Wolfgang Meiske; Klaus R. Schneider

Existence, persistence and structure of integral manifolds in the neighbourhood of a periodic solution of autonomous differential systems

Časopis pro pěstování matematiky, Vol. 111 (1986), No. 3, 304--313

Persistent URL: `http://dml.cz/dmlcz/108154`

Časopis pro pěstování matematiky, roč. 111 (1986), Praha

EXISTENCE, PERSISTENCE AND STRUCTURE OF INTEGRAL MANIFOLDS IN THE NEIGHBOURHOOD OF A PERIODIC SOLUTION OF AUTONOMOUS DIFFERENTIAL SYSTEMS

WOLFGANG MEISKE, KLAUS R. SCHNEIDER, Berlin

Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

(Received August 26, 1985)

1. INTRODUCTION

In the frame of the qualitative theory of evolution equations the method of integral manifolds takes a central position [2, 4, 6, 7, 11, 20, 21]. Integral manifolds provide a powerful tool to justify the averaging principle [5, 14] and to reduce bifurcation problems (including stability questions) as for certain types of partial differential systems to finite dimensional bifurcation problems as for systems of ordinary differential equations to corresponding problems for systems of lower order [7, 12, 13, 19, 21, 25]. Concerning bifurcation problems the question for persistence of integral manifolds is of crucial importance [10, 24, 25]. With respect to exponentially stable integral manifolds such problems has been succesfully treated by J. Kurzweil in the 1960's [15, 16, 17]. By our view, the papers of J. Kurzweil distinguish themselves by the class of admissible perturbations (it permits averaging too) and the degree of generality (it admits results for partial differential systems, functional-differential equations and diffeomorphisms on Banach manifolds).

The class of center manifolds plays a crucial role in studying stability and bifurcation problems [12, 13, 19, 24, 25]. Using center manifolds we can reduce the Hopf bifurcation problem for a system of evolution equations to the corresponding one for a system of two ordinary differential equations in case of codimension one.

Concerning bifurcations of a nontrivial periodic solution the problem under consideration is usually reduced to the problem of bifurcation of a fixed point of the corresponding Poincaré map [8, 18]. Our aim is to investigate bifurcations of a periodic solution of autonomous differential equations by means of integral manifolds. In this paper we study existence and structure of integral manifolds near a periodic solution of a system of autonomous ordinary differential equations

$$\frac{dx}{dt} = f(x, \lambda) \tag{1.1}$$

and their persistence when λ changes in a metric space. In section 2 we introduce local coordinates in a neighbourhood of the family of periodic solutions under consideration. This coordinate system differs from analogous ones used by other authors [3, 4, 22]. By this way we obtain a nonautonomous system of differential equations whose qualitative behaviour near the stationary solution at the origin uniquely defines the topological structure of the trajectories of the system (1.1) near the considered periodic solution. In section 3 we investigate existence, persistence and structure of integral manifolds of the derived nonautonomous system. We show that these integral manifolds are homeomorphic either to $\mathbb{R}^k \times S^1$ where S^1 denotes the unit circle in $\mathbb{R}^2$, or to $\mathbb{R}^1 \times \mathscr{M}^2$ where $\mathscr{M}^2$ is the Möbius strip. The problem of existence of submanifolds is also considered.

2. INTRODUCTION OF LOCAL COORDINATES. REDUCTION TO A NONAUTONOMOUS SYSTEM

Let $\mathscr{G}$ be an open set of $\mathbb{R}^{n+1}$, Λ a metric space. We consider the autonomous differential system (1.1) assuming

(F_1). $f \in C^{r0}_{x\lambda}(\mathscr{G} \times \Lambda, \mathbb{R}^{n+1})$, $r \geqq 1$, that is, $f: \mathscr{G} \times \Lambda \to \mathbb{R}^{n+1}$ is continuous and r-times continuously differentiable with respect to x where all derivatives continuously depend on (x, λ).

(F_2). There is a subset Λ_1 of Λ such that (1.1) has a periodic solution $p(t, \lambda)$ with the primitive period $\omega(\lambda)$ for $\lambda \in \Lambda_1$ satisfying $p \in C^{r+1}_{t}{}^{0}_{\lambda}(\mathbb{R} \times \Lambda_1, \mathbb{R}^{n+1})$, $\omega \in C(\Lambda_1, \mathbb{R})$. Let λ^0 be an element of Λ_1.

We denote by γ_λ the orbit belonging to the periodic solution $p(t, \lambda)$, that is $\gamma_\lambda := \{x \in \mathbb{R}^{n+1}: x = p(t, \lambda),\ 0 \leqq t \leqq \omega(\lambda)\}$. Our purpose is to study existence and structure of local integral manifolds of (1.1) passing through γ_{λ^0} and their dependence on λ. To this end we first introduce a local coordinate system by means of γ_λ. Our way to construct such a local coordinate system differs from that one given by Diliberto [3], Hale [4], Reizinš [22]. The advantages of our coordinate system consist in the facts that it immediately leads to a system with constant linear part and that it is more appropriate to investigate the dependence on parameters.

Let $\Phi(t, \lambda)$ be the fundamental matrix of the linear system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f_x(p(t, \lambda), \lambda)y$$

satisfying $\Phi(0, \lambda) = I_{n+1}$ where I_l is the $l \times l$-unit matrix. For $\lambda = \lambda^0$ the monodromy matrix $P(\lambda) := \Phi(\omega(\lambda), \lambda)$ let have the distinct complex eigenvalues $\nu_1^0, \bar{\nu}_1^0, \ldots, \nu_j^0, \bar{\nu}_j^0$ and the distinct real eigenvalues $\nu_{j+1}^0, \ldots, \nu_k^0$ ($\nu_k^0 = 1$) with the multiplicities $n_1/2, \ldots, n_j/2, n_{j+1}, \ldots, n_k$. From the hypotheses ($F_1$) and ($F_2$) we get $P \in C(\Lambda_1, L(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$, and according to the perturbation theory of linear operators [9] there are a $\varrho^1 > 0$ and a function D continuously mapping $\mathscr{B}_{\varrho^1}(\lambda^0) := \{\lambda \in \Lambda_1: |\lambda - \lambda^0| \leqq \varrho^1\}$

into $L(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ with the properties

(i) $D(\lambda)$ is regular $\forall \lambda \in \mathscr{B}_{\varrho^1}(\lambda^0)$.

(ii) $J(\lambda) := D^{-1}(\lambda)\, P(\lambda)\, D(\lambda)$ has the structure $J(\lambda) = \operatorname{diag}(J_1(\lambda), \ldots, J_k(\lambda))$ where $J_i \in C(\Lambda_1, L(\mathbb{R}^{n_i}, \mathbb{R}^{n_i})$, and $J_i(\lambda^0)$ has the eigenvalues $v_i^0, \bar{v}_i^0$ with the multiplicity $n_i/2$ for $i \leqq j$ and the eigenvalue v_i^0 with the multiplicity n_i for $i > j$.

Because $\mathrm{d}p/\mathrm{d}t\,(0, \lambda)$ is an eigenvector of $P(\lambda)$ to the eigenvalue $v = 1$ $\forall \lambda \in \mathscr{B}_{\varrho^1}(\lambda^0)$ we may choose $D(\lambda)$ in such a way that

$$D(\lambda)\, \mathrm{e}^{n+1} = \frac{\mathrm{d}p}{\mathrm{d}t}(0, \lambda),$$

here e^i denotes the i-th column unit vector in $\mathbb{R}^{n+1}$; that means, the last column of $D(\lambda)$ consists of $\mathrm{d}p/\mathrm{d}t\,(0, \lambda)$.

Let $Q_i := I_{n_i}$ for $i = 1, \ldots, j$, let

$$Q_i := \begin{cases} -I_{n_i} & \text{if } v_i^0 < 0, \\ \;\;\, I_{n_i} & \text{if } v_i^0 > 0 \end{cases}$$

for $i = j + 1, \ldots, k$. We put

$$\tilde{Q} := \operatorname{diag}(Q_1, \ldots, Q_k).$$

The matrix $Q_i\, J_i(\lambda^0)$ has no negative eigenvalue $\forall i$. Thus, there is a $\varrho^2 \in (0, \varrho^1)$ such that $Q_i\, J_i(\lambda)$ can be represented in the form

$$Q_i\, J_i(\lambda) = \mathrm{e}^{\omega(\lambda) K_i(\lambda)}$$

for $\lambda \in \mathscr{B}_{\varrho^2}(\lambda^0)$ where $K_i \in C(\mathscr{B}_{\varrho^2}(\lambda^0), L(R^{n_i}, R^{n_i}))$ for all i.

We define

$$\begin{aligned} \hat{K}(\lambda) &:= \operatorname{diag}(K_1(\lambda), \ldots, K_k(\lambda)), \\ K(\lambda) &:= D(\lambda)\, \hat{K}(\lambda)\, D^{-1}(\lambda), \\ W(t, \lambda) &:= \Phi(t, \lambda)\, \mathrm{e}^{-K(\lambda)} D(\lambda). \end{aligned} \tag{2.1}$$

Obviously we have

$$W \in C^{r,0}_{t,\lambda}(\mathbb{R} \times \mathscr{B}_{\varrho^2}(\lambda^0), L(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})).$$

Finally, we define a function $Z^\lambda\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ for $\lambda \in \mathscr{B}_{\varrho^2}(\lambda^0)$ by

$$Z^\lambda(v, s) := v_1 W(\omega(\lambda)\, s, \lambda)\, e^1 + \ldots + v_n W(\omega(\lambda)\, s, \lambda)\, e^n + p(\omega(\lambda)\, s, \lambda).$$

The hypotheses (F_1) and (F_2) imply that Z^λ is r-times continuously differentiable where all derivatives continuously depend on λ, additionally all derivatives of Z^λ are differentiable with respect to v of any order.

It is obvious that Z^λ maps $\{0\} \times \mathbb{R}$ onto γ_λ $\forall \lambda \in \mathscr{B}_{\varrho^2}(\lambda^0)$. Hence, there are $\varrho^3 \in (0, \varrho^2)$ and $\mu^1 > 0$ such that $Z^\lambda\colon \mathscr{B}^n_{\mu^1}(0) \times \mathbb{R} \to \mathscr{G}$. In what follows we want to use v, s as local coordinates near γ_λ. To this end we state some properties of Z^λ.

(zi) There are $\varrho^4 \in (0, \varrho^3)$, $\mu^2 \in (0, \mu^1)$ such that the Jacobian $Z^\lambda_{(v,s)}(v, s)$ is regular for $(v, s) \in \mathscr{B}^n_{\mu^2}(0) \times \mathbb{R}$ and $\lambda \in \mathscr{B}_{\varrho^4}(\lambda^0)$. Let Q be the $n \times n$-matrix which we obtain from the $(n + 1) \times (n + 1)$-matrix $\tilde{Q}$ by deleting the last row and the last column.

(zii) $Z^\lambda(Qv, s+1) \equiv Z^\lambda(v, s)\ \forall (v, s) \in \mathscr{B}^n_{\mu^2}(0) \times \mathbb{R}$ and $\lambda \in \mathscr{B}_{\varrho^4}(\lambda^0)$.

From (zi) and (zii) we obtain that Z^λ is only locally one-to-one for $\lambda \in \mathscr{B}_{\varrho^4}(\lambda^0)$. Let $\mathscr{G}_\lambda$ be the image of $\mathscr{B}^n_{\mu^2}(0) \times \mathbb{R}$ by Z^λ. For $\lambda \in \mathscr{B}_{\varrho^4}(\lambda^0)$ we associate to the system (1.1) restricted to $\mathscr{G}_\lambda$ the differential system

$$(2.2)\qquad \begin{pmatrix} \dfrac{dv}{dt} \\[2ex] \dfrac{ds}{dt} \end{pmatrix} = \begin{pmatrix} \tilde{g}(v, s, \lambda) \\[2ex] \tilde{h}(v, s, \lambda) \end{pmatrix} := (Z^\lambda_{(v,s)}(v, s))^{-1} f(Z^\lambda(v, s)),$$

$$(v, s) \in \mathscr{B}^n_{\mu^2}(0) \times \mathbb{R}.$$

It is clear that Z^λ maps the trajectories of (2.2) onto the trajectories of (1.1), vice versa the multivalued mapping $(Z^\lambda)^{-1}$ maps the trajectories of (1.1) onto the trajectories of (2.2).

(ziii) The matrix $\hat{K}(\lambda)$ defined in (2.1) is of the type

$$\hat{K}(\lambda) = \begin{pmatrix} \hat{A}(\lambda) & 0_{n,1} \\ a^{\mathrm{T}}(\lambda) & 0 \end{pmatrix}$$

where $\hat{A} \in C(\mathscr{B}_{\varrho^4}(\lambda^0), L(\mathbb{R}^n, \mathbb{R}^n))$, $a \in C(B_{\varrho^4}(\lambda^0), \mathbb{R}^n)$.

The functions $\tilde{g}$ and $\tilde{h}$ defined in (2.2) have the form

$$\tilde{g}(v, s, \lambda) = \hat{A}(\lambda)\, v + \hat{g}(v, s, \lambda),$$

$$\tilde{h}(v, s, \lambda) = \omega(\lambda)^{-1} + \hat{h}(v, s, \lambda)$$

where $\hat{g}$ and $\hat{h}$ satisfy

$$\hat{g}(0, s, \lambda) \equiv 0, \left.\frac{\partial \hat{g}(v, s, \lambda)}{\partial v}\right|_{v=0} \equiv 0, \quad \hat{h}(0, s, \lambda) \equiv 0,$$

$$\hat{g}(Qv, s+1, \lambda) \equiv Q\hat{g}(v, s, \lambda), \quad \hat{h}(Qv, s+1, \lambda) \equiv \hat{h}(v, s, \lambda).$$

Hence, the system (2.2) can be written in the form

$$(2.3)\qquad \frac{dv}{dt} = \hat{A}(\lambda)\, v + \hat{g}(v, s, \lambda),$$

$$\frac{ds}{dt} = \omega(\lambda)^{-1} + \hat{h}(v, s, \lambda).$$

In virtue of $\omega(\lambda^0)^{-1} + h(0, s, \lambda^0) = \omega(\lambda^0)^{-1} > 0$ there are $\varrho^5 \in (0, \varrho^4)$, $\mu^3 \in (0, \mu^2)$ such that $ds/dt \geqq \omega(\lambda^0)^{-1}/2$ on $\mathscr{B}^n_{\mu^3}(0) \times \mathbb{R} \times \mathscr{B}_{\varrho^5}(\lambda^0)$. Consequently, the system (2.3) has the same trajectories as the system

$$(2.4)\qquad \frac{dv}{dt} = A(\lambda)\, v + g(v, s, \lambda), \quad \frac{ds}{dt} = 1$$

which we obtain by multiplying the right hand side of (2.3) by $(\omega(\lambda)^{-1} + \hat{h}(v, s, \lambda))^{-1}$.

Here,

$$(2.5)\qquad A(\lambda) := \omega(\lambda)\, \hat{A}(\lambda)\,, \quad g(0, s, \lambda) \equiv 0\,, \quad \frac{\partial g(v, s, \lambda)}{\partial v}_{|v=0} \equiv 0\,,$$

$$g(Qv, s + 1, \lambda) \equiv Qg(v, s, \lambda)\,.$$

It is clear that the phase space of (2.4) coincides with the (t, x)-space (space of motion) of the nonautonomous system

$$(2.6)\qquad \frac{\mathrm{d}v}{\mathrm{d}t} = A(\lambda)\, v + g(v, t, \lambda)$$

and that the graph of the solution $v(t; v^0, t^0, \lambda)$ of (2.6) corresponds to the trajectory of (2.4) passing through (v^0, s^0).

By $(Qv, s + 1) \sim (v, s)$ we define an equivalence relation on $\mathbb{R}^n \times \mathbb{R}$, the set of corresponding equivalence classes is denoted by $\mathcal{M}$. The canonical map $\pi\colon \mathbb{R}^n \times \times \mathbb{R} \to \mathcal{M}$ which orders to each $(v, s) \in \mathbb{R}^n \times \mathbb{R}$ its equivalence class induces in $\mathcal{M}$ a topology. Since π is locally one-to-one we can conclude that $\mathcal{M}$ is a smooth manifold [27].

From (zii) it follows that we may interpret Z^λ as mapping from $\pi(\mathcal{B}^n_{\mu^3}(0) \times \mathbb{R})$ into $\mathcal{G}$ for $\lambda \in \mathcal{B}_{\varrho^5}(\lambda^0)$. It is easy to show that there are numbers $\mu^4 \in (0, \mu^3)$, $\varrho^6 \in \in (0, \varrho^5)$ such that for $\lambda \in \mathcal{B}_{\varrho^6}(\lambda^0)$ the mapping Z^λ is a one-to-one mapping on $\pi(\mathcal{B}^n_{\mu^3}(0) \times \mathbb{R})$. From (2.5) we get that for any solution $v = \psi(t; v^0, t^0, \lambda)$ of (2.6) $\pi(\psi(t; v^0, t^0, \lambda), t)$ depends only on $\pi(v^0, t^0)$, t, λ. Therefore, we may consider (2.6) as a system with $\pi(\mathcal{B}^n_{\mu^4}(0) \times \mathbb{R}) \subset \mathcal{M}$ as space of motion for $\lambda \in \mathcal{B}_{\varrho^6}(0)$.

Summarizing our results we get

Theorem 2.1. *Suppose the hypotheses* (F_1) *and* (F_2) *are valid. Then there are constants* $\mu > 0$, $\varrho > 0$, *a differential system* (2.6) *defined on* $\mathcal{B}^n_\mu(0) \times \mathbb{R}$ *for* $\lambda \in \mathcal{B}_\varrho(\lambda^0)$, *and a family of mappings* $Z^\lambda\colon \mathcal{B}^n_\mu(0) \times \mathbb{R} \to \mathcal{G}$ *such that the following properties are valid:*

(i) $A \in C(\mathcal{B}_\varrho(\lambda^0), L(\mathbb{R}^n, \mathbb{R}^n))$. $A(\lambda) = \operatorname{diag}(A_1(\lambda), \ldots, A_k(\lambda))$ *where the spectrum of* $A_i(\lambda)$ *consists either of a pair of conjugate complex eigenvalues or of a real eigenvalue and where* $\sigma(A_i(\lambda)) \cap \sigma(A_j(\lambda)) = \emptyset$ *for* $i \neq j$. $\mathrm{e}^{\sigma(A(\lambda))} \cup \{1\} = \sigma(Q\, P(\lambda))$ *where* $P(\lambda)$ *is a monodromy matrix of* γ_λ.

(ii) $g\colon \mathcal{B}^n_\mu(0) \times \mathbb{R} \times \mathcal{B}_\varrho(\lambda^0) \to \mathbb{R}^n$ *is continuous and has continuous derivatives with respect to* v *and* t *up to the order* r, *possibly except the derivative* $\partial^r g/\partial t^r$, *satisfying*

$$(2.7)\qquad g(0, t, \lambda) \equiv 0\,, \quad \frac{\partial g(v, t, \lambda)}{\partial v}_{|v=0} \equiv 0 \quad \forall (t, \lambda) \in \mathbb{R} \times \mathcal{B}_\varrho(\lambda^0)\,,$$

$$g(Qv, t + 1, \lambda) \equiv Qg(v, t, \lambda) \quad \forall (v, t, \lambda) \in \mathcal{B}^n_\mu(0) \times \mathbb{R} \times \mathcal{B}_\varrho(\lambda^0)\,.$$

(iii) Z^λ *has the regularity properties mentioned above.*

(iv) *Z^λ as mapping of $\pi(\mathscr{B}^n_\mu(0) \times \mathbb{R})$ on $\mathscr{G}_\lambda$ is one-to-one, where $(Z^\lambda)^{-1}$ is r-times continuously differentiable and all its derivatives continuously depend on λ.*

Z^λ maps the graph of the solution of (2.6) passing through (v^0, t^0) on the trajectory of (1.1) through $x^0 = Z^\lambda(v^0, t^0)$ $\forall \lambda \in \mathscr{B}_\varrho(\lambda^0)$. Especially we have

$$Z^\lambda(\pi(\{0\} \times \mathbb{R})) = \gamma_\lambda .$$

Using Theorem 2.1 we are able to determine the topological structure of the trajectories of (1.1) near γ_λ by investigating the behaviour of the solutions of (2.6) near $v = 0$.

The manifold $\mathscr{M}$ is generated by identifying the points $(Qv, t + 1)$ and (v, t) in $\mathbb{R}^n \times \mathbb{R}$. Since we have $Q = \operatorname{diag}(q_1, \ldots, q_n)$ where $q_i = \pm 1$ this identifying proceeds coordinate wise that means we identify (v_i, t) and $(v_i, t + 1)$ or $(-v_i, t + 1)$ resp. Therefore, the plane spanned by $e^i \in \mathbb{R}^n$ and $1 \in \mathbb{R}$ is mapped by π onto the cylinder $\mathbb{R} \times \mathbb{S}^1$ or the Möbius strip $\mathscr{M}^2$.

Let us now consider a $(m + 1)$-dimensional subspace $\mathbb{H}$ of $\mathbb{R}^n \times \mathbb{R}$ spanned by m of the vectors $e^1, \ldots, e^n \in \mathbb{R}^n$ and $1 \in \mathbb{R}$ $(m \leqq n)$. Without loss of generality we assume that $\mathbb{H}$ is spanned by $e^1, \ldots, e^m \in \mathbb{R}$ and $1 \in \mathbb{R}$. If all corresponding $q_1, \ldots, q_m$ are equal to 1 then it is obvious that $\pi(\mathbb{H})$ is homeomorphic to $\mathbb{R}^m \times \mathbb{S}^1$ and if exactly one of the q_i $(i = 1, \ldots, m)$ equals -1 then $\pi(\mathbb{H})$ is homeomorphic to $\mathbb{R}^{m-1} \times \mathscr{M}^2$. We now assume $l \geqq 2$ of the q_i to be -1, without loss of generality be $q_1 = q_2 = -1$. Together with Q we consider $Q' = \operatorname{diag}(1, 1, q_3, \ldots, q_n)$ and the corresponding manifold $\mathscr{M}'$. It is easy to see that

$$v_1 \to v_1 \cos \pi t + v_2 \sin \pi t\,, \quad v_2 \to -v_1 \sin \pi t + v_2 \cos \pi t\,,$$

$$v_3 \to v_3, \ldots, v_n \to v_n\,, \quad t \to t$$

defines a homeomorphism of $\mathscr{M}$ on $\mathscr{M}'$. By this way we get: $\pi(\mathbb{H})$ is homeomorphic to $\mathbb{R}^m \times S^1$ if the number of elements being -1 in Q corresponding to $\mathbb{H}$ is even else $\pi(\mathbb{H})$ is homeomorphic to $\mathbb{R}^{m-1} \times \mathscr{M}^2$.

From the property $\det P(\lambda^0) > 0$ we obtain that an even number of elements of Q equals -1, that is, $\mathscr{M} = \pi(\mathbb{R}^n \times \mathbb{R})$ is homeomorphic to $\mathbb{R}^n \times \mathbb{S}^1$.

3. EXISTENCE, PERSISTENCE AND STRUCTURE OF INTEGRAL MANIFOLDS NEAR γ_{λ^0}

According to Theorem 2.1 we may assume

$$A(\lambda) = \operatorname{diag}(A^-(\lambda), A^0(\lambda), A^+(\lambda)) \tag{3.0}$$

where $\operatorname{Re} \sigma(A^-(\lambda^0)) \leqq -\varkappa$, $\operatorname{Re} \sigma(A^0(\lambda^0)) = 0$, $\operatorname{Re} \sigma(A^+(\lambda^0)) \geqq \varkappa$ for some $\varkappa > 0$. By the perturbation theory of linear operators [9] there is $\varrho^7 \in (0, \varrho)$ such that

$$\operatorname{Re} \sigma(A^-(\lambda)) \leqq -\frac{\varkappa}{2} < -\frac{\varkappa}{4} \leqq \operatorname{Re} \sigma(A^0(\lambda)) \leqq \frac{\varkappa}{4} < \frac{\varkappa}{2} \leqq \operatorname{Re} \sigma(A^+(\lambda)) \tag{3.1}$$

for $\lambda \in \mathscr{B}_{\varrho 7}(\lambda^0)$. Using (3.0) we may rewrite (2.6) in the form

$$\frac{dv_1}{dt} = A^-(\lambda)\, v_1 + g_1(v, t, \lambda)\,, \tag{3.2}$$

$$\frac{dv_2}{dt} = A^0(\lambda)\, v_2 + g_2(v, t, \lambda)\,, \tag{3.2}$$

$$\frac{dv_3}{dt} = A^+(\lambda)\, v_3 + g_3(v, t, \lambda)$$

where the matrices $A^-(\lambda)$, $A^0(\lambda)$, $A^+(\lambda)$ satisfy (3.1) and the function $g(v, t, \lambda)$ obeys (2.7). Modifying the function g outside some sufficiently small neighbourhood of $v = 0$ we may additionally suppose that g is defined for all $v \in \mathbb{R}^n$ and fulfils for some δ_1, δ_2

$$|g(v, t, \lambda)| \leqq \delta_1\,, \quad \|g_v(v, t, \lambda)\| \leqq \delta_2 \qquad \forall (v, t, \lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathscr{B}_{\varrho 7}(\lambda^0)\,. \tag{3.3}$$

Let $\mathfrak{M}_s(\lambda)$ $[\mathfrak{M}_u(\lambda)]$ be the set of all $(v^*, t^*) \in \mathbb{R}^n \times \mathbb{R}$ satisfying $|\bar{v}\,(t; t^*, v^*, \lambda)| = 0\,(e^{-\varkappa|t|/2})$ as $t \to +\infty$ $(t \to -\infty)$. Let $\mathfrak{M}_c(\lambda)$ be the set of all solutions of (3.2) whose v_1-component is bounded for decreasing t and whose v_3-component is bounded for increasing t.

Using the same techniques as in [24, 25] we can prove:

Theorem 3.1. *Assume the following hypotheses hold*:

(A) $A^-, A^0, A^+ \in C(\mathscr{B}_{\varrho 7}(\lambda^0), L(\mathbb{R}^{m_i}, \mathbb{R}^{m_i}))$ *satisfy* (3.1).

(G_1) $g \in C^{r}_{(v,t)}\,{}^{\lambda}_{0}(\mathbb{R}^n \times \mathbb{R} \times \mathscr{B}_{\varrho 7}(\lambda^0), \mathbb{R}^n)$ *satisfies* (2.8) *and* (3.3) *where* δ_2 *is sufficiently small.*

Then the sets $\mathfrak{M}_s$, $\mathfrak{M}_u$, *and* $\mathfrak{M}_c$ *are integral manifolds of the system* (3.2) *in* (t, x)*-space having the representations*

$$\mathfrak{M}_s(\lambda)\colon v_2 = S_2(v_1, t, \lambda)\,, \quad v_3 = S_3(v_1, t, \lambda)\,,$$

$$\mathfrak{M}_u(\lambda)\colon v_2 = U_2(v_3, t, \lambda)\,, \quad v_1 = U_1(v_3, t, \lambda)\,,$$

$$\mathfrak{M}_c(\lambda)\colon v_1 = C_1(v_2, t, \lambda)\,, \quad v_3 = C_3(v_2, t, \lambda)$$

where the functions $S_2, S_3, U_1, U_2, C_1, C_3$ *possess the same regularity properties as* g *and vanish together with their derivative with respect to* v *at* $v = 0$.

Remark 3.1.1. In case $\lambda = \lambda^0$ the manifold $\mathfrak{M}_s(\lambda)$ $[\mathfrak{M}_u(\lambda)]$ is said to be the stable (unstable) manifold, $\mathfrak{M}_c(\lambda)$ is referred to as center manifold. For $\lambda \neq \lambda^0$ $\mathfrak{M}_s(\lambda)$, $\mathfrak{M}_u(\lambda)$, and $\mathfrak{M}_c(\lambda)$ resp. are called the perturbed stable, unstable, and center manifold resp. Corresponding to the partition $v = (v_1, v_2, v_3)$ we have $Q = \operatorname{diag}(Q^1, Q^2, Q^3)$. By (2.8) the relations

$$g_i(Qv, t + 1, \lambda) = Q^i g_i(v, t, \lambda), \quad i = 1, 2, 3$$

hold. The following theorem says that the functions S_j, U_j, C_j resp. describing the manifolds $\mathfrak{M}_s(\lambda)$, $\mathfrak{M}_u(\lambda)$, $\mathfrak{M}_c(\lambda)$ resp. satisfy the corresponding relation.

Theorem 3.2. *Assume the hypotheses of Theorem* 3.1 *to be valid. Then we have for* $(v, t, \lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathscr{B}_{\varrho_7}(\lambda^0)$

$$\begin{aligned} S_j(Q^1 v_1, t + 1, \lambda) &= Q^j S_j(v_1, t, \lambda), \quad j = 2, 3, \\ U_j(Q^3 v_3, t + 1, \lambda) &= Q^j U_j(v_3, t, \lambda), \quad j = 1, 2, \\ C_j(Q^2 v_2, t + 1, \lambda) &= Q^j C_j(v_2, t, \lambda), \quad j = 1, 3. \end{aligned}$$

Remark 3.2.1. Theorem 3.2 implies that the sets $\pi(\mathfrak{M}_s(\lambda))$, $\pi(\mathfrak{M}_u(\lambda))$, and $\pi(\mathfrak{M}_c(\lambda))$ are also integral manifolds of the system (3.2) in $\mathscr{M}$ as space of motion. Consequently, the images of these manifolds by $(Z^\lambda)^{-1}$ lying in a small neighbourhood of γ_λ are local integral manifolds of (1.1).

Corollary 3.3.1. *In case* $Q = I_n$ *the functions* S_j, U_j, C_j *are* 1*-periodic in t, that is, the integral manifolds* $\pi(\mathfrak{M}_s)$, $\pi(\mathfrak{M}_u)$, $\pi(\mathfrak{M}_c)$ *are homeomorphic to a cylinder. In case that* Q^1, Q^2, Q^3 *resp. contains an odd number of* (-1) *on the main diagonal the stable, unstable, center manifold resp, is homeomorphic to a manifold of type* $\mathbb{R}^l \times \mathscr{M}^2$.

Remark 3.3.2. The case $Q = I$ has been treated by Kelley [10], Knobloch [12], Pliss [21]. Example of dynamical systems whose stable and unstable manifolds are homeomorphic to a Möbius strip has been given by Reizinš [22] and Hale [4]. The question for existence of submanifold can be easily answered for the stable and unstable manifold.

Theorem 3.4. *Assume the hypotheses of Theorem* 3.1 *hold. Suppose further that* $\operatorname{Re} \sigma(A^+(\lambda^0))$ $[\operatorname{Re} \sigma(A^-(\lambda^0))]$ *can be separated into* $r^+[r^-]$ *disjoint sets. Then the manifold* $\mathfrak{M}_s(\lambda)$ $[\mathfrak{M}_u(\lambda)]$ *contains* $r^+ - 1$ $[r^- - 1]$ *submanifolds characterized by different velocities of approaching the stationary solutions* $v = 0$ *by their solutions.*

Remark 3.4.1. The underyling idea for the proof of Theorem 3.4 essentially is the same as for the proof of the manifolds $\mathfrak{M}_s(\lambda)$ and $\mathfrak{M}_u(\lambda)$.

Concerning center manifolds passing through an equilibrium point of an autonomous system the problem for existence of submanifolds has been treated by one of the authors for the first time [26]. The underlying method applies also in case of system (3.2). We formulate the prototyp of results in case of a three-dimensional system.

Consider the system

$$\text{(3.4)} \qquad \frac{dv_1}{dt} = A_1(\lambda)\, v_1 + h_1(v_1, v_2, t, \lambda)\,,$$

$$\frac{dv_2}{dt} = A_2(\lambda)\, v_2 + h_2(v_1, v_2, t, \lambda)$$

under the assumptions

(3.5)

(V_1) $A_1 \in C^1(\Lambda_1, L(\mathbb{R}^2, \mathbb{R}^2))\,, \quad A_2 \in C^1(\Lambda, \mathbb{R})$
where $\Lambda_1 := (-\lambda_1, \lambda_1)\,, \quad \lambda_1 > 0\,.$

(V_2) $\sigma(A_1(\lambda)) = \alpha_1(\lambda) \pm i\beta_1(\lambda)$ for $\lambda \in \Lambda_1\,,$
$\alpha_1(0) = A_2(0) = 0\,, \quad \beta_1(0) > 0\,,$
$\alpha_1'(0) > A_2'(0)\,.$

(V_3) $h = (h_1, h_2)^{\mathrm{T}} \in C^1(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \Lambda_1, \mathbb{R}^3)\,,$
$h(0, t, \lambda) \equiv 0\,, \quad h_v(0, t, \lambda) \equiv 0 \quad \forall (t, \lambda) \in \mathbb{R} \times \Lambda_1\,,$
$h(Q^1 v_1, Q^2 v_2, t + 1, \lambda) = Q h(v, t, \lambda) \quad \forall (v, t, \lambda) \in \mathbb{R}^3 \times \mathbb{R} \times \Lambda_1\,,$
where $Q = \operatorname{diag}(Q^1, Q^2)\,, \quad Q^l = \operatorname{diag}(q_1^l, q_2^l)\,, \quad q_i^l = \pm 1\,,$
$|h(v, t, \lambda)| = o(|\lambda|)\,, \quad \|h_v(v, t, \lambda)\| = o(|\lambda|) \quad \forall (v, t) \in \mathbb{R}^3 \times \mathbb{R}\,.$

Then the following result holds.

Theorem 3.5. *Suppose* (V_1), (V_2), (V_3) *hold. Then there exists a unique integral manifold of the system* (3.4) *having the representation*

$$v_1 = \varphi(v_2, t, \lambda) \quad (v_2, t, \lambda) \in \mathbb{R} \times \mathbb{R}\,, \quad \lambda_2 < \lambda_1\,,$$

where φ *possesses the same regularity properties as* h *and satisfies*

$$\varphi(0, t, \lambda) \equiv 0\,, \quad \varphi_{v_2}(0, t, \lambda) \equiv 0 \quad \forall (t, \lambda) \in \mathbb{R} \times \Lambda_2\,,$$

$$\varphi(Q^2 v_2, t + 1, \lambda) = Q^1 \varphi(v_2, t, \lambda) \quad \forall (v_2, t, \lambda) \in \mathbb{R} \times \mathbb{R} \times \Lambda_2\,.$$

References

[1] *N. N. Bogoljubov, Ju. A. Mitropol'skij:* Asymptotic methods in the theory of nonlinear oscillations. (Russian) Moscow 1955.

[2] *E. A. Coddington, N. Levinson:* Theory of ordinary differential equations. McGraw-Hill, New York 1955.

[3] *S. Diliberto, G. Hufford:* Perturbation theorems for nonlinear differential equations. Ann. Math. Stud. *30* (1956), 207–236.

[4] *J. K. Hale:* Ordinary differential equations. John Wiley & Sons, New York.

[5] *J. K. Hale:* Integral manifolds for perturbed differential systems. Ann. Math. *73* (1961), 496–531.

[6] *Ph. Hartman:* Ordinary differential equations. Birkhäuser, Boston 1982.

[7] *D. Henry:* Geometric theory of semilinear parabolic equations. Lect. Not. Math. 840, Springer, Berlin 1981.

[8] *G. Iooss:* Bifurcation of maps and applications. North-Holland, Amsterdam 1979.
[9] *T. Kato:* Perturbation theory for linear operators. Springer, Berlin 1966.
[10] *A. Kelley:* The stable, center-stable, center, center-unstable, unstable manifolds. J. Diff. Eqs. *3* (1967), 546—570.
[11] *H. W. Knobloch, F. Kappel:* Gewöhnliche Differentialgleichungen. Teubner, Stuttgart 1974.
[12] *H. W. Knobloch:* Hopf bifurcation via integral manifolds. Abh. AdW DDR 1977, 3N, 413—419.
[13] *H. W. Knobloch, B. Aulbach:* The role of center manifolds in ordinary differential equations. Teubner-Texte zur Mathematik *47*, 179—189. Leipzig 1982.
[14] *N. M. Krylov, N. N. Bogoljubov:* Application of methods from nonlinear mechanics to the theory of stationary oscillations. (Russian) Kiev 1934.
[15] *J. Kurzweil:* Exponentially stable integral manifolds, averaging principle and continuous dependence on a parameter. Czech. Mat. J. *16* (1966), 380—423, 463—491.
[16] *J. Kurzweil:* Invariant sets of differential systems. (Russian) Diff. Uravn. IV (1968), 785—797.
[17] *J. Kurzweil:* Invariant manifolds. Comm. Math. Univ. Carol. *11* (1970), 309—336.
[18] *O. E. Landford:* Bifurcation of periodic solutions into invariant tori: the work of Ruelle and Takens. Lect. Not. Math. *322* (1973), 159—192.
[19] *J. E. Marsden, M. McCracken:* The Hopf bifurcation and its applictions. Springer, New York 1976.
[20] *Ju. A. Mitropol'skij, O. B. Lykova:* Integral manifolds in the nonlinear mechanics. (Russian) Nauka, Moscow 1973.
[21] *V. A. Pliss:* Integral manifolds of systems of periodic differential equations. (Russian) Nauka, Moscow 1977.
[22] *L. E. Reizinš:* Systems of differential equations in local coordinates in the neighbourhood of a closed trajectory. (Russian) Izv. AN Latv. SSR, ser. fiz. techn. nauk *1*, (1964), 59—66.
[23] *L. E. Reizinš:* Local equivalence of differential equations. (Russian) Zinatne, Riga 1971.
[24] *K. R. Schneider:* Hopf bifurcation and center manifolds. Coll. Math. Soc. J. Bolyai *30* (1979), 953—970.
[25] *K. R. Schneider:* On the application of integral manifolds to Hopf bifurcation. Math. Nachr. *97* (1980), 313—323.
[26] *K. R. Schneider:* On the existence of perturbed center submanifolds of a class of autonomous differential systems. Preprint P-26/80 ZIMM der AdW der DDR.
[27] *A. H. Wallace:* Differential Topology. Benjamin. New York 1968.

Authors' addresses: W. Meiske, AdW der DDR, Zentralinstitut für Molekularbiologie, Robert-Rössle-Str. 10, 1115 Berlin-Buch, DDR; K. R. Schneider, AdW der DDR, Karl-Weierstraß-Institut für Mathematik, Mohrenstr. 39, 1086 Berlin, DDR.