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ON ALMOST DISCRETE SPACE

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Abstract. Let $C(X)$ be the ring of real continuous functions on a completely regular Hausdorff space. In this paper an almost discrete space is determined by the algebraic structure of $C(X)$. The intersection of essential weak ideal in $C(X)$ is also studied.

1. Introduction

For every topological space of X, let $I(X)$ be the set of all isolated points of X and $D(X) = X \setminus I(X)$. A topological space X is called *scattered* if each nonempty set $A \subseteq X$ contains an (in A) isolated point. Since $\operatorname{int}_X D(X) = X \setminus \operatorname{cl}_X I(X)$ and $D(X)$ is closed, then the $D(X)$ of every scattered space is nowhere dense.

A topological space X is called *almost discrete* if $I(X)$ is dense in X, see [6]. Let Y be the subset of plane consisting of all points $\left(\frac{m}{n}, \frac{1}{n}\right)$, where $n \gneqq 0$ and the greatest common divisor of m and n is 1. Clearly Y is discrete. Let $X = \operatorname{cl}_{\mathbb{R}^2} Y$. Since $[0,1] \times \{0\} \subseteq X$ has no isolated points, then X is almost discrete space, but it is not a scattered space.

In what follows, X will denote a completely regular Hausdorff space. We denote $C(X)$ the ring of real continuous functions on a topological space of X. As usual, if $f \in C(X)$, its zero set $f^{\leftarrow}(0)$ is denoted by $Z(f)$, its cozero set $X \setminus Z(f)$ is denoted by $\operatorname{Coz}(f)$, and if $S \subseteq C(X)$, $Z[S] = \{Z(f) : f \in S\}$ and $\operatorname{Coz}[S] = \{\operatorname{Coz}(f) : f \in S\}$. Recall that βX is the Stone-Čech compactification of X and υX is the Hewitt realcompactification of X. For undefined terms and notations, see [4].

Let R always denote a commutative ring with identity. For $S \subseteq R$, the ideal

$$\{a \in R : aS = \{0\}\} = \{a \in R : ab = 0 \text{ for all } b \in S\}$$

is called the annihilator of S and is also denoted by $\operatorname{Ann}(S)$ or $\operatorname{Ann}_R(S)$.

M. R. Ahmadi Zand studied $S.B.$ space that is a topological space X that for every real-valued function f on X there exists an open dense subset D of X such that $f|_D$ is continuous, he showed that every dense subset and open subset of $S.B.$ space is $S.B.$ space (see [8]).

In 1995, essential ideal in $C(X)$ were studied first by F. Azarpanah (see [1]). Also he studied the countable intersection of essential ideals in $C(X)$ (see [2]).

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In view of the connection between weak ideal and ideal in rings, in this paper, we study ideal on semigroup $(C(X), \cdot)$ by using the techniques similar to those used in ring $C(X)$. We introduce the concepts Weak ideal and essential weak ideal. The study of these concepts and the union of all minimal ideal of $C(X)$ are our main objects. These objects are important tools to study almost discrete spaces.

2. Almost discrete space

In the theory of rings, many structure results were obtained with the help of minimal ideals, and the socle of a ring seems to be most efficient. The sum of all minimal ideals of R is the *socle* of R, see [5]. In this article we want to study the relation between the union of all minimal ideal of $C(X)$ and the topological space of X.

If x is an isolated point of X, then we define

$$I_x = \{f \in C(X) : \mathrm{Coz}(f) = \{x\}\} \cup \{0\} \subseteq C(X)$$

and put $WC_F(X) = \bigcup_{x \in I(X)} I_x$, if $I(X) \neq \emptyset$ and $WC_F(X) = \{0\}$, if $I(X) = \emptyset$. The minimal ideal in $C(X)$ is characterized in [7], it follows that for every $x \in I(X)$, I_x is a minimal ideal, and $WC_F(X)$ is the union of all minimal ideal of $C(X)$. Finally we have $WC_F(X) = \{f \in C(X) : |\mathrm{Coz}(f)| = 0 \text{ or } 1\}$.

But we must begin at the beginning, with the basic definitions.

Definition 2.1. A nonempty subset I of R is called a *weak ideal* of R if $RI \subseteq I$.

Remark. For a topological space of X, $WC_F(X)$ is a weak ideal in $C(X)$, but it is not necessarily ideal in $C(X)$.

Definition 2.2. A proper weak ideal P in R is called *prime* if for every $a, b \in R$, we have that $a \in P$ or $b \in P$ whenever $ab \in P$.

By the following proposition, for a topological space of X, if $WC_F(X)$ is a prime weak ideal, then X is an almost discrete space.

Proposition 2.1. *For a topological space of X, the following statements are equivalent:*

(1) *$WC_F(X)$ is a prime weak ideal.*

(2) $|X| = 2$.

Proof. (1) $\Rightarrow$ (2) Let $I(X) \geqslant 2$ and $a, b \in I(X)$ be different elements. There exists $f, g \in C(X)$ such that $f[X \setminus I(X)] = g[\{a, b\}] = \{1\}$ and $f(\{a\}) = g[X \setminus \{a, b\}] = \{0\}$. Then $fg \in WC_F(X)$, and $g \notin WC_F(X)$, it follows that $f \in WC_F(X)$. Hence $|X| = 2$. Now we suppose that $I(X) \lneqq 2$. If X is a finite space, then $X = I(X)$. Hence $C(X) = WC_F(X)$ and we have a contradiction with $WC_F(X)$ is a prime weak ideal. Therefore we may suppose that X is an infinite space. Let $a, b \in X \setminus I(X)$ be different elements. By complete regularity of X, there exists $f, g \in C(X)$ such that $f(a) = g(b) = 0$ and $f(b) = g(a) = 0$. Define $h = f^2 - g^2$, and consider $(h - |h|)(h + |h|) = 0$. Since $WC_F(X)$ is a prime weak ideal of $C(X)$, $h - |h| \in WC_F(X)$ or $h + |h| \in WC_F(X)$. If $h + |h| \in WC_F(X)$, then

$f^2 - g^2 + |h| \in M_b$ implies that $f^2 + |h| \in M_b$. But M_b is absolutely convex, and therefore this would imply that $f \in M_b$, a contradiction. Thus $h - |h| \in WC_F(X)$. But $f^2 - g^2 - |h| \in M_b$ implies that $g^2 + |h| \in M_b$ and hence $g \in M_b$ a contradiction.

(2) $\Rightarrow$ (1) It is evident. □

Proposition 2.2. *For a topological space of X, the following statements are equivalent:*

(1) *X is an almost discrete space.*

(2) $\operatorname{Ann}\big(WC_F(X)\big) = \{0\}$.

Proof. (1) $\Rightarrow$ (2) Let $e \in \operatorname{Ann}(WC_F(X))$, then $I(X) \subseteq Z(e)$, it follows that $X = \operatorname{cl}_X(I(X)) \subseteq Z(e)$, i.e., $e = 0$

(2) $\Rightarrow$ (1) We suppose that $x \in X \setminus \operatorname{cl}_X I(X)$ and get a contradiction. Then by complete regularity of X, there exists $g \in C(X)$ such that $g[\operatorname{cl}_X I(X)] = \{0\}$ and $g(x) = 1$. Hence it is clear that $0 \neq g \in \operatorname{Ann}(WC_F(X))$ and we get a contradiction. □

Lemma 2.1. *If D is a dense subset of topological space X such that for all $f \in F(X, \mathbb{R})\big(f|_D \in C(D)\big)$ then $D = I(X)$ and it is a discrete subspace of X.*

Proof. Let f be an arbitrary function on D. We can extend f to a function on X, say g, then by hypothesis $f = g|_D \in C(D)$. Thus $F(D, \mathbb{R}) = C(D)$, it follows that D is a discrete subspace of X.

Since D is a discrete subspace of X, for every $d \in D$ there exists an open subset V of X such that $V \cap D = \{d\}$. Thus

$$V \subseteq \operatorname{cl}_X V = \operatorname{cl}_X(V \cap D) = \operatorname{cl}_X\{d\} = \{d\} \subseteq V$$

it follows that $V = \{d\}$ is an open subset of X, this means that $D \subseteq I(X)$ and D is an open subset of X. Since D is a dense subset of X, therefore $I(X) \subseteq D$ and finally $D = I(X)$. □

Proposition 2.3. *For a topological space X, the following statements are equivalent:*

(1) *X is an almost discrete space.*

(2) *There exists an unique dense subset D of X such that*

$$\forall f \in F(X, \mathbb{R})\big(f|_D \in C(D)\big)\,.$$

Proof. (1) $\Rightarrow$ (2) It is clear that $I(X)$ is a discrete subspace of X. Hence if $f \in F(X, \mathbb{R})$, then $f|_{I(X)} \in C\big(I(X)\big)$. By Lemma 2.1, we are through.

(2) $\Rightarrow$ (1) By Lemma 2.1, $D = I(X)$ and we are through. □

Lemma 2.2. *If D is a dense subset of X, then $I(X) = I(D)$.*

Proof. Since D is a dense subset of X, then $I(X) \subseteq I(D) \subseteq D$. If $d \in I(D)$, then there exists an open subset V of X such that $D \cap V = \{d\}$, it follows that $\big(X \setminus \{d\}\big) \cap D \cap V = \emptyset$. Since D is a dense subset of X, then $\big(X \setminus \{d\}\big) \cap V = \emptyset$ and therefore $V = \{d\}$ is an open subset of X. Thus $d \in I(X)$ and we conclude that $I(X) = I(D)$. □

Proposition 2.4. *For a topological space of X, the following statements are equivalent:*

(1) *X is an almost discrete space.*

(2) *For every dense subset D in X, D is an almost discrete space.*

(3) *There exists a dense subset D in X such that D is an almost discrete space.*

Proof. By Lemma 2.2, it is evident. □

By the above proposition, X is an almost discrete space if and only if βX is an almost discrete space if and only if υX is an almost discrete space.

Proposition 2.5. *For a topological space X, the following statements are equivalent:*

(1) *X is an almost discrete space.*

(2) *For every open subset D in X, D is an almost discrete subspace of X.*

Proof. (1) ⇒ (2) Let U be an open subset of X. Then $I(U) = I(X) \cap U$ is a dense subset of U.

(2) ⇒ (1) It is clear. □

Proposition 2.6. *For a topological space X, if every proper closed subset in X is an almost discrete subspace of X, then X is an almost discrete space.*

Proof. If X is finite, then X is discrete space, it follows that $I(X) = X$. Now we suppose that X is infinite. By hypothesis $I(X) \neq \emptyset$. Let U be an open subset of X. If $\mathrm{cl}_X(U) = X$, then by Lemma 2.2, $I(U) = I(X) \neq \emptyset$, it follows that X is an almost discrete space. Therefore we may suppose that $\mathrm{cl}_X(U) \subsetneq X$. By hypothesis $V = \mathrm{cl}_X(U)$ is an almost discrete subspace of X, hence U has an isolated point x in V. It is clear that $x \in I(X)$. Thus $\mathrm{cl}_X I(X) = X$, i.e., X is an almost discrete space. □

Proposition 2.7. *Let $\{X_i\}_{i=1}^n$ be a family topological spaces and $X = \prod_{i=1}^n X_i$ be a product space. If for every $1 \leq i \leq n$, X_i is an almost discrete space, then X is an almost discrete space.*

Proof. If $I(X_i)$ and $I(X_j)$ are dense in X_i and X_j respectively, then $I(X_i \times X_j) = I(X_i) \times I(X_j)$ is dense in $X_i \times X_j$ and by induction on n, we are through. □

Remark. If for every $i \in \mathbb{N}$, $X_i = \{0, 1\}$ is a discrete space, then $X = \prod_{i \in \mathbb{N}} X_i$ is not almost discrete space, in fact $I(X) = \emptyset$.

It is clear that every ideal of R is a weak ideal and conversely is false. It is natural to ask: when $WC_F(X)$ is an ideal.

Proposition 2.8. *For a topological space of X, the following statements are equivalent:*

(1) *$WC_F(X)$ is an ideal of $C(X)$.*

(2) $|I(X)| \leq 1$.

(3) $WC_F(X) = \bigcap_{x \in D(X)} O_x$.

Proof. (1) $\Rightarrow$ (2) Let $a, b \in I(X)$ be different points of X. Then we define $f, g \in C(X)$ such that $f(a) = g(b) = 1$ and $f\big[X \setminus \{a\}\big] = g\big[X \setminus \{b\}\big] = \{0\}$. Hence $\operatorname{Coz}(f^2+g^2) = \{a, b\}$ and by hypothesis $f^2+g^2 \in WC_F(X)$, i.e., $|\operatorname{Coz}(f^2+g^2)| \leq 1$ which we have a contradiction.

(2) $\Rightarrow$ (3) If $I(X) = \emptyset$, then $WC_F(X) = \bigcap_{x \in D(X)} O_x = \{0\}$.

Let $I(X) = \{a\}$. If $0 \neq f \in WC_F(X)$, then $Z(f) = X \setminus \{a\}$ is an open subset of X, it follows that $f \in \bigcap_{x \in D(X)} O_x$. So that if $0 \neq f \in \bigcap_{x \in D(X)} O_x$, then $D(X) \subseteq Z(f) \neq X$, i.e., $\operatorname{Coz}(f) = \{a\}$ and $f \in WC_F(X)$.

(3) $\Rightarrow$ (1) It is clear. $\square$

By Lemma 2.2 and Proposition 2.8, the following statements are equivalent:

(1) $WC_F(X)$ is an ideal of $C(X)$.

(2) $WC_F(\beta X)$ is an ideal of $C(\beta X)$.

(3) $WC_F(\upsilon X)$ is an ideal of $C(\upsilon X)$.

3. Essential weak ideals of $C(X)$

An ideal of R is called *essential* if it intersects every nonzero ideal nontrivially. In the theory of rings, many structure results were obtained with the help of essential ideals, and the socle of a commutative ring is the intersection of all essential ideals, see [5].

One of the main aims of this section is to show that $WC_F(X)$ is an essential weak ideal of $C(X)$ if and only if X is an almost discrete space and also we study the intersection essential weak ideals of $C(X)$.

But we must begin with the basic definition, such as essential weak ideal.

Definition 3.1. A weak ideal of a ring R is called *essential* if it intersects every nonzero weak ideal nontrivially.

Proposition 3.1. *If A is a nonzero weak ideal in $C(X)$, then the following statements are equivalent:*

(1) *A is essential weak ideal in $C(X)$.*

(2) $\operatorname{Ann}(A) = \{0\}$.

(3) *$\bigcap Z[A]$ is a nowhere dense subset of X.*

Proof. (1) $\Rightarrow$ (2) It is clear that $(\operatorname{Ann}(A) \cap A)^2 = \{0\}$, implies that $\operatorname{Ann}(A) \cap A = \{0\}$. Hence $\operatorname{Ann}(A) = \{0\}$.

(2) $\Rightarrow$ (3) Suppose the interior of $\bigcap Z[A]$ is nonempty set. If $x \in \operatorname{int}_X \bigcap Z[A]$, then by the complete regularity of X, there is $g \in C(X)$ such that $g(x) = 1$ and $g\big[X \setminus \operatorname{int}_X \bigcap Z[A]\big] = \{0\}$. Thus for every $f \in A$ we have $fg = 0$, i.e., $\operatorname{Ann}(A) \neq \{0\}$, a contradiction.

(3) $\Rightarrow$ (1) Let B be a nonzero weak ideal in $C(X)$ and $0 \neq g \in B$. It is clear that $X \setminus \bigcap Z[A]$ is open and dense in X. Then $\big(X \setminus Z[g]\big) \cap \big(X \setminus \bigcap Z[A]\big) \neq \emptyset$, it follows that there is a $f \in A$ for which $\big(X \setminus Z[g]\big) \cap \big(X \setminus Z[f]\big) \neq \emptyset$. Therefore $Z[fg] \neq X$, i.e., $0 \neq fg \in A \cap B$. Hence A is essential weak ideal in $C(X)$. $\square$

Corollary 3.1. *The ideal* (*weak ideal*) E *is essential ideal* (*weak ideal*) *in* $C(X)$ *if and only if* $\operatorname{int}_X \bigcap Z[E] = \emptyset$.

Proposition 3.2. *For a topological space of* X*, the following statements are equivalent:*

(1) $WC_F(X)$ *is an essential weak ideal of* $C(X)$.

(2) X *is an almost discrete space.*

Proof. (1) $\Rightarrow$ (2) Let G be a proper nonempty open subset of X. Then

$$I = \{f \in C(X) : X \setminus G \subseteq Z(f)\}$$

is a nonzero ideal of $C(X)$ and by hypothesis there exists $0 \neq f \in WC_F(X) \cap I$. Hence $\operatorname{Coz}(f) \subseteq I(X) \cap G$ and we are through.

(2) $\Rightarrow$ (1) Let I be a nonzero weak ideal and $0 \neq f \in I$, then by hypothesis

$$\operatorname{Coz}(f) \cap \Big(X \setminus \bigcap Z\big[WC_F(X)\big]\Big) = \operatorname{Coz}(f) \cap I(X) \neq \emptyset$$

this implies that there exists $g \in WC_F(X)$ such that $\operatorname{Coz}(f) \cap \operatorname{Coz}(g) \neq \emptyset$. Hence $Z(fg) \neq X$, i.e., $0 \neq fg \in WC_F(X) \cap I$. □

Proposition 3.3. *For a topological space of* X*, the following statements are equivalent:*

(1) $\bigcup_{x \in X} O_x$ *is an essential weak ideal in* $C(X)$.

(2) *For every* $0 \neq f \in C(X)$*, if* f *is not unit then there exists* $0 \neq g \in C(X)$ *such that* $\operatorname{int}_X Z(fg) \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2) Let $0 \neq f \in C(X)$ and it is not unit. Then $fC(X) \cap \big(\bigcup_{x \in X} O_x\big) \neq \{0\}$, it follows that there exists $g \in C(X)$ such that $fg \in \bigcup_{x \in X} O_x$. Therefore there exists $x \in X$ such that $x \in \operatorname{int}_X Z(fg) \neq \emptyset$.

(2) $\Rightarrow$ (1) Let $I \neq \{0\}$ be a proper weak ideal in $C(X)$. If $0 \neq f \in I$, then there exists $0 \neq g \in C(X)$ and $x \in X$ such that $x \in \operatorname{int}_X Z(fg)$, it follows that $fg \in I \cap \big(\bigcup_{x \in X} O_x\big)$. Hence $\bigcup_{x \in X} O_x$ is an essential weak ideal in $C(X)$. □

In the following example we show that there exists weak ideal in $C(X)$ such that $Z[I]$ is closed under finite intersection, but it is not ideal in $C(X)$.

Example 1. Let

$$f(x) = \begin{cases} \frac{1}{\ln(x)} & x \gneqq 0 \\ x & x \leqq 0 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x & x \geqq 0 \\ \frac{1}{\ln(-x)} & x \lneqq 0 \end{cases}.$$

Then $\lim_{x \to 0^+} \frac{f(x)}{g(x)} = -\infty$ and $\lim_{x \to 0^-} \frac{g(x)}{f(x)} = +\infty$. Thus $f \notin gC(X)$ and $g \notin fC(X)$. Let $I = fC(X) \cup gC(X)$, then I is a weak ideal in $C(X)$, but I is not ideal in $C(X)$, for if $f + g \in I$, then there exists $h \in C(X)$ such that $f + g = fh$ or $f + g = gh$, it follows that $g = (h-1)f \in fC(X)$ or $f = (h-1)g \in gC(X)$, which we have a contradiction. Also since $Z(ff_1) \cap Z(gg_1) = Z\big(f(f_1^2 + g_1^2)\big)$, hence $Z[I]$ is closed under finite intersection.

Lemma 3.1. *Let J be a weak ideal of $C(X)$ and $A = \bigcap_{f\in J} \mathrm{cl}_{\beta X} Z(f)$. If $Z[J]$ is closed under finite intersection then $O^A \subseteq J$, where $O^A = \{f \in C(X) : A \subseteq \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X} Z(f)\}$.*

Proof. Let $g \in O^A$. By Lemma 1.1 in [3], there exists an open neighborhood U of A with $U \cap X \subseteq Z(g)$. For each $y \in \beta X \setminus U$. We can find an $f_y \in J$ so that $y \notin \mathrm{cl}_{\beta X} Z(f_y)$. Since βX is regular we may choose a neighborhood U_y of y disjoint from $\mathrm{cl}_{\beta X} Z(f_y)$. The U_y's cover the compact set $\beta X \setminus U$ so for some $y_1, \dots, y_n \in \beta X$, $\beta X \setminus U \subseteq \bigcup_{i=1}^n U_{y_i}$. By hypothesis there exists $f \in J$ such that $Z(f) = \bigcup_{i=1}^n Z(f_{y_i})$ so that $\mathrm{cl}_{\beta X} Z(f) = \bigcup_{i=1}^n \mathrm{cl}_{\beta X} Z(f_{y_i})$ and hence

$$(\beta X\setminus U)\cap \mathrm{cl}_{\beta X} Z(f) \subseteq \Big(\bigcup_{i=1}^n U_{y_i}\Big)\cap\Big(\bigcup_{i=1}^n \mathrm{cl}_{\beta X} Z(f_{y_i})\Big) \subseteq \bigcup_{i=1}^n \big(U_{y_i}\cap \mathrm{cl}_{\beta X} Z(f_{y_i})\big) = \emptyset\,.$$

This means $Z(f) \subseteq X\cap \mathrm{cl}_{\beta X} Z(f) \subseteq U\cap X \subseteq Z(g)$, it follows that $Z(f) \subseteq \mathrm{int}_X Z(g)$. By Problem $1D(1)$ in [4], there exists $h \in C(X)$ with $g = fh \in J$, so $O^A \subseteq J$. □

Corollary 3.2. *If J is an ideal of $C(X)$ and $A = \bigcap_{f\in J} \mathrm{cl}_{\beta X} Z(f)$, $O^A \subseteq J \subseteq M^A$, where $M^A = \{f \in C(X) : A \subseteq \mathrm{cl}_{\beta X} Z(f)\}$.*

Proof. Since $Z[J]$ is closed under finite intersection, be Lemma 3.1, we are through. □

We need the following lemma which is proved in [3].

Lemma 3.2. *For a topological space of X, if $A \subseteq \beta X$,*

$$\bigcap_{f\in O^A} \mathrm{cl}_{\beta X} Z(f) = \mathrm{cl}_{\beta X} A = \bigcap_{f\in M^A} \mathrm{cl}_{\beta X} Z(f)$$

and if $O^A \subseteq M^B$, $\mathrm{cl}_{\beta X} B \subseteq \mathrm{cl}_{\beta X} A$.

Proposition 3.4. *Let λ be a cardinal number and X be a compact space. If every intersection of a family $\mathcal{A}$ of essential weak ideals in $C(X)$ with $|\mathcal{A}| \le \lambda$ is an essential ideal in $C(X)$, then every union of a family $\mathcal{V}$ of nowhere dense subset in X with $|\mathcal{V}| \le \lambda$ is nowhere dense subset of X.*

Proof. Let $\{V_i\}_{i\in I}$ be a family of nowhere dense subset of X with $|I| \le \lambda$ and $V = \bigcup_{i\in I} V_i$. By Lemma 3.2, $\bigcap Z[O^{V_i}] = \mathrm{cl}_X V_i$. Since $\mathrm{int}_X\, \mathrm{cl}_X V_i = \emptyset$, by Corollary 3.1, O^{V_i} is an essential weak ideal of $C(X)$. So that by our hypothesis $E = \bigcap_{i\in I} O^{V_i} = O^V$ is an essential weak ideal of $C(X)$, hence again by Lemma 3.2 and Corollary 3.1, $\bigcap Z[E] = \mathrm{cl}_X V$ and $\mathrm{int}_X\, \mathrm{cl}_X V = \emptyset$, it follows that V is nowhere dense subset of X. □

Proposition 3.5. *Let λ be a cardinal number and X be a compact space. If every union of a family $\mathcal{V}$ of nowhere dense subset in X with $|\mathcal{V}| \le \lambda$ is nowhere dense subset of X, then every intersection of a family $\{A_i\}_{i\in I}$ of essential weak ideals in $C(X)$ with $|I| \le \lambda$ such that for every $i \in I$, $Z[A_i]$ is closed under finite intersection is an essential ideal in $C(X)$.*

Proof. Let $\{A_i\}_{i\in I}$ be a family of essential weak ideals in $C(X)$ with $|I| \leq \lambda$ such that for every $i \in I$, $Z[A_i]$ is closed under finite intersection. We put for each $i \in I$, $V_i = \bigcap Z[A_i]$ and $V = \bigcup_{i\in I} V_i$. Hence by Corollary 3.1, for each $i \in I$, $\operatorname{int}_X \operatorname{cl}_X V_i = \operatorname{int}_X \bigcap Z[A_i] = \emptyset$, i.e., V_i is a nowhere dense subset of X. By Lemma 3.1, $O^{V_i} \subseteq A_i$ and hence $O^V = \bigcap_{i\in I} O^{V_i} \subseteq \bigcap_{i\in I} A_i$. Now we have by Lemma 3.2, $\bigcap Z[O^V] = \operatorname{cl}_X V$ and since by our hypothesis V is nowhere dense subset of X, then O^V is an essential weak ideal of $C(X)$, it follows that $\bigcap_{i\in I} A_i$ is an essential weak ideal of $C(X)$. □

The following result the consequence of Proposition 3.4 and 3.5.

Corollary 3.3. *Let λ be a cardinal number. For a compact space X, the following statements are equivalent:*

(1) *If $\{A_i\}_{i\in I}$ is a family of essential ideals in $C(X)$ and $|I| \leq \lambda$, then $\bigcap_{i\in I} A_i$ is essential ideal in $C(X)$.*

(2) *If $\{V_i\}_{i\in I}$ is a family of nowhere dense subset of X and $|I| \leq \lambda$, then $\bigcup_{i\in I} V_i$ is nowhere dense subset of X.*

By the above proposition, for a compact space X, every countable intersection of essential ideals in $C(X)$ is an essential ideal in $C(X)$ if and only if every first category subset of X is nowhere dense subset in X.

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