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ON THE EXISTENCE OF CONTINUOUS SOLUTIONS OF OPERATOR EQUATIONS IN BANACH SPACES

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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1. INTRODUCTION

Usually the proofs of existence theorems for functional, functional-differential, integro-differential and other equations are based on some fixed point theorems for mappings in topological spaces. This often leads us to a fixed point problem for an appropriate operator in continuous functions spaces $C(G, B)$, where G is a compact space with a metric d and B is a Banach space with a norm $\|\cdot\|$.

In literature we can find many particular results concerned with the problem, including those obtained by the classical Banach contraction mapping principle, the comparison method, the Schauder fixed point theorem and by the more recent results employing the notions of measure of noncompactness or of degree of a mapping.

The results of the present paper are obtained by an adaptation of the Schauder fixed point theorem and generalize the results of papers [5] and [8].

The present paper essentially develops an idea which in particular cases can be found in [2]–[8] and [10].

2. THE MAIN RESULT AND APPLICATIONS

Let T be a fixed nonempty set, $\mathbb{R}_+ \doteq [0, \infty)$ and let $M(T, \mathbb{R}_+)$ be the set of all functions from T into $\mathbb{R}_+$. In $C(G, \mathbb{R}_+)$ and $M(T, \mathbb{R}_+)$ we introduce partial order $\leqq$ induced by the natural order in $\mathbb{R}_+$.

Consider functional equations in the fixed point form:

$$u(x) = (Fu)(x), \quad x \in G, \tag{1}$$

where $F: C(G, B) \to C(G, B)$ is a fixed continuous operator.

Introduce the following

Assumption H_1. Suppose that

1° there exists a function $u_0 \in C(G, B)$ and a nondecreasing operator $\Gamma: C(G, R_+) \to$

$\to C(G, R_+)$ such that

$$\|(Fu)(x) - u_0(x)\| \leqq \Gamma(\|u - u_0\|(\cdot))(x) \tag{2}$$

for $u \in C(G, B)$, $x \in G$, where $\|z\|(y) = \|z(y)\|$ for $z \in C(G, B)$, $y \in G$,

2° there exists a function $g \in C(G, R_+)$ such that

$$\Gamma g \leqq g\,. \tag{3}$$

It is obvious that the set

$$K(u_0, g) = \{u \in C(G, B): \|u - u_0\| \leqq g\}$$

is bounded, closed and convex in $C(G, B)$.

Moreover, $F(K(u_0, g)) \subset K(u_0, g)$ if Assumption H_1 is fulfilled. We shall also use the following

Assumption H_1'. Assume that there exists a closed, bounded and convex set $W \subset C(G, B)$ such that $F(W) \subset W$.

Assumption H_1 is used only in order to find a set W such that Assumption H_1' holds true.

Let a set $V \subset M(T, R_+)$ be fixed.

Assumption H_2. Suppose that

1° there exist functions $\beta: T \times G \to G$ and $\omega: T \times I_\delta \to I_\delta$ ($I_\delta = [0, \delta]$, $\delta > 0$) such that for all $x, y \in G$, $d(x, y) \leqq \delta$, $u \in W$ the following conditions are fulfilled:

$$\|u(\beta(\cdot, x)) - u(\beta(\cdot, y))\| \in V, \tag{4}$$

$$d(\beta(\tau, x), \beta(\tau, y)) \leqq \omega(\tau, d(x, y))\,, \quad \tau \in T;$$

2° there exists an operator $\Omega: R_+ \times V \to R_+$ nondecreasing with respect to the second variable, such that $\Omega(0, 0) = 0$ and for every $u \in W$, $x, y \in G$, $d(x, y) \leqq \delta$, the following inequality is satisfied:

$$\|(Fu)(x) - (Fu)(y)\| \leqq \Omega(d(x, y), \|u(\beta(\cdot, x)) - u(\beta(\cdot, y))\|)\,; \tag{5}$$

3° there exists a nondecreasing function $\gamma: I_\delta \to R_+$ such that

$$\gamma(t) \geqq \Omega(t, \gamma(\omega(\cdot, t)))\,, \quad t \in I_\delta \tag{6}$$

and $\lim_{t \to 0+} \gamma(t) = 0$;

4° there exists $u \in W$ such that the function γ is a modulus of continuity of the function u.

Suppose that α and $\bar{\alpha}$ are the Kuratowski measures of noncompactness in B and $C(G, B)$, respectively. The essential properties of the measure of noncompactness are found for instance in [1].

Assumption H_3. Assume that for every subset $U \subset W$ and $x \in G$ the inequality

$$\alpha((FU)(x)) < \alpha(U(x))$$

is fulfilled if $\alpha(U(x)) \neq 0$, where

$$U(x) = \{u(x): u \in U\}.$$

Now we state

Theorem 1. *Suppose that the operator F is continuous and that Assumptions H_1', H_2, H_3 are satisfied. Then in the set W there exists at least one solution of equation* (1).

Proof. Put

$$S_\gamma = \{u \in W: \|u(x) - u(y)\| \leqq \gamma(d(x, y)),\ x, y \in G,\ d(x, y) \leqq \delta\},$$

where the function γ fulfils Assumption H_2. It follows from the condition 4° of Assumption H_2 that $S_\gamma \neq \emptyset$. It is also obvious that the set S_γ is convex, closed and bounded in $C(G, B)$. We prove that $F(S_\gamma) \subset S_\gamma$. Let $u \in S_\gamma$, then $Fu \in W$ and

$$\|(Fu)(x) - (Fu)(y)\| \leqq \Omega(d(x, y), \|u(\beta(\cdot, x)) - u(\beta(\cdot, y))\|) \leqq$$
$$\leqq \Omega(d(x, y), \gamma(\omega(\cdot, d(x, y)))) \leqq \gamma(d(x, y))$$

for $x, y \in G$, $d(x, y) \leqq \delta$, so $Fu \in S_\gamma$.

For an arbitrary set $U \subset C(G, B)$ of uniformly bounded and equicontinuous functions (such set is called regular) the following formula is true:

$$\bar{\alpha}(U(\cdot)) = \sup_{x \in G} \alpha(U(x)).$$

This may be proved in the same way as in [1], where the case $G \subset R$ is considered.

Notice that for a regular set $U \subset C(G, B)$ the function $\alpha(U(\cdot))$ is continuous on G. From Assumption H_3 we obtain

$$\bar{\alpha}(FU) = \sup_{x \in G} \alpha((FU)(x)) < \sup_{x \in G} \alpha(U(x)) = \bar{\alpha}(U)$$

or $\bar{\alpha}(U) = 0$ for $U \subset S_\gamma$. It follows from the Sadovskii fixed point theorem for α-condensing operators (see Theorem 3.4.3 [9]) that there exists a fixed point of the operator F in S_γ. Thus the proof is complete.

Some considerations and examples of applications of Theorem 1 are given in [5] for $G \subset \mathbb{R}$ and in [8] for $G \subset \mathbb{R}^k$. In the present paper we consider some applications of the general result to integro-differential equations with the Stieltjes integral.

Example 1. Consider an equation of the form

$$(7)\quad \begin{cases} x'(t) = f\left(t, x(\alpha(t)), \int_0^1 h(t, \tau, x'(\beta(\tau, t)))\, d_\tau K(\tau, t)\right), & t \in G, \\ x(0) = 0, \end{cases}$$

where $G = [0, b]$, $b > 0$, $f \in C(G \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n)$, $\alpha \in C(G, G)$, $h \in C(G \times [0, 1] \times \times \mathbb{R}^n, R^p)$, $\beta \in C([0, 1] \times G, G)$, $K: [0, 1] \times G \to \mathbb{R}$, $K(\cdot, t)$ is a function with bounded variation for every $t \in G$. Put

$$x'(t) = u(t) .$$

Then the equation (7) assumes the form

$$u = Fu ,$$

where

$$(Fz)(t) = f\left(t, \int_0^{\alpha(t)} z(s)\, ds, \int_0^1 h(t, \tau, z(\beta(\tau, t)))\, d_\tau K(\tau, t)\right).$$

Assume that $F(K(0, g)) \subset K(0, g)$ for some $g \in C(G, R_+)$ and there exists a function $v \in C(R_+, R_+)$ such that $v(0) = 0$ and

$$\left\| \int_0^1 h(t, \tau, z(\beta(\tau, t)))\, d_\tau(K(\tau, t) - K(\tau, s)) \right\| \leqq v(|t - s|)$$

for $t, s \in G$ and $z \in K(0, g)$. Then we obtain the following form of the operator Ω, for which the inequality (5) should be fulfilled:

$$\Omega(t, z(\cdot)) = \omega_f\left(\sup_{s \in G} \int_0^1 \omega_h(z(\tau))\, d_\tau\, V(\tau, s)\right) + \eta(t),$$

where the function η is defined by v and the moduli of continuity of the functions α, f and h, functions ω_f, ω_h are moduli of continuity of the functions f and h with respect to the last variable and $V(\tau, s)$ is the variation of $K(\cdot, s)$ on $[0, \tau]$. This leads to the following form of the inequality (6):

(8)
$$\gamma(t) \geqq \omega_f\left(\sup_{s \in G} \int_0^1 \omega_h(\gamma(\omega(\tau, t)))\, d_\tau V(\tau, s)\right) + \eta(t).$$

In general it is difficult to solve the inequality (8) but it is rather simple in some special cases. Suppose that

$$\omega_f(z) = L_f z , \quad \omega_h(z) = L_h z .$$

This means that the functions f and h fulfil the Lipschitz condition with respect to the last variable. Now the inequality (8) takes the form

(9)
$$\gamma(t) \geqq L \sup_{s \in G} \int_0^1 \gamma(\omega(\tau, t))\, d_\tau V(\tau, s) + \eta(t)$$

where $L = L_f L_h$. Moreover, assume that

$$\eta(t) = \bar{\eta} t^r , \quad \omega(\tau, t) = \bar{\omega}(\tau) t, \quad \bar{\omega}(\tau) \leqq 1$$

and

$$(10) \qquad q = L \sup_{s \in G} \int_0^1 [\bar{\omega}(\tau)]^r \, \mathrm{d}_\tau V(\tau, s) < 1 .$$

Then the inequality (9) has the solution

$$\gamma(t) = \bar{\eta}(1 - q)^{-1} \, t^r .$$

Notice that for the function K defined by the formula

$$K(\tau, t) = \begin{cases} 0 , & \tau = 0 \\ 1 , & 0 < \tau \leqq 1 \end{cases}$$

and for $h(t, \tau, z) = z$ the equation (7) has the form

$$x'(t) = f(t, x(\alpha(t)), x'(\tilde{\beta}(t))) ,$$

where $\tilde{\beta}(t) = \beta(0, t)$, and the condition (10) assumes the form

$$L[\bar{\omega}(0)]^r < 1$$

(cf. [5], Example 2).

Example 2. Consider the boundary value problem of the form

$$(11) \qquad \begin{cases} x'(t) = A(t) \, x(t) + f(t, x(\cdot)) , & t \in G , \\ Lx = Nx , \end{cases}$$

where $A(\cdot)$ is a continuous $n \times n$ – matrix on $G = [0, b]$, $b > 0$, $f: G \times \times C^1(G, R^n) \to R^n$, $N: C^1(G, R^n) \to C(G_1, R^m)$ are continuous functions, $G_1 \subset R$ is an interval, $L: (G, R^n) \to C(G_1, R^m)$ is a continuous and linear mapping.

Suppose that there exists a linear continuous operator $L_A: C(G_1, R^m) \to C(G, R^n)$ such that

$$L_A L y = y \quad \text{if} \quad y'(t) = A(t) \, y(t) , \quad t \in G .$$

Now the existence problem for (11) leads to the following equation of the fixed point form (see [3]):

$$u = Fu ,$$

where

$$Fx = L_A N x - L_A L H x + H x ,$$

$$(Hx)(t) = \int_0^t U(t) \, U^{-1}(s) f(s, x(\cdot)) \, \mathrm{d}s ,$$

$$U'(t) = A(t) \, U(t) , \quad U(0) = I ,$$

and I is the identity mapping in R^n.

Assume that the functions N and f are bounded with respect to the norm in $C^1(G, R^n)$,

$$(12) \qquad \|f(t_1, x) - f(t_2, x)\| \leqq \eta(|t_1 - t_2|) +$$

$$+ \int_0^1 \|x'(\beta(\tau, t_1)) - x'(\beta(\tau, t_2))\| \, \mathrm{d}_\tau K(\tau, |t_1 - t_2|)$$

where $K(\cdot, t)$ is a nondecreasing and bounded function, and

$$|\beta(\tau, t_1) - \beta(\tau, t_2)| \leqq \omega(\tau, |t_1 - t_2|)$$

for any $t_1, t_2 \in G$ and $\tau \in [0, 1]$. This leads to an operator Ω of the form

$$\Omega(t, u(\cdot)) = \tilde{\eta}(t) + \int_0^1 u(\tau) \, \mathrm{d}_\tau K(\tau, t)$$

with $\tilde{\eta}$ defined by η and the moduli of continuity of the functions $A(\cdot)$ and $f(\cdot, x)$.

Now the inequality (6) assumes the form

$$\gamma(t) \geqq \tilde{\eta}(t) + \int_0^1 \gamma(\omega(\tau, t)) \, \mathrm{d}_\tau K(\tau, t) \, .$$

As in Example 1 we find that this inequality has the solution

$$\gamma(t) = \bar{\eta}(1 - q)^{-1} \, t^r$$

if $\tilde{\eta}(t) = \bar{\eta} t^r$, $\omega(\tau, t) = \bar{\omega}(\tau) \, t$, $\bar{\omega}(\tau) \leqq 1$ and

$$q = \sup_{s \in [0, \delta]} \int_0^1 [\bar{\omega}(\tau)]^r \, \mathrm{d}_\tau K(\tau, s) < 1 \, .$$

Example 3. Consider the initial-value problem of the form

$$(13) \qquad \begin{cases} u_{xy}(x, y) = f\Big(x, y, u(\alpha_1^1(x, y), \alpha_1^2(x, y)), u_x(\alpha_2^1(x, y), \alpha_2^2(x, y)) \, , \\ u_y(\alpha_3^1(x, y), \alpha_3^2(x, y)), \int_0^1 \int_0^1 h(x, y, \tau_1, \tau_2, u_{xy}(\beta^1(\tau_1, \tau_2, x, y) \, , \\ \beta^2(\tau_1, \tau_2, x, y))) \, \mathrm{d}_{\tau_2} K_2(\tau_1, \tau_2, x, y) \, \mathrm{d}_{\tau_1} K_1(\tau_1, x, y)\Big) \, , \\ u(x, 0) = u(0, y) = 0 \end{cases}$$

for $(x, y) \in G = [0, a] \times [0, b]$, $a, b > 0$, where $f \in C(G \times \mathbb{R}^{3n} \times \mathbb{R}^k, \mathbb{R}^n)$, $h \in C(G \times [0, 1]^2 \times \mathbb{R}^n, \mathbb{R}^k)$, $\alpha_i^1 \in C(G, [0, a])$, $\alpha_i^2 \in C(G, [0, b])$, $i = 1, 2, 3$, $\beta^1 \in C([0, 1]^2 \times G, [0, a])$, $\beta^2 \in C([0, 1]^2 \times G, [0, b])$, $K_2(\tau_1, \cdot, x, y)$ and $K_1(\cdot, x, y)$ are real- valued functions with bounded variation.

Put

$$u_{xy} = z \, .$$

Then the problem (13) assumes the form

$$z = Fz \, ,$$

$$(Fz)(x, y) = f\Bigg(x, y, \int_0^{\alpha_1{}^1(x,y)} \int_0^{\alpha_1{}^2(x,y)} z(s, t)\, \mathrm{d}t\, \mathrm{d}s\,,$$

$$\int_0^{\alpha_2{}^2(x,y)} z(\alpha_2^1(x, y), t)\, \mathrm{d}t, \int_0^{\alpha_3{}^1(x,y)} z(s, \alpha_3^2(x, y))\, \mathrm{d}s\,,$$

$$\int_0^1 \int_0^1 h(x, y, \tau_1, \tau_2, z(\beta^1(\tau_1, \tau_2, x, y),\ \beta^2(\tau_1, \tau_2, x, y)))$$

$$\mathrm{d}_{\tau_2} K_2(\tau_1, \tau_2, x, y)\, \mathrm{d}_{\tau_1} K_1(\tau_1, x, y)\Bigg)\,.$$

Suppose that $F(K(0, g)) \subset K(0, g)$ for some $g \in C(G, R_+)$. Under a certain assumption on the continuity of the Stieltjes integral we obtain the estimate

$$\|(Fz)(x, y) - (Fz)(\bar{x}, \bar{y})\| \leqq \eta(|(x - \bar{x}, y - \bar{y})|) +$$

$$+ \eta_2\left(\int_0^{\alpha_2{}^2(x,y)} \|z(\alpha_2^1(x, y), t) - z(\alpha_2^1(\bar{x}, \bar{y}), t)\|\, \mathrm{d}t\right) +$$

$$+ \eta_3\left(\int_0^{\alpha_3{}^1(x,y)} \|z(s, \alpha_3^2(x, y)) - z(s, \alpha_3^2(\bar{x}, \bar{y}))\|\, \mathrm{d}z\right) +$$

$$+ \omega_f\Bigg(\int_0^1 \int_0^1 \omega_h(\|z(\beta(\tau_1, \tau_2, x, y)) -$$

$$- z(\beta(\tau_1, \tau_2, \bar{x}, \bar{y}))\|)\, \mathrm{d}_{\tau_2} V_2(\tau_1, \tau_2, x, y)\, \mathrm{d}_{\tau_1} V_1(\tau_1, x, y)\Bigg)\,,$$

where $|\cdot|$ is a norm in R^2, $\beta(q) = (\beta^1(q), \beta^2(q))$ for $q \in [0, 1]^2 \times G$, η is a modulus of continuity dependent on f, h, K_1, K_2, the functions η_2, η_3 are moduli of continuity of f with respect to the fourth and fifth variables, the functions ω_f, ω_h are moduli of continuity of f and h with respect to the last variable and V_1, V_2 are variations of functions $K_1(\cdot, x, y)$ and $K_2(\tau_1, \cdot, x, y)$, respectively.

If we assume that

$$\eta_2(u) = L_2 u\,, \quad \eta_3(u) = L_3 u\,, \quad \omega_f(u) = L_f u\,, \quad \omega_h(u) = L_h u\,,$$

$$|\alpha_2^1(x, y) - \alpha_2^1(\bar{x}, \bar{y})| \leqq \bar{\alpha}_2 |(x - \bar{x},\ y - \bar{y})|\,,$$

$$|\alpha_3^2(x, y) - \alpha_3^2(\bar{x}, \bar{y})| \leqq \bar{\alpha}_3 |(x - \bar{x},\ y - \bar{y})|\,,$$

$$|\beta(\tau_1, \tau_2, x, y) - \beta(\tau_1, \tau_2, \bar{x}, \bar{y})| \leqq \omega(\tau_1, \tau_2,\ |(x - \bar{x},\ y - \bar{y})|)\,,$$

then the inequality (6) assumes the form

$$\gamma(t) \geqq \eta(t) + L_2 b\, \gamma(\bar{\alpha}_2 t) + L_3 a\, \gamma(\bar{\alpha}_3 t) + \tag{14}$$

$$+ L \sup_{(x,y)\in G} \int_0^1 \int_0^1 \gamma(\omega(\tau_1, \tau_2, t))\, \mathrm{d}_{\tau_2} V_2(\tau_1, \tau_2, x, y)\, \mathrm{d}_{\tau_1} V_1(\tau_1, x, y)\,,$$

where $L = L_f L_h$. Moreover, if we assume that

$$\eta(t) = \bar{\eta} t^p, \quad \omega(\tau_1, \tau_2, t) = \bar{\omega}(\tau_1, \tau_2)\, t,$$
$$\bar{\omega}(\tau_1, \tau_2) \leqq 1, \quad \bar{\alpha}_2, \bar{\alpha}_3 \leqq 1,$$

then the inequality (14) has the solution

$$\gamma(t) = \bar{\eta}(1 - q)^{-1}\, t^p$$

if

(15)
$$q = L_2 b \bar{\alpha}_2 + L_3 a \bar{\alpha}_3 +$$
$$+ L \sup_{(x,y)\in G} \int_0^1 \int_0^1 [\bar{\omega}(\tau_1, \tau_2)]^p \, d_{\tau_2} V_2(\tau_1, \tau_2, x, y)\, d_{\tau_1} V_1(\tau_1, x, y) < 1.$$

Unfortunately, we obtain only a local existence theorem for the problem (13), because the condition (15) may be fulfilled only for sufficiently small a and b.

3. A GENERALIZATION OF THE MAIN RESULT

First we show a simple equation for which the existence of solution does not follow from Theorem 1.

Example 4. Consider the equation

$$x = Fx,$$

where $F: C(G, \mathbb{R}) \to C(G, \mathbb{R})$, $G = [0, a]$, $a = \frac{1}{2}(3 + \sqrt{5})$,

(16)
$$(Fx)(t) = k(t)\, x(\beta(t)) + h(t), \; t \in G,$$
$$\beta(t) = \sqrt{t} + 1, \quad h(t) = \sqrt{t}, \quad t \in G,$$
$$k(t) = \begin{cases} 1 - \sqrt{t} & \text{for} \quad t \in [0, 1], \\ 0 & \text{for} \quad t \in (1, a]. \end{cases}$$

Let

$$\Gamma = F|_{C(G, R_+)}, \quad u_0 = 0, \quad g = 2.$$

Now, Assumption H_1 is satisfied. Assumption H_3 is also fulfilled, because $U(x)$ is a finite-dimensional and bounded set, hence precompact, for every $U \subset K(u_0, g)$ and $x \in G$.

We get

(17)
$$|(Fx)(t_1) - (Fx)(t_2)| \leqq |k(t_1) - k(t_2)|\, |x(\beta(t_1))| +$$
$$+ k(t_2)\, |x(\beta(t_1)) - x(\beta(t_2))| + |h(t_1) - h(t_2)|.$$

We need an estimate depending on $|t_1 - t_2|$, so we derive

$$|(Fx)(t_1) - (Fx)(t_2)| \leqq \eta_k(|t_1 - t_2|) \sup_{t \in G} g(t) +$$
$$+ \sup_{t \in G} k(t) |x(\beta(t_1)) - x(\beta(t_2))| + \eta_h(|t_1 - t_2|) =$$
$$= 3\sqrt{|t_1 - t_2|} + |x(\beta(t_1)) - x(\beta(t_2))| ,$$

where

$$\eta_k(t) = \sqrt{t}\,, \quad \eta_h(t) = \sqrt{t}$$

are moduli of continuity of the functions k and h. Since

$$|\beta(t_1) - \beta(t_2)| \leqq \sqrt{|t_1 - t_2|}\,,$$

the condition (4) is satisfied for

$$\omega(t) = \sqrt{t}$$

(we omit the variable τ because $T = \{1\}$ and $V = R$ in this case). The inequality (6) assumes the form

$$\gamma(t) \geqq 3\sqrt{t} + \gamma(\sqrt{t})\,. \tag{18}$$

Notice that (18) has sense only on an interval not smaller than $[0, 1)$. Suppose that (18) has a nonnegative solution on $[0, 1)$, then by induction we get

$$\gamma(t) \geqq 3t^{1/2} + \ldots + 3t^{2^{-n}} + \gamma(t^{2^{-n}})$$

for $n = 1, 2, \ldots$, whence

$$\gamma(t) \geqq 3\sum_{i=1}^{\infty} t^{2^{-i}}\,.$$

Since $t^{2^{-n}} \to 1$, $n \to \infty$, we have $\gamma(t) = \infty$ for $t \in (0, 1)$. It means that the inequality (18) has no finite solution. Consequently, Assumption H_2 is not satisfied in the case considered.

Below we prove a generalization of Theorem 1, which implies the existence of a solution of the equation $x = Fx$ with F defined by (16).

Assumption H_2'. Suppose that

1° there exists a function $\beta: T \times G \to G$ and $\delta > 0$ such that

$$\|u(\beta(\cdot, x)) - u(\beta(\cdot, y))\| \in V$$

for all $u \in W$ and $(x, y) \in I_\delta$, where

$$I_\delta = \{(t, s): t, s \in G,\ d(t, s) \leqq \delta\}\,;$$

2° there exists an operator $\bar{\Omega}: I_\delta \times V \to \mathbb{R}_+$ nondecreasing with respect to the second variable such that $\bar{\Omega}(x, x, 0) = 0$ for $x \in G$, and for every $u \in W$ and $(x, y) \in I_\delta$ the following inequality is satisfied:

$$\|(Fu)(x) - (Fu)(y)\| \leqq \bar{\Omega}(x, y, \|u(\beta(\cdot, x)) - u(\beta(\cdot, y))\|)\,; \tag{19}$$

3° there exists a function $\bar{\gamma}: I_\delta \to R_+$ such that

$$\bar{\gamma}(x, y) \geqq \bar{\Omega}(x, y, \bar{\gamma}(\beta(\cdot, x), \beta(\cdot, y))) \tag{20}$$

for $(x, y) \in I_\delta$ and

$$\lim_{x,y \to z} \bar{\gamma}(x, y) = 0 \tag{21}$$

for $z \in G$;

4° there exists $u \in W$ such that

$$\|u(x) - u(y)\| \leqq \bar{\gamma}(x, y)$$

for all $(x, y) \in I_\delta$.

In what follows we shall need

Lemma. *Suppose that a function $\bar{\gamma}: I_\delta \to R_+$ fulfils the condition* (21). *Then*

$$S = \{u \in C(G, B): \|u(x) - u(y)\| \leqq \bar{\gamma}(x, y), (x, y) \in I_\delta\}$$

as a set of equicontinuous functions.

Proof. Assume that the assertion is not true, then there exists $\varepsilon > 0$ and sequences $\{u_n\} \subset S$, $\{(x_n, y_n)\} \subset I_\delta$ such that

$$d(x_n, y_n) \to 0 , \quad n \to \infty$$

and

$$\bar{\gamma}(x_n, y_n) \geqq \|u_n(x_n) - u_n(y_n)\| \geqq \varepsilon .$$

Since G is a compact metric space, there exist $z \in G$ and a subsequence $\{(\bar{x}_i, \bar{y}_i)\} \subset \subset \{(x_n, y_n)\}$ such that

$$\bar{x}_i \to z , \quad \bar{y}_i \to z , \quad i \to \infty .$$

We obtain

$$\bar{\gamma}(\bar{x}_i, \bar{y}_i) \geqq \varepsilon > 0$$

and the condition (21) yields

$$\lim_{i \to \infty} \bar{\gamma}(\bar{x}_i, \bar{y}_i) = 0 .$$

We get a contradiction which completes the proof.

Theorem 2. *Suppose that the operator F is continuous and Assumptions* H_1', H_2', H_3 *are satisfied. Then there exists in the set W at least one solution of the equation* (1).

Proof. Let

$$S_{\bar{\gamma}} = \{u \in W: \|u(x) - u(y)\| \leqq \bar{\gamma}(x, y), (x, y) \in I_\delta\} ,$$

where the function $\bar{\gamma}$ fulfils Assumption H_2'. It is obvious that $S_{\bar{\gamma}}$ is a nonempty, convex, closed and bounded subset of $C(G, B)$. We prove that $F(S_{\bar{\gamma}}) \subset S_{\bar{\gamma}}$. Let $u \in S_{\bar{\gamma}}$, then $Fu \in W$ and

$$\|(Fu)(x) - (Fu)(y)\| \leqq \bar{\Omega}(x, y, \|u(\beta(\cdot, x)) - u(\beta(\cdot, y))\|) \leqq$$
$$\leqq \bar{\Omega}(x, y, \bar{\gamma}(\beta(\cdot, x), \beta(\cdot, y))) \leqq \bar{\gamma}(x, y)$$

for $(x, y) \in I_\delta$, so $Fu \in S_{\bar{\gamma}}$.

Since Assumption H_3 is fulfilled and in view of Lemma the set $S_{\bar{\gamma}}$ is regular, we can complete our proof in the same way as that of Theorem 1.

Remark. Suppose that the assumptions of Theorem 1 are fulfilled and

$$\bar{\Omega}(x, y, z(\cdot)) = \Omega(d(x, y), z(\cdot)),$$
$$\bar{\gamma}(x, y) = \gamma(d(x, y)).$$

Then Assumption H_2' is satisfied, so Theorem 2 is a generalization of Theorem 1.

Return to Example 4. Assume that

$$\bar{\Omega}(t_1, t_2, z) = |k(t_1) - k(t_2)| \,.\, 2 + |h(t_1) - h(t_2)| + z,$$

then the inequality (19) is satisfied since (17) is true. Define

$$\bar{\gamma}(t_1, t_2) = 6|\sqrt{t_1} - \sqrt{t_2}|,$$

then

$$\bar{\Omega}(t_1, t_2, \bar{\gamma}(\beta(t_1), \beta(t_2))) = 2|k(t_1) - k(t_2)| + |h(t_1) - h(t_2)| +$$
$$+ \bar{\gamma}(\sqrt{t_1} + 1, \sqrt{t_2} + 1) \leqq 3|\sqrt{t_1} - \sqrt{t_2}| + 6|\sqrt{(\sqrt{t_1} + 1)} - \sqrt{(\sqrt{t_2} + 1)}| =$$
$$= 3|\sqrt{t_1} - \sqrt{t_2}| + 6(\sqrt{(\sqrt{t_1} + 1)} + \sqrt{(\sqrt{t_2} + 1)})^{-1} |\sqrt{t_1} - \sqrt{t_2}| \leqq$$
$$\leqq 6|\sqrt{t_1} - \sqrt{t_2}| = \bar{\gamma}(t_1, t_2),$$

so the inequality (20) has a solution. It follows from Theorem 2 that there exists a solution of the equation $x = Fx$ with F defined by (16). This means that Theorem 2 is a proper generalization of Theorem 1.

Consider the equation (11) again. This equation was studied in [3] under the assumption that

$$(22) \qquad \|f(t_1, x) - f(t_2, x)\| \leqq \eta(|t_1 - t_2|) + \sum_{i=1}^{n(t_1,t_2)} \lambda_i(t_1, t_2) \| x'(t_1 + v_i(t_1, t_2)) - x'(t_2 + v_i(t_1, t_2))\|$$

and there exists $\lambda \in [0, 1)$ such that

$$(23) \qquad \sum_{i=1}^{n(t_1,t_2)} \lambda_i(t_1, t_2) \leqq \lambda\,.$$

In general, the condition (22) cannot be written in the form (12), so Theorem 1 is not a generalization of Theorem 2 from [3]. Nonetheless, if we assume that

$$(24) \qquad \|f(t_1, x) - f(t_2, x)\| \leqq \varrho(t_1, t_2) +$$

$$+\int_0^1 \|x'(\beta(\tau, t_1)) - x'(\beta(\tau, t_2))\| \, \mathrm{d}_\tau K(\tau, t_1, t_2),$$

then from Theorem 2 we can obtain the result from [3]. Indeed, suppose that

$$v_1(t_1, t_2) < \ldots < v_{n(t_1,t_2)}(t_1, t_2),$$

$$\sigma(s) = \begin{cases} 0, & s < 0, \\ s, & s \in [0, b], \\ b, & s > b, \end{cases}$$

$$\beta(\tau, t) = \sigma(t + (2\tau - 1)\, b),$$

$$v_{n(t_1,t_2)+1} = b,$$

$$K(\tau, t_1, t_2) = \begin{cases} 0, & (2\tau - 1)\, b \in [-b, v_1(t_1, t_2)], \\ \sum_{i=1}^{k} \lambda_i(t_1, t_2), & (2\tau - 1)\, b \in (v_k(t_1, t_2), v_{k+1}(t_1, t_2)], \\ & k = 1, \ldots, n(t_1, t_2), \end{cases}$$

$$\varrho(t_1, t_2) = \eta(|t_1 - t_2|).$$

Then the inequalities (22) and (24) coincide. If we assume (23), then the inequality (20) assumes the form

$$\bar{\gamma}(t_1, t_2) \geqq \bar{\eta}(|t_1 - t_2|) + \lambda \bar{\gamma}(t_1, t_2)$$

and has a solution

$$\bar{\gamma}(t_1, t_2) = (1 - \lambda)^{-1}\, \bar{\eta}(|t_1 - t_2|).$$

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