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A CLASS OF ONE-DIMENSIONAL DEGENERATE PARABOLIC EQUATIONS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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1. INTRODUCTION

In this paper we use self similar solutions to study the one-dimensional degenerate Cauchy problem

$$\text{(P)} \qquad \begin{aligned} v_t &= \phi(v)_{xx}\,, \quad x \in \mathbb{R}\,, \quad t > 0\,, \\ v(x, 0) &= g(x) \end{aligned}$$

for the constitutive function $\phi(v) = \max(v, 0)$. We assume that the initial data g are smooth on $\mathbb{R} \setminus \{0\}$ with at most exponential growth at infinity and satisfy

$$\text{(g)} \qquad x\, g(x) \geqq 0\,, \quad x \neq 0\,, \quad g(0) = 0\,,$$

and the inequality is strict near $x = 0$. The principal issue is the regularity of the free boundary near $t = 0$.

Problems of this type arise as convexifications of diffusion equations with non-monotone constitutive functions as has been discussed in [HN 1, H]. The behavior of solutions for (P) is similar to those of the one phase Stefan problem in which $g(x) \equiv -1$ for $x < 0$, and g has a jump discontinuity at $x = 0$. However, (P) and the Stefan problem exhibit different regularity properties of the free boundary near $t = 0$.

Existence and uniqueness of weak solutions of (P) can be proved using nonlinear semigroup theory [BCPa, E]. Indeed, Benilan, Crandall, and Pierre [BCPi] have obtained optimal existence and uniqueness results for the porous medium equation (i.e. problem (P) with $\phi(v) = \max(v^m, 0)$, $m > 1$) in $\mathbb{R}^n$. However, their results do not apply directly to the case $m = 1$. Details of the proofs in the present context are given in Vázquez [V1] and the results (Theorems 3.1, 3.2) are stated in Section 3. Using standard approximation arguments and the comparison method discussed in [V1] (see Section 3) one establishes the existence of a continuous monotone decreasing free boundary $t \to s(t)$, $s(0) = 0$ and $v(s(t)^+, t) = 0$. Moreover, the pair (v, s) satisfies the free boundary problem

$$(\bar{\mathrm{P}}) \qquad \begin{aligned} &v_t = v_{xx}\,, \quad x > s(t)\,, \quad t > 0\,,\\ &v(x, 0) = g(x)\,, \quad x > 0\,,\\ &v(s(t)^+, t) = 0\,,\\ &g(s(t))\, s'(t) = v_x(s(t)^+, t)\,, \quad t > 0\,,\\ &s(0) = 0. \end{aligned}$$

Conversely, the solution v of $(\bar{\mathrm{P}})$ extended by $v(x, t) = g(x)$ for $x < s(t)$ is a weak solution of (P).

In the special case of (P), (g) with the function g satisfying the additional assumption: $g'(0^+), g'(0^-) \neq 0$, we have shown by solving a singular integral equation [HN2, 3] that the problem $(\bar{\mathrm{P}})$ has a unique solution (v, s). Moreover,

$$(1.1) \qquad s(t) = -\kappa \sqrt{t} + 0(t^{\alpha + 1/2}) \quad (t \to 0^+)\,,$$

for any $0 < \alpha < 1/2$, where κ is the uniquely defined monotone function of $p := := g'(0^+)/g'(0^-)$ determined implicity by the equation

$$(1.2) \qquad p = \frac{\kappa^2}{2} + \frac{\kappa^3}{4} \mathrm{e}^{\kappa^2/4} \int_{-\kappa}^{\infty} \mathrm{e}^{-y^2/4}\, \mathrm{d}y\,.$$

For $t \geqq \varepsilon > 0$ the free boundary is smooth. It should be observed that a formal expansion of the ODE $g(s(t))\, s'(t) = v_x(s(t)^+, t)$ yields

$$(p_-)\, s(t)\, s'(t) = (p_+) + \dots\,,$$

so that

$$s(t) = -\sqrt{(2pt)} + \dots\,,$$

where ... denote terms of higher degree powers of t. However, if $p = 1$, then (1.2) implies $\kappa = .9034\ldots \neq \sqrt{2}$. The reason for this apparent inconsistency is that all derivatives of v become singular at (x, t) at $(0, 0)$ and in particular, v_x is discontinuous at $(0, 0)$. By contrast, for the one phase Stefan problem [FP 1, 2; KN; S] the solution is smooth on the set $\{(x, t) : x \geqq s(t), t \in [0, T]\}$.

The purpose of this note is to discuss the regularity and qualitative behavior for small t of the free boundary of (P) for more general initial data g. Of particular interest is the model datum

$$(\mathrm{D}) \qquad g(x) = \begin{cases} p_+ x^\gamma\,, & x \geqq 0\,,\\ -p_- |x|^\gamma\,, & x < 0\,, \end{cases}$$

where $p_\pm$, $\gamma > 0$ are given constants, for which the integral equation method of [HN2, 3] breaks down if $\gamma \neq 1$, i.e. if the datum g is not piecewise linear. It will be shown in Section 2 that (P) with the model initial datum (D) can be solved explicitly for all $\gamma > 0$ using self similar solutions, and the free boundary is determined explicitly for all $t \geqq 0$ by proving the following result.

Proposition 1.1. *For the model datum* (D) *problem* ($\bar{\text{P}}$) *has the unique self similar solution*

$$v(x, t) = t^{\gamma/2}\psi\left(\frac{x}{\sqrt{t}}\right), \quad x > s(t), \quad t > 0$$

where $\psi(\cdot)$ *is the unique solution of the ordinary differential equation*

$$2\,\psi''(\xi) + \alpha\,\psi'(\xi) - \gamma\,\psi(\xi) = 0\,, \quad \xi > -\kappa \tag{1.3}$$

subject to the initial conditions

$$\psi(-\kappa) = 0\,, \quad \psi'(-\kappa) = (p_-)\frac{\kappa^{\gamma+1}}{2}\,, \tag{1.4}$$

and κ *is related to* $p_\pm$ *via the condition*

$$\lim_{\xi \to +\infty} \xi^{-1}\,\psi(\xi) = p_+\,. \tag{1.5}$$

The free boundary is $s(t) = -\kappa\sqrt{t}$, $t \geqq 0$. *Moreover, given* $p = p_+/p_- > 0$, $\kappa > 0$ *is the unique solution of the equation*

$$p = \left(\frac{\kappa}{\sqrt{2}}\right)^{\gamma+1} D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right), \tag{1.6}$$

where $D_{-\gamma-1}(\cdot)$ *is the parabolic cylinder function of index* $(-\gamma - 1)$ [B, p. 119].

It is readily verified that (1.6) reduces to (1.2) if $\gamma = 1$. The regularity of the free boundary for problem (P) with initial data (g) more general data than (D) can then be discussed using the above explicit solutions as comparison functions. This will be done briefly in Section 3 using ideas of Vázquez [V1, 2]. The principal result is

Theorem 1.2. *Let* v *be the solution of* (P), *where the smooth datum* g *satisfies assumptions* (g), $x\,g(x) > 0$ *for* x *near zero, and there exists* $\gamma > 0$ *such that*

$$\begin{aligned} -\lim_{x \to 0^-} g(x)\,|x|^{-\gamma} &= a > 0 \\ \lim_{x \to 0^+} G(x)\,x^{-(\gamma+1)} &= b > 0\,, \end{aligned} \tag{1.7}$$

where $G(x) = \int_0^x g(\xi)\,d\xi$, $x \geqq 0$. *Let* $p := b(\gamma + 1)/a$. *Then*

$$s(t) = -\,\kappa t^{1/2}(1 + o(1)) \quad (t \to 0^+)\,, \tag{1.8}$$

where $\kappa = \kappa(p) > 0$ *is the uniquely defined inverse of the function* p *defined by equation* (1.6); *for* $t \geqq \varepsilon > 0$ *the free boundary is smooth for as long as it exists.*

As already remarked in the special case $\gamma = 1$ the integral equation approach yields the stronger result (1.1). The proof of Theorem 1.2 is sketched in Section 3.

Proposition 1.1 and the comparison method used in the proof of Theorem 1.2 can be used to discuss the regularity of the free boundary near $t = 0$ for initial data more general than those satisfying (1.7). For example, if, in place of (D),

$$g(x) = \begin{cases} ax^{\alpha}, & x \geqq 0 \\ -|x|^{\beta}, & x < 0, \end{cases} \tag{1.9}$$

where $a, \alpha > 0$, $\beta \geqq 0$, and $\alpha > \beta$, then there exist constants $c_1, c_2 > 0$ which depend on α, β such that

$$c_1 a^{1/(\beta+1)} t^{\lambda} \leqq -s(t) \leqq c_2 a^{1/(\beta+1)} t^{\lambda}, \tag{1.10}$$

where $\lambda = (\alpha + 1)/2(\beta + 1)$, for $t \geqq 0$ sufficiently small. On the other hand, if $\beta > \alpha$ in (1.9),

$$\lim_{t \to 0^+} \frac{-s(t)}{t^{1/2}|\log t|^{1/2}} = (2(\beta - \alpha))^{1/2}.$$

If $\beta = \alpha$ in (1.9), $\lambda = \frac{1}{2}$ and the situation is covered by Proposition 1.1.

The constitutive function $\phi(v) = \max(v, 0)$ in (P) is piecewise linear. However, self similar solutions of (P, D) exist for more general constitutive functions ϕ,but they cannot in general be found explicitly. We expect to investigate the regularity of the free boundary near $t = 0$ for such problems in future work. This question is not discussed in [BMPe] where other qualitative aspects of several such problems are investigated.

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2. SELF SIMILAR SOLUTIONS

In this section we consider the Cauchy problem (P) for the model datum (D). Let $\bar{v} = \bar{v}(x, t; \gamma, p_+, p_-)$ be the solution of (P), (D), and define the transformation $v \to Tv$ by

$$Tv(x, t) = \mu v(\lambda x, \lambda^2 t), \tag{2.1}$$

where $\mu, \lambda > 0$ are constants. T transforms solutions into solutions and in particular, it is clear that $(p_-)\,\bar{v}(x, t; \gamma, p, 1)$, where $p = p_+/p_-$, is a solution of (P), (D). Denoting $\bar{v}(x, t, \gamma; p, 1) := \bar{v}(x, t; \gamma, p)$, uniqueness (Theorem 3.2) implies that

$$\bar{v}(x, t; \gamma, p_+, p_-) = (p_-)\,\bar{v}(x, t; \gamma, p). \tag{2.2}$$

Since $T\bar{v}(x, 0; \gamma, p) = \bar{v}(x, 0; \gamma, p)$ if and only if $\mu\lambda^{\gamma} = 1$, uniqueness of solutions of (P), (D) further implies that

$$\bar{v}(x, t; \gamma, p) = (\lambda^{-\gamma})\, \bar{v}(\lambda x, \lambda^2 t; \gamma, p) \tag{2.3}$$

for any $\lambda > 0$. Fixing the point (x, t) and choosing $\lambda = t^{-1/2}$ in (2.3) implies that $\bar{v}$ is the self similar solution

$$\bar{v}(x, t; \gamma, p) := t^{\gamma/2}\psi(xt^{-1/2}) \tag{2.4}$$

of the Cauchy problem (P), (D) for any $\gamma > 0$. By the comparison method (Section 3) the free boundary s of (P), (D) is monotone decreasing and $s'(t) < 0$ for $t > 0$, $s(0) = 0$. Moreover, by the equivalence of problems (P), (D) and ($\bar{\text{P}}$), (D) the free boundary $s = \{(x, t) : xt^{-1/2} = -\kappa,\ t > 0\}$, where $\kappa > 0$ is the constant uniquely determined in Proposition 1.1, and by Proposition 3.3

$$\bar{v}(x, t; \gamma, p) > 0\,, \quad x > s(t)\,, \quad t \geqq 0\,.$$

Proof of Proposition 1.1. Substituting $\bar{v}$ defined by (2.4) in ($\bar{\text{P}}$) one sees that ψ must satisfy the linear differential equation (1.3), and conditions (1.4) and (1.5).

Equation (1.3) can be solved explicitly. Put $x = \xi/\sqrt{2}$ and $w(\xi) = \psi(x)$. Then (1.3) becomes

$$w''(x) + x\, w'(x) - \gamma\, w(x) = 0\,. \tag{2.5}$$

Setting $w(x) := y(x) \exp(-x^2/4)$ we obtain

$$y''(x) - \left(\frac{1}{2} + \gamma + \frac{x^2}{4}\right) y(x) = 0\,. \tag{2.6}$$

This differential equation has the general solution [B, p. 116–117]

$$y(x) = b_1\, D_{-\gamma-1}(x) + b_2\, D_\gamma(\mathrm{i}x) \quad (-\infty < x < \infty,\ \gamma > 0)\,,$$

where $D_\nu(\cdot)$ is the parabolic cylinder function of index ν. Thus the general solution of (1.3) is

$$\psi(\xi) = \left[b_1 D_{-\gamma-1}\left(\frac{\xi}{\sqrt{2}}\right) + b_2 D_\gamma\left(\frac{\mathrm{i}\xi}{\sqrt{2}}\right)\right] \exp\left(\frac{-\xi^2}{8}\right) \tag{2.7}$$

for $-\infty < \xi < \infty$ and $\gamma > 0$. To impose the initial conditions (1.4) we need the formulae [B, p. 119]

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[D_{-\gamma-1}\left(\frac{\xi}{\sqrt{2}}\right) \exp\left(\frac{-\xi^2}{8}\right)\right] = \frac{-1}{\sqrt{2}}\, D_{-\gamma}\left(\frac{\xi}{\sqrt{2}}\right) \exp\left(\frac{-\xi^2}{8}\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[D_\gamma\left(\frac{\mathrm{i}\xi}{\sqrt{2}}\right) \exp\left(\frac{-\xi^2}{8}\right)\right] = \frac{\mathrm{i}\gamma}{\sqrt{2}}\, D_{\gamma-1}\left(\frac{\mathrm{i}\xi}{\sqrt{2}}\right) \exp\left(\frac{-\xi^2}{8}\right).$$

Then the initial conditions (1.4) yield the pair of equations

$$
(2.8)\qquad
\begin{aligned}
&b_1 D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right) + b_2 D_\gamma\left(-\frac{i\kappa}{\sqrt{2}}\right) = 0 \\
&-b_1 D_\gamma\left(-\left(\frac{\kappa}{\sqrt{2}}\right) + i\gamma b_2 D_{\gamma-1}\left(-\frac{i\kappa}{\sqrt{2}}\right) = p_- \frac{\kappa^{\gamma+1}}{\sqrt{2}} \exp\left(\frac{\kappa^2}{8}\right).
\end{aligned}
$$

Because (2.6) is of self-adjoint form the Wronskian of $D_\gamma(\cdot)$, $D_{-\gamma-1}(\cdot)$ is constant,

$$
W(D_{-\gamma-1}(\cdot), D_\gamma(\cdot)) \equiv -\mathrm{i}\exp\left[\left(\frac{\gamma+1}{2}\right)\pi \mathrm{i}\right].
$$

Thus

$$
(2.9)\qquad
\begin{aligned}
b_1(\kappa) &= \frac{(p_-)\,\kappa^{\gamma+1} D_\gamma\left(-\frac{i\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right)}{\mathrm{i}\sqrt{2}\exp\left[\left(\frac{\gamma+1}{2}\right)\pi\mathrm{i}\right]} \\
b_2(\kappa) &= -\frac{(p_-)\,\kappa^{\gamma-1} D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right)}{\mathrm{i}\sqrt{2}\exp\left[\left(\frac{\gamma+1}{2}\right)\pi\mathrm{i}\right]},
\end{aligned}
$$

and (2.7) with b_1, b_2 given by (2.9) is the solution of (1.3) satisfying the initial conditions (1.4). To compute the limit in (1.5) we use [B, p. 122]:

$$
(2.10)\qquad D_\nu(z) = z^\nu \exp\left(-\frac{z^2}{4}\right)[1 + 0(|z|^{-2})] \quad \text{as} \quad |z| \to \infty,
$$

which is valid for $-3\pi/4 < \arg z < 3\pi/4$. Thus for $\xi \in R$, $\gamma > 0$,

$$
(2.11)\qquad
\begin{aligned}
&D_{-\gamma-1}\left(\frac{\xi}{\sqrt{2}}\right) = \exp\left(-\frac{\xi^2}{8}\right)\left(\frac{\xi}{\sqrt{2}}\right)^{-\gamma-1}[1 + O(|\xi|^{-2})], \quad \xi \to +\infty, \\
&\left|D_\gamma\left(\frac{i\xi}{\sqrt{2}}\right)\right| = \exp\left(\frac{\xi^2}{8}\right)\left(\frac{\xi}{\sqrt{2}}\right)^\gamma \left|\exp\left(\frac{i\gamma\pi}{2}\right)[1 + O(|\xi|^{-2})], \quad \xi \to +\infty.\right.
\end{aligned}
$$

Substitution of (2.11) and (2.9) into the general solution (2.7) yields

$$
(2.12)\qquad \psi(\xi) \simeq b_2(\kappa)\exp\left(\frac{i\gamma\pi}{2}\right)\left(\frac{\xi}{\sqrt{2}}\right)^\gamma [1 + O(1)], \quad \xi \to +\infty.
$$

From formula (2.9) we see that

$$
(2.13)\qquad b_2(k)\exp\left(\frac{i\gamma\pi}{2}\right) = \frac{p_-}{\sqrt{2}}\kappa^{\gamma+1} D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right).
$$

Imposing the asymptotic condition (1.5) and using (2.12), (2.13) we finally obtain

$$\lim_{\xi \to +\infty} \frac{\psi(\xi)}{\xi^{\gamma}} = p_+ = p_- \left(\frac{\kappa}{\sqrt{2}}\right)^{\gamma+1} D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right) \exp\left(\frac{\kappa^2}{8}\right) \tag{2.14}$$

which is equation (1.6).

To complete the proof of the Proposition we have to show that given any $p > 0$, (1.6) is uniquely solvable for κ. This is a consequence of uniqueness (Theorem 3.2). However, it can also be seen directly as follows. From [BO, p. 573]

$$\begin{aligned} D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right) &= \frac{\sqrt{\pi}}{2^{(\gamma+1)/2}\Gamma\left(1+\frac{\gamma}{2}\right)} \sum_{n=0}^{\infty} \frac{a_{2n}}{(2n)!}\left(\frac{\kappa}{\sqrt{2}}\right)^{2n} + \\ &+ \frac{\sqrt{\pi}}{2^{\gamma/2}\Gamma\left(\frac{1+\gamma}{2}\right)} \sum_{n=0}^{\infty} \frac{a_{2n+1}}{(2n+1)!}\left(\frac{\kappa}{\sqrt{2}}\right)^{2n+1}, \end{aligned} \tag{2.15}$$

is an analytic function of κ, $-\infty < \kappa < \infty$, $\gamma > -1$; $a_0 = a_1 = 1$,

$$a_{n+2} = \left(\gamma + \frac{1}{2}\right) a_n + \frac{n}{4}(n-1)\, a_{n-2}, \quad \text{and} \quad D_{-\gamma-1}(0) = \frac{\sqrt{\pi}}{2^{(\gamma+1)/2}\Gamma(1+\gamma/2)}.$$

The coefficients a_ν are positive, $D_{-\gamma-1}(-\kappa/\sqrt{2})$ is a positive, strictly increasing function of κ for $0 \leqq \kappa < \infty$, and by (1.6) so is $p(\kappa)$. Moreover $p(0) = 0$. This completes the proof of Proposition 1.1.

We conclude this section by establishing two useful asymptotic estimates. From (2.15) and (1.6)

$$p(\kappa) \simeq \frac{\sqrt{\pi}}{2^{(\gamma+1)/2}\Gamma(1+\gamma/2)}\left(\frac{\kappa}{\sqrt{2}}\right)^{\gamma+1} (\kappa \to 0^+). \tag{2.16}$$

Moreover, [BO, p. 574]

$$D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right) \simeq \frac{\sqrt{2\pi}}{\Gamma(1+\gamma)}\left(\frac{\kappa}{\sqrt{2}}\right)^{\gamma} \exp\left(\frac{\kappa^2}{8}\right) (\kappa \to +\infty),$$

and therefore, from (1.6)

$$p(\kappa) \simeq \frac{\sqrt{(2\pi)}}{\Gamma(1+\gamma)}\left(\frac{\kappa}{\sqrt{2}}\right)^{2\gamma+1} \exp\left(\frac{\kappa^2}{4}\right) (\kappa \to +\infty). \tag{2.17}$$

3. EXISTENCE UNIQUENESS AND COMPARISON OF SOLUTIONS OF (P)

In this section we state general existence and uniqueness results for (P), briefly discuss the comparison method, and sketch the proof of Theorem 1.2.

Consider the Cauchy problem (P) in $Q_T = \mathbb{R} \times [0, T)$ where $T > 0$, and assume that the datum $g \in L^\infty_{\text{loc}}(\mathbb{R})$ and satisfies

$$g(x) \leqq c \exp(hx^2), \quad x \in \mathbb{R}, \tag{3.1}$$

for some constants c, $h > 0$. Define $g^-(x) := \max(-g(x), 0)$. For existence of solutions of (P) one has the following result proved in Vázquez [V1].

Theorem 3.1. *There exists $T = T(g) \geqq (4\pi h)^{-1}$ and a function $v \in C([0, T] : L^1_{\text{loc}}(\mathbb{R}) \cap L^\infty_{\text{loc}}(Q_T))$ with the following properties:*

(i) *$v(\cdot, t) \to g$ in $L^1_{\text{loc}}(\mathbb{R})$ as $t \to 0^+$.*

(ii) *$v_t - \phi(v)_{xx} = 0$ in $\mathscr{D}'(Q_T)$.*

(iii) *For every $t_1 < T$ there exists constants $h_1 \geqq h$ and $c_1 > 0$ such that*

$$-g^-(x) \leqq v(x, t) \leqq c_1 \exp(-h_1 x^2) \tag{3.2}$$

for every $0 < t \leqq t_1$ and a.e. *for $x \in \mathbb{R}$.*

Uniqueness of solutions of (P) can be established in a larger class of functions which includes the class of solutions in Theorem 3.1; the proof is given Vázquez [V1].

Theorem 3.2. *Let $u, v \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}) \cap L^\infty_{\text{loc}}(Q_T))$ satisfy the following properties:*

(i) *$u_t - \phi(u)_{xx} = v_t - \phi(v)_{xx}$ in $\mathscr{D}'(Q_T)$;*

(ii) *$u(\cdot, t) - v(\cdot, t) \to 0$ in $L^1_{\text{loc}}(\mathbb{R})$ as $t \to 0^+$;*

(iii) *for every $t_1 \leqq T$ there exist constants $c_1 > 0$, $h_1 > 0$ such that*

$$u(x, t), v(x, t) \leqq c_1 \exp(h_1 x^2) \quad \text{a.e. in} \quad \mathbb{R} \times (0, t_1). \tag{3.3}$$

(iv) *$u^-(x, t), v^-(x, t) \in L^\infty_{\text{loc}}(Q_T)$.*

Then $u(x, t) = v(x, t)$ a.e. *in Q_T.*

As a consequence of Theorems 3.1 and 3.2 the Cauchy problem (P) has a unique solution in Q_T for some $T > 0$ if the datum g satisfies (3.1). If the datum g has exponential growth as $|x| \to \infty$, the solution v of (P) will become infinite in finite time. This is evident because nonnegative solutions of the heat equation $v_t = v_{xx}$ have this property for exponentially growing data. On the other hand, if the datum g has at most polynomial growth at infinity, then by Theorem 3.1 the solution v of (P) exists on $0 \leqq t < \infty$.

Next, we turn to the comparison method in the form it is needed in the proof of Theorem 1.2, see Vázquez [V1]. To state it we define

$$G(x) = \int_{-\infty}^{x} g(\xi)\, \mathrm{d}\xi, \quad x \in \mathbb{R},$$

$$V(x, t) = \int_{-\infty}^{x} v(\xi, t)\, \mathrm{d}\xi, \quad x \in \mathbb{R}, \quad t \geqq 0.$$

Proposition 3.3. *Let v_1, v_2 be two solutions of* (P) *corresponding to data g_1, g_2*

respectively, each satisfying assumptions (g), *and* $g_1, g_2 \in L^1(\mathbb{R})$. *If* $G_1(x) \geqq G_2(x)$ *for every* $x \in \mathbb{R}$, *then* $V_1(x, t) \geqq V_2(x, t)$ *for every* $x \in \mathbb{R}$, $t > 0$.

Remark. Proposition 3.3 holds for solutions v of (P) whenever ϕ is a continuous nondecreasing function: $\mathbb{R} \to \mathbb{R}$, and $\phi(0) = 0$. Observe that V satisfies the P.D.E. $V_t = \phi(V_x)_x$ and Proposition 3.3 is the maximum principle for this equation.

Sketch of Proof of Theorem 1.2. We assume that assumptions (g) are satisfied. Since the interest is in the regularity of the free boundary near $t = 0$, we assume without loss of generality that g has at most polynomial growth at infinity. Then the Cauchy problem (P), (g) exhibits a unique free boundary $x = s(t)$ defined on $0 < < t < \infty$, $s(0) = 0$, $s \in C^1(0, \infty)$, and $s'(t) < 0$ for $t > 0$. In what follows we denote by $G(x)$ the primitive $\int_0^x g(\xi)\, d\xi$. An essential ingredient in the proof is the following refinement of Proposition 3.3. An analogous result for the porous medium equation has been proved by Vázquez [V2]; the proof is similar.

Lemma 3.4. *Let* v_1, v_2 *be two solutions of* (P) *corresponding to initial data* g_1, g_2 *respectively, satisfying assumptions* (g). *Let* s_1, s_2 *be the corresponding free boundaries,* $s_1(0) = s_2(0) = 0$. *Assume that*

$$0 > g_1(x) \geqq g_2(x) \tag{3.4}$$

for small $x < 0$, *and that*

$$G_1(x) \geqq G_2(x) \tag{3.5}$$

for every x *on some interval* $(0, \alpha)$, $\alpha > 0$, $G_1(\alpha) > 0$ *and* $G_1 \not\equiv G_2$ *on* $(0, \alpha)$. *Then*

$$s_1(t) \leqq s_2(t) \tag{3.6}$$

for $t > 0$ *sufficiently small.*

To complete the proof of Theorem 1.2 we apply Lemma 3.4. Let $v_1 = v$, the solution of (P), (g) where the initial datum g satisfies assumptions (g) and (1.7). Let v_2 be the self similar solution of Proposition 1.1 corresponding to the model datum (D) with $p_+ = b(\gamma + 1) + \varepsilon$, $p_- = a - \varepsilon$, where $0 < \varepsilon < a$. Then $G_2(x) \geqq G_1(x)$ for small $x \geqq 0$, $G_1 \not\equiv G_2$, and $0 > g_2(x) \geqq g_1(x)$ for $x < 0$. By Lemma 3.4 $s_2(t) \leqq \leqq s_1(t)$ for small $t \geqq 0$. Thus identifying s_1 with s and using the result of Proposition 1.1,

$$\liminf_{t \to 0^+} (-s(t)\, t^{-1/2}) \geqq \kappa \left(\frac{b(\gamma + 1) + \varepsilon}{a - \varepsilon} \right)$$

for every $0 < \varepsilon < a$. Since κ is a continuous function of p_+/p_-, we let $\varepsilon \to 0^+$ to obtain

$$\liminf_{t \to 0^+} (-s(t)\, t^{-1/2}) \geqq \kappa \left(\frac{b(\gamma + 1)}{a} \right). \tag{3.7}$$

By a similar argument, letting v_1 be the self similar solution of (P), (D) with p_+ and p_-

defined as above, and letting v_2 be the solution of (P), (D) we obtain

$$\limsup_{t \to 0^+} \left(-s(t)\, t^{-1/2}\right) \leqq \kappa\left(\frac{b(\gamma+1)}{a}\right). \tag{3.8}$$

The result (1.8) is a consequence of (3.7), (3.8). This completes the sketch of the proof.

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