

О ЧИСЛЕ ПОЛНОТЫ КОНЕЧНОГО ГРАФА

ИВАН ГАВЕЛ (Ivan Havel), Прага

В заметке описан определенный способ сопоставления заданному конечному неориентированному графу G без изолированных вершин графа G', и доказано, что имеет место равенство $\omega(G) = \chi(G')$, где $\omega(G)$ — число полноты графа G и $\chi(G')$ — хроматическое число графа G'. $\omega(G)$ определяется как наименьшая мощность системы $\mathfrak{S}$ полных подграфов графа G, покрывающей G (т. е. все вершины и ребра G).