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Rod A. Gover; Josef Šilhan

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# COMMUTING LINEAR OPERATORS AND ALGEBRAIC DECOMPOSITIONS

A. ROD GOVER AND JOSEF ŠILHAN

Abstract. For commuting linear operators $P_0, P_1, \dots, P_\ell$ we describe a range of conditions which are weaker than invertibility. When any of these conditions hold we may study the composition $P = P_0 P_1 \cdots P_\ell$ in terms of the component operators or combinations thereof. In particular the general inhomogeneous problem $Pu = f$ reduces to a system of simpler problems. These problems capture the structure of the solution and range spaces and, if the operators involved are differential, then this gives an effective way of lowering the differential order of the problem to be studied. Suitable systems of operators may be treated analogously. For a class of decompositions the higher symmetries of a composition $P$ may be derived from generalised symmmetries of the component operators $P_i$ in the system.

## 1. Introduction

Given a vector space $\mathcal{V}$ and a system $P_0, P_1, \dots, P_\ell$ of mutually commuting endomorphims of $\mathcal{V}$ we study the composition $P := P_0 P_1 \cdots P_\ell$. It is natural to ask whether we can reduce the questions of null space and range of $P$ to the similar questions for the component operators $P_i$. If these component operators are each invertible then of course one trivially has a positive answer to this question. On the other hand experience with, for example, constant coefficient linear ordinary differential equations shows that this is too much to hope for in general. Here we review, discuss, and extend a recent work [7] in which we introduce a range of conditions which are significantly weaker than invertibility of $P$ and yet which, in each case, enables progress along these lines.

Each condition we describe on the system $P_0, P_1, \dots, P_\ell$ is termed an $\alpha$–decomposition (where $\alpha$ is a subset of the power set of the index set $\{0, \dots, \ell\}$). The case that the operators $P_i$ are each invertible is one extreme. Of course one may ask that some of $P_i$ are invertible but, excluding an explicit assumption along these

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lines, the next level is what we term as simply a decomposition. This is described explicitly in Section 2 below, but intuitively the main point is that each pair $P_i$ and $P_j$, for $i \neq j$ in $\{0, \ldots, \ell\}$, consistes of operators which are relatively invertible in the sense that for example $P_i$ is invertible on the null space of $P_j$ and vice versa. In the case that we have a decomposition then one obtains very strong results: the null space of $P$ is exactly the direct sum of the null spaces for the factors $P_i$; the range of $P$ is precisely the intersection of the range spaces for the factors; and one may explicitly decompose the general inhomogeneous problem $Pu = f$ into an equivalent system of "lower order" problems $P_i u_i = f$, $i = 0, \ldots, \ell$. (Note that the fact that the same inhomogeneous term $f \in \mathcal{V}$ appears in $Pu = f$ and in each of the $P_i u_i = f$ problems is one signal that the construction we discuss is not the trivial manouever of renaming variables.)

At the other extreme of the $\alpha$–decompositions we ask only that the operator $(P_0, \ldots, P_\ell)\colon \mathcal{V} \to \oplus^{\ell+1}\mathcal{V}$ is injective with left inverse given by a system of $\mathcal{V}$ endomorphisms $Q_i$, $i = 0, \ldots, \ell$, which commute with the $P_j$s. Remarkably this is sufficient for obtaining results along a similar line to the case of a decomposition, but the extent of simplification is less dramatic: issues of null space and range for $P$ are subordinated only to similar questions for the operators $P^i := P/P_i$. The full summary result for $\alpha$-decompositions is given in Theorem 2.1 below.

A natural setting for the use of these results is in the study of operators $P$ which are polynomial in a mutually commuting system of linear operators $\mathcal{D}_j\colon \mathcal{V} \to \mathcal{V}$, $j = 1, \ldots, k$. This is the subject of Section 3. Given a $P$ which is suitably factored, or alternatively working over an algebraically closed field, one sees that generically some algebraic $\alpha$–decomposition is available. The main point here is the word "algebraic". The $\alpha$–decompositions of compositions $P = P_0 P_1 \cdots P_\ell$ involve identities which involve operators $Q_i$, $i = 0, \ldots, \ell$, which invert some subsystem of $(P_0, \ldots, P_\ell)$. In the case of an algebraic decomposition the $Q_i$s are also polynomial in the $\mathcal{D}_j$s. So, for example, if the factors $P_i$ of $P$ are differential and polynomial in the differential operators $\mathcal{D}_i$ then the $Q_i$ are also differential and are given in terms of the $\mathcal{D}_i$ by explicit algebraic formulae. In particular pseudo-differential calculus is avoided. These results are universal in the sense that they are independent of any details of the operators $\mathcal{D}_j$.

The idea behind the decompositions of the equation $P_0 \cdots P_\ell u = f$ is rather universal; for example one can extend it to systems of equations. In this direction our aim is mainly to demonstrate the technique, so we shall treat, as an example (see Section 4), only one specific situation where the idea applies. This is a system of $k+1$ equations where the first equation $P_0 \cdots P_\ell u = f$ is factored and the remaining ones are of the form $R^{(j)} u = g^j$, $1 \leq j \leq k$ for given $f, g^j \in \mathcal{V}$ and $R^{(j)} \in \mathrm{End}(\mathcal{V})$. This may be viewed as a problem where one wants to solve the problem $P_0 \cdots P_\ell u = f$, subject to the conditions $R^{(j)} u = g^j$. The difference, in comparison to the single equation problem, is that now the operators $R^{(j)}$ feature in the relative invertibility of the factors $P_i$ of $P$. The result is that provided one has a suitable decomposition at hand, the original system is equivalent to a family of "lower order" systems of the same type as the original one.

Finally in Section 5 we discuss symmetries of operators. We define a formal symmetry of an operator $P : \mathcal{V} \to \mathcal{V}$ to be an operator $S : \mathcal{V} \to \mathcal{V}$ such that $PS = S'P$ for some other operator $S'$ on $\mathcal{V}$. For $P$ a Laplacian (or Laplacian power) type operator differential operator, and $S$, $S'$ differential, such symmetries are central in the separation of variable techniques [1, 9]. For $P = P_0 P_1 \ldots P_\ell$, as above, the tools we develop earlier are used to show that the formal symmetry algebra of $P$ is generated by the formal symmetry operators, and appropriate generalizations thereof, for the component operators $P_0, P_1, \ldots, P_\ell$.

## 2. Decompositions and $\alpha$-decompositions

Let $\mathcal{V}$ denote a vector space over a field $\mathbb{F}$ and consider linear operators $P_0, \ldots, P_\ell$ (i.e. endomorphisms of $\mathcal{V}$) which mutually commute. In [7] we study properties of the operator

$$P = P_0 P_1 \cdots P_\ell \,. \tag{1}$$

For example, an obvious question is: what can we say about the kernel and the image of $P$ in terms of related data for the component operators $P_i$? This is clearly straightforward if the operators $P_i$ are invertible, but the point of our studies is that much weaker assumptions are sufficient to obtain quite striking results. These assumptions are captured in the notion of various "decompositions"; the different possible decompositions are parametrised by a nonempty system of subsets of $L := \{0, \ldots, \ell\}$. We shall use the notation $P_J := \prod_{j \in J} P_j$ for $\emptyset \neq J \subseteq L$ and we set $P_\emptyset := \mathrm{id}_\mathcal{V}$. Further we put $P^J := P_{L \setminus J}$, $P^j := P^{\{j\}}$ and write $2^L$ to denote the power set of $L$. Also $|J|$ will denote the cardinality of a set $J$.

**Definition.** For a linear operator $P : \mathcal{V} \to \mathcal{V}$, an expression of the form (1) will be said to be an *$\alpha$–decomposition* of $P$, with $\emptyset \neq \alpha \subseteq 2^L$, $L \notin \alpha$, if there exist operators $Q_J \in \mathrm{End}(\mathcal{V})$, $J \in \alpha$ such that

$$\mathrm{id}_\mathcal{V} = \sum_{J \in \alpha} Q_J P^J, \quad [P_i, P_k] = [Q_J, P_i] = 0\,, \quad i, k \in L, J \in \alpha\,. \tag{2}$$

The choice $\alpha := \{\emptyset\}$ means that $P$ (hence also each of the $P_i$) is invertible. The other possible decompositions involve weaker assumptions on the component operators. At the next level is the $\alpha$–decomposition with $\alpha := \{J \subseteq L \mid |J| = 1\}$ which will be termed simply a **decomposition** of $P$. In this case we still obtain, for example, that $\mathcal{N}(P) = \bigoplus \mathcal{N}(P_i)$. Therefore the problem $Pu = 0$, $u \in \mathcal{V}$ is reduced to the system $P_i u_i = 0$, $u_i \in \mathcal{V}$ for $i \in L$. In the case that the $P_i$ are differential operators, this result shows that, given a decomposition, the equation $Pu = 0$ reduces to the lower order system $P_i u_i = 0$.

For the general $\alpha$–decomposition we do not generally obtain a direct sum analogous to $\mathcal{N}(P) = \bigoplus \mathcal{N}(P_i)$ as above, however we still get a reduction to a "lower order" problem. The key is the following theorem which is a central result in [7]. (See the latter for the proof and more details.)

**Theorem 2.1** ([7])**.** *Assume $P : \mathcal{V} \to \mathcal{V}$ as in (1) is an $\alpha$–decomposition. Let us fix $f \in \mathcal{V}$. There is a surjective mapping $B$ from the space of solutions $(u_J)_{J\in\alpha} \in \oplus^{|\alpha|}\mathcal{V}$ of the problem*

$$P_J u_J = f\,, \quad J \in \alpha\,. \tag{3}$$

*onto the space of solutions $u \in \mathcal{V}$ of $Pu = f$.*

*Writing $\mathcal{V}_P^f$ for the solution space of $Pu = f$ and (for $J \in \alpha$) $\mathcal{V}_J^f$ for the solution space of $P_J\tilde{u} = f$. The map $B\colon \times_{J\in\alpha}\mathcal{V}_J^f \to \mathcal{V}_P^f$ is given by*

$$(u_J)_{J\in\alpha} \mapsto \sum_{J\in\alpha} Q_J u_J\,.$$

*A right inverse for this is $F\colon \mathcal{V}_P^f \to \times_{J\in\alpha}\mathcal{V}_J^f$ given (component-wise) by*

$$u \mapsto P^J u\,;$$

*on $\mathcal{V}$ we have $B \circ F = id_{\mathcal{V}}$.*

*If $\alpha$ satisfies $I \cap J = \emptyset$, for all $I \neq J \in \alpha$, then $F$ is a 1-1 mapping and $F \circ B$ is the identity on the solution space to (3).*

**Remark 2.2.** 1. The important feature of the decomposition of inhomogeneous problems is that in $Pu = f$ and (3) it is the *same* $f \in \mathcal{V}$ involved. So (3) describes the range of $P$, $\mathcal{R}(P)$, in terms of the ranges of the $P_J$: $\mathcal{R}(P) = \cap_{J\in\alpha}\mathcal{R}(P_J)$.

2. The condition in the last paragraph in the theorem is satisfied by a decomposition, but is easy to construct other examples. In any case where this is satisfied the mappings $F$ and $B$ are bijections and we do get a direct sum decomposition of $\mathcal{N}(P)$. The point, which is easily verified, is that from (2) it follows that for each $J \in \alpha$

$$Q_J P^J \colon \mathcal{N}(P) \to \mathcal{N}(P_J)$$

is a projection.

Although the $\alpha$–decomposition (2) is what is directly employed in the previous theorem and its proof, there is a distinct, but related, notion which shows what it really means for the commuting operators $P_i$. The following definition introduces an idea of a decomposition which turns about to be in a suitable sense "dual" to the previous one.

**Definition.** For a linear operator $P : \mathcal{V} \to \mathcal{V}$, an expression of the form (1) will be said to be a *dual $\beta$–decomposition* of $P$, $\emptyset \neq \beta \subseteq 2^L$, $\{\emptyset\} \neq \beta$ if for every $J \in \beta$ there exist operators $Q_{J,j} \in \mathrm{End}(\mathcal{V})$, $j \in J$ such that

$$\mathrm{id}_{\mathcal{V}} = \sum_{j\in J} Q_{J,j} P_j\,, \quad [P_i, P_k] = [Q_{J,j}, P_i] = 0\,, \quad i,k \in L, j \in J\,. \tag{4}$$

To describe the suggested duality (see Proposition 2.4 below), first observe that each system $\alpha \subseteq 2^L$ is partially ordered by restricting the poset structure of $2^L$. The sets of minimal and maximal elements in $\alpha$ will be denoted by $\mathrm{Min}(\alpha)$ and $\mathrm{Max}(\alpha)$, respectively. We say the system $\beta \subseteq 2^L$ is a **lower set**, if it is closed under taking a subset. (That is, if $I \in \beta$ and $J \subseteq I$ then $J \in \beta$.) The *upper set* is defined dually. The lower set and upper set generated by a system $\alpha \subseteq 2^L$ will be denoted by $\mathcal{L}(\alpha) := \{J \subseteq I \mid I \in \alpha\}$ and $\mathcal{U}(\alpha) := \{J \supseteq I \mid J \subseteq L \text{ and } I \in \alpha\}$, respectively.

The proof of the following is obvious.

**Lemma 2.3.** *Let $\alpha \subseteq 2^L$. Then $P = P_0 \cdots P_\ell$ satisfies the following:*
(i) *it is an $\alpha$–decomposition $\iff$ it is a* $\mathrm{Max}(\alpha)$*–decomposition $\iff$ it is an $\mathcal{L}(\alpha)$–decomposition*
(ii) *it is a dual $\alpha$–decomposition $\iff$ it is a dual* $\mathrm{Min}(\alpha)$*–decomposition $\iff$ it is a dual $\mathcal{U}(\alpha)$–decomposition.*

To formulate the relation between $\alpha$– and dual $\alpha$–decompositions, we need the following notation. We put $\alpha^u := 2^L \setminus \mathcal{L}(\alpha)$ and $\alpha^l := 2^L \setminus \mathcal{U}(\alpha)$. Clearly $(\alpha^u)^l = \mathcal{L}(\alpha)$ and $(\alpha^l)^u = \mathcal{U}(\alpha)$. Also it is easily seen that

$$\begin{aligned} \alpha^u &= \{J \subseteq L \mid \forall I \in \alpha\colon J \setminus I \neq \emptyset\} \\ \alpha^l &= \{J \subseteq L \mid \forall I \in \alpha\colon I \setminus J \neq \emptyset\}\,. \end{aligned} \tag{5}$$

**Proposition 2.4** (The duality)**.** (1) *is an $\alpha$–decomposition if and only if it is a dual $\alpha^u$–decomposition. Equivalently,* (1) *is a dual $\beta$–decomposition if and only if it is a $\beta^l$–decomposition.*

*In particular,* (1) *is a decomposition if and only if it is a dual $\beta$–decomposition for $\beta := \{J \subseteq L \mid |J| = 2\}$. This means*

$$\mathrm{id}_{\mathcal{V}} = Q_{i,j}P_i + Q_{j,i}P_j \tag{6}$$

*where $Q_{i.j} \in \mathrm{End}(\mathcal{V})$ and satisfy $[Q_{i,j}, P_k] = 0$ for every triple of integers $(i, j, k)$ such that $0 \le i, j, k \le \ell$ and $i \neq j$.*

See [7] for the proof. When (6) is satisfied we shall say that the operators $P_i$ and $P_j$ are *relatively invertible.* The dual version of a (true) decomposition is the dual $\beta$–decomposition for $\beta = \{J \subseteq L \mid |J| = 2\}$; this will be termed simply a *dual decomposition.* The general dual $\beta$–decomposition means that for every $J \in \beta$, the operators $P_j$, $j \in J$ are relatively invertible.

**Remark 2.5.** 1. The proof of Proposition 2.4 in [7] is constructive in the sense that starting with an $\alpha$–decomposition (2), there is a simple recipe which describes how to construct the operators $Q_{J,j}$ from (4), as required for the dual $\alpha^u$–decomposition, and then vice versa.

2. The operators $Q_J$ in (2) and $Q_{J,j}$ in (4) are not given uniquely. Also we can see an obvious duality between $Q_{i,j}$, $Q_{j,i}$ and $P_i$, $P_j$ in (6). This is discussed in [7] where notion of (dual) $\alpha$–decompositions is formulated in the language of Koszul complexes.

From the practical point of view, given an operator $P\colon \mathcal{V} \to \mathcal{V}$ as in (1), to apply Theorem 2.1 one needs to show whether $P$ is a (dual) $\alpha$–decomposition and also to determine explicitly the corresponding operators $Q_J$ (or $Q_{J,j}$ in the dual case). Also, one can ask which choice of $\alpha$ yields the most suitable $\alpha$–decomposition. Another strategy might be to "regroup" the operators $P_i$ (e.g. to consider the product $P_iP_j$ as a single factor) and then to seek a better $\alpha$–decomposition. In the case that the operator $P$ is polynomial in other mutually commuting operators

$\mathcal{D}_j : \mathcal{V} \to \mathcal{V}$, there is a category of decompositions which arise algebraically from the formula for $P$. Within this category all these questions can all be solved in a completely algorithmic way.

## 3. Operators polynomial in commuting endomorphisms and algebraic decompositions

Writing $\mathcal{V}$ to denote a vector space over some field $\mathbb{F}$, suppose that $\mathcal{D}_i : \mathcal{V} \to \mathcal{V}$, $i = 1, \dots, k$, are non-trivial linear endomorphisms that are mutually commuting: $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$ for $i, j \in \{1, \dots, k\}$. We obtain a commutative algebra $\mathbb{F}[\boldsymbol{\mathcal{D}}]$ consisting of those endomorphisms $\mathcal{V} \to \mathcal{V}$ which may be given by expressions polynomial (with coefficients in $\mathbb{F}$) in the $\mathcal{D}_i$. We write $\boldsymbol{x} = (x_1, \dots, x_k)$ for the multivariable indeterminate, and $\mathbb{F}[\boldsymbol{x}]$ for the algebra of polynomials in the variables $x_1, \dots, x_k$ over the field $\mathbb{F}$. There is a unital algebra epimorphism from $\mathbb{F}[\boldsymbol{x}]$ onto $\mathbb{F}[\boldsymbol{\mathcal{D}}]$ given by formally replacing each variable $x_i$, in a polynomial, with $\mathcal{D}_i$.

The simplest case is when $k = 1$, that is operators polynomial in a single operator $\mathcal{D}$. We write $\mathbb{F}[\mathcal{D}]$ for the algebra of these. Since any linear operator $\mathcal{D} : \mathcal{V} \to \mathcal{V}$ is trivially self-commuting there is no restriction on $\mathcal{D}$. Thus this case is an important specialisation with many applications. In this setting we may quickly find algebraic decompositions. Let us write $\mathbb{F}[x]$ for polynomials in the single indeterminate $x$ and illustrate the idea with a very simple case. Consider a polynomial $P[x] = (x + \lambda_0)(x + \lambda_1) \dots (x + \lambda_\ell)$ where for $i = 0, \dots, \ell$, the $\lambda_i \in \mathbb{F}$ are are mutually distinct (i.e. $i \neq j \Rightarrow \lambda_i \neq \lambda_j$). Related to $P[x]$ are the polynomials obtained by omitting a factor

$$P^i[x] = \prod_{i \neq j = 0}^{\ell} (x + \lambda_j)\,.$$

Then we associate to $P[x]$ the following decomposition of the unit in $\mathbb{F}[x]$.

**Lemma 3.1.**

$$1 = \alpha_0 P^0[x] + \alpha_1 P^1[x] + \cdots + \alpha_\ell P^\ell[x]\,,$$

*where*

$$\alpha_i = \prod_{i \neq j = 0}^{j = \ell} \frac{1}{\lambda_j - \lambda_i}\,.$$

**Proof.** The polynomial on the right hand side of the claimed equality has degree $\ell$ thus it is sufficient to evaluate this polynomial in the $\ell + 1$ different points $-\lambda_0, \dots, -\lambda_\ell$. Since

$$P^i[-\lambda_k] = \begin{cases} \frac{1}{\alpha_i} & k = i \\ 0 & k \neq i \end{cases}$$

for $i, k \in \{0, \dots, \ell\}$, the lemma follows. □

Thus we have the following.

**Proposition 3.2.** *For $P[\mathcal{D}] = (\mathcal{D}+\lambda_0)(\mathcal{D}+\lambda_1)\cdots(\mathcal{D}+\lambda_\ell)$ we have a decomposition given by*

$$\mathrm{id}_{\mathcal{V}} = Q_0 P^0[\mathcal{D}] + \cdots + Q_\ell P^\ell[\mathcal{D}]\,,$$

*where $P^i = \prod_{i\neq j=0}^{\ell}(\mathcal{D}+\lambda_j)$ and $Q_i = \alpha_i$ for $i = 0,\ldots,\ell$.*

Thus we may immediately apply Theorem 2.1, and in fact the stronger variants for decompositions as in [7], to reduce homogeneous or inhomogeneous problems for $R$ to corresponding problems of the form $(\mathcal{D}+\lambda_i)u_i = f$.

**Remark 3.3.** However the point we wish to emphasise heavily is that we used no information about the operator $\mathcal{D}$ to obtain the decomposition in Proposition 3.2; $\mathcal{D}$ can be any linear operator $\mathcal{D}\colon \mathcal{V}\to\mathcal{V}$ on any vector space $\mathcal{V}$. Thus we will say that Proposition 3.2 is an *algebraic decomposition* of $P[\mathcal{D}]$. For specific operators $\mathcal{D}$ there may be other decompositions (or $\alpha$–decompositions) that do use information about $\mathcal{D}$.

For operators polynomial in mutually commuting operators $\mathcal{D}_i$ we generically may obtain $\alpha$–decompositions that are algebraic in this way; that is they arise, via the algebra epimorphism $\mathbb{F}[\boldsymbol{x}]\to\mathbb{F}[\boldsymbol{\mathcal{D}}]$, from a polynomial decomposition of the unit in $\mathbb{F}[\boldsymbol{x}]$. These are universal $\alpha$–decompositions that are independent of the details of the $\mathcal{D}_i$, $i = 1,\ldots,k$.

Via the Euclidean algorithm, and related tools more powerful for these purposes, we may easily generalise Lemma 3.1 to obtain decompositions for operators more interesting than $P[\mathcal{D}]$ as in the Proposition above. The case of a operators polynomial in a single other operator is treated in some detail in [7] so let us now turn our attention to some general features which appear more in the multivariable case $k\geq 2$.

Given polynomials $P_0[\boldsymbol{x}], P_1[\boldsymbol{x}],\ldots,P_\ell[\boldsymbol{x}]\in\mathbb{F}[\boldsymbol{x}]$ consider the product polynomial

$$P[\boldsymbol{x}] = P_0[\boldsymbol{x}]P_1[\boldsymbol{x}]\cdots P_\ell[\boldsymbol{x}]\,. \tag{7}$$

With $L = \{0,1,\ldots,L\}$, we carry over, in an obvious way, the labelling from Section 2 via elements of the power set $2^L$; products of the polynomial $P_i[\boldsymbol{x}]$ are labelled by the corresponding subset of $L$. For example for $J\subseteq L$, $P_J[\boldsymbol{x}]$ means $\prod_{j\in J}P_j[\boldsymbol{x}]$, while $P^J[\boldsymbol{x}]$ means $P_{L\setminus J}[\boldsymbol{x}]$.

Considering the dual $\beta$–decompositions, we need to verify that for each $I\in\beta$ we have

$$1\in\langle P_i[\boldsymbol{x}]\ :\ i\in I\rangle \tag{8}$$

where $\langle ..\rangle$ denotes the ideal in $\mathbb{F}[\boldsymbol{x}]$ generated by the enclosed polynomials. It is useful to employ algebraic geometry to shed light on this problem, in particular to use the "algebra – geometry dictionary", see for example [2, Chapter 4]. Let us write $\mathcal{N}(S[\boldsymbol{x}]) := \{\boldsymbol{x}\in\mathbb{F}^k \mid S[\boldsymbol{x}] = 0\}$ for the algebraic variety determined by the polynomial $S[\boldsymbol{x}]$. The ideal $\langle P_i[\boldsymbol{x}]\rangle$ corresponds to the variety $\mathcal{N}_i := \mathcal{N}(P_i[\boldsymbol{x}])$ and the previous display clearly requires $\bigcap_{i\in I}\mathcal{N}_i = \emptyset$. In fact if $\mathbb{F}$ is algebraically closed then the latter condition is equivalent to (8). (This follows from the Hilbert Nullstellensatz, see [2].) Since generically $\bigcap_{i\in I}\mathcal{N}_i$ has codimension $|I|$, we conclude that (for $\mathbb{F}$ algebraically closed) if $|I| \geq k+1$ then in the generic case (8) will be satisfied.

## (Dual) decompositions and $\alpha$–decompositions

Aside from invertible $P$, the decompositions are the "best possible" among all $\alpha$–decompositions (and similarly for the dual versions). However they require $2 \geq k+1$ in the generic case (we need $|I| = 2$ in (8)) which holds only for one variable polynomials. On the other hand there is always a chance that we obtain a decomposition by a suitable "regrouping" of the polynomials in (7). So we can proceed as follows.

Any polynomial $P[\boldsymbol{x}] \in \mathbb{F}[\boldsymbol{x}]$ can be decomposed into irreducibles. If we were to take $P_i[\boldsymbol{x}]$ in (7) as such irreducibles then (7) would not be generally the decomposition in the multivariable case. To obtain the decomposition one can consider products $P_I[\boldsymbol{x}]$ as single factors in (7) for suitable $I \subseteq L$. This reduces the number of factors (i.e. $\ell$); to find an optimal (i.e. with $\ell$ maximal) version of this procedure we use the following lemma.

**Lemma 3.4.** (i) *Assume $P[\boldsymbol{x}]$ has the form* (7) *satisfying $1 \in \langle P_i[\boldsymbol{x}], P_j[\boldsymbol{x}]\rangle$ for all $0 \leq i < j \leq \ell$ and $P[\boldsymbol{x}] = R_0[\boldsymbol{x}] \cdots R_r[\boldsymbol{x}]$ is a decomposition of $P[\boldsymbol{x}]$ into irreducible polynomials $R_i[\boldsymbol{x}]$. If $1 \notin \langle R_p[\boldsymbol{x}], R_q[\boldsymbol{x}]\rangle$ for some $0 \leq p, q \leq r$ then there exists $0 \leq i \leq \ell$ such that $P_i[\boldsymbol{x}] = R_p[\boldsymbol{x}]R_q[\boldsymbol{x}]P'_i[\boldsymbol{x}]$ for a polynomial $P'_i[\boldsymbol{x}]$.*

(ii) *Assume the polynomials $S_0[\boldsymbol{x}], \dots, S_s[\boldsymbol{x}]$ and $T_0[\boldsymbol{x}], \dots, T_t[\boldsymbol{x}]$ satisfy $1 \in \langle S_i[\boldsymbol{x}], T_j[\boldsymbol{x}]\rangle$ for all $0 \leq i \leq s$ and $0 \leq j \leq t$. Then $1 \in \langle S[\boldsymbol{x}], T[\boldsymbol{x}]\rangle$ where $S[\boldsymbol{x}] = S_0[\boldsymbol{x}] \cdots S_s[\boldsymbol{x}]$ and $T[\boldsymbol{x}] = T_0[\boldsymbol{x}] \cdots T_t[\boldsymbol{x}]$.*

**Proof.** (i) Assume the case $P_i[\boldsymbol{x}] = R_p[\boldsymbol{x}]P''_i[\boldsymbol{x}]$ and $P_j[\boldsymbol{x}] = R_q[\boldsymbol{x}]P''_j[\boldsymbol{x}]$ for some $i \neq j$. Then $1 \in \langle P_i[\boldsymbol{x}], P_j[\boldsymbol{x}]\rangle$ implies $1 \in \langle R_p[\boldsymbol{x}], R_q[\boldsymbol{x}]\rangle$.

(ii) We use the induction with respect to $s+t$. Clearly the lemma holds for $s+t = 0$ so assume $s+t \geq 1$. Then e.g. $t \geq 1$ so by the inductive hypothesis we get $1 \in \langle S[\boldsymbol{x}], \tilde{T}[\boldsymbol{x}]\rangle$ and $1 \in \langle S[\boldsymbol{x}], T_t[\boldsymbol{x}]\rangle$ where $\tilde{T}[\boldsymbol{x}] = T_0[\boldsymbol{x}] \cdots T_{t-1}[\boldsymbol{x}]$. This means

$$1 = a[\boldsymbol{x}]S[\boldsymbol{x}] + b[\boldsymbol{x}]\tilde{T}[\boldsymbol{x}] \quad \text{and} \quad 1 = c[\boldsymbol{x}]S[\boldsymbol{x}] + d[\boldsymbol{x}]T_t[\boldsymbol{x}]$$

for some polynomials $a[\boldsymbol{x}]$, $b[\boldsymbol{x}]$, $c[\boldsymbol{x}]$ and $d[\boldsymbol{x}]$. Now multiplying the right hand sides of these two equalities and using $T[\boldsymbol{x}] = \tilde{T}[\boldsymbol{x}]T_t[\boldsymbol{x}]$, the lemma follows. □

We will use this lemma as follows. We start with the decomposition of $P[\boldsymbol{x}] = R_0[\boldsymbol{x}] \cdots R_r[\boldsymbol{x}]$ into irreducibles. Consider the graph with vertices $v_0, \dots, v_r$, and an edge $\{v_p, v_q\}$ for every $0 \leq p,\ q \leq r$ such that $1 \notin \langle R_p[\boldsymbol{x}], R_q[\boldsymbol{x}]\rangle$. Denote the number of connected components by $\ell+1$ and the set of vertices in the $i$th component by $G_i$, $0 \leq i \leq \ell$. We put

$$P_i[\boldsymbol{x}] := \prod_{v_u \in G_i} R_u[\boldsymbol{x}]\,, \quad i = 0, \dots, \ell$$

which yields the form (7) of $P[\boldsymbol{x}]$. This satisfies $1 \in \langle P_i[\boldsymbol{x}], P_j[\boldsymbol{x}]\rangle$ for all $0 \leq i < j \leq \ell$ according to Lemma 3.4 (ii) and thus (7) is the decomposition. Moreover, it follows from the part (i) of the lemma that no form $P[\boldsymbol{x}] = P'_0[\boldsymbol{x}] \dots P'_{\ell'}[\boldsymbol{x}]$ with $\ell' > \ell$ can satisfy the condition $1 \in \langle P_i[\boldsymbol{x}], P_j[\boldsymbol{x}]\rangle$ for all $0 \leq i < j \leq \ell'$. (The discussed graph has $\ell + 1$ connected components and we need to "regroup" the vertices (corresponding to irreducible components) into $\ell'+1$ groups corresponding to $\ell'+1$ polynomials $P_i'[\boldsymbol{x}]$. If $\ell' > \ell$ then there is a pair of irreducible polynomials $R_p[\boldsymbol{x}], R_q[\boldsymbol{x}]$ such that $1 \notin \langle R_p[\boldsymbol{x}], R_q[\boldsymbol{x}]\rangle$ which satisfy that $R_p[\boldsymbol{x}]$ is a factor of $P_i'[\boldsymbol{x}]$ and $R_p[\boldsymbol{x}]$ is a factor of $P_j'[\boldsymbol{x}]$ for some $0 \le i < j \le \ell'$. This is a contradiction with Lemma 3.4 (i).)

**Remark 3.5.** From the geometrical point of view, if (7) is a decomposition then $\mathcal{N} := \mathcal{N}(P) = \bigcup \mathcal{N}_i$ is the disjoint union. The previous paragraph describes how to find such decomposition for the variety $\mathcal{N}$ corresponding to any $P[\boldsymbol{x}]$ given by (7). Moreover, the obtained decomposition is minimal in the sense that $\mathcal{N}_i$ in $\mathcal{N} = \bigcup \mathcal{N}_i$ cannot be disjointly decomposed into smaller (nonzero) varieties.

Generically, the (dual) decompositions are not available in the multivariable case. The "optimal" choice among all possible $\alpha$–decompositions (in the sense of [7]) is as follows. The subsets $\beta \subseteq 2^L$ are partially ordered by inclusion (i.e. now we use the poset structure of $2^{2^L}$). Given an operator $P$ in the form (7) consider the family $\Gamma$ of systems $\beta$ such that (7) is a dual $\beta$–decomposition. Then $\Gamma$ has the greatest element $\beta_P = \bigcup_{\beta\in\Gamma} \beta$. Then an "optimal" choice for the dual $\beta$–decomposition of $P$ is $\beta := \mathrm{Min}(\beta_P)$. (We want to have in $\beta$ to the smallest possible subsets of $L$. So if the $P_i$s are not invertible then the case of a dual decomposition may be regarded as the best we can do. With this philosophy we thus take $\beta_P$. Then using Lemma 2.3 we take $\beta := \mathrm{Min}(\beta_P)$ as it is easier to work with a smaller number of subsets.) Consequently, we obtain the optimal choice $\alpha := \mathrm{Max}((\beta_P)^l)$ for the $\alpha$–decomposition of $P$.

## Algorithmic approach and the Gröbner basis

Summarising, starting with $P[\boldsymbol{x}]$, the problem of obtaining a factoring (7) which is the decomposition or a suitable $\alpha$–decomposition boils down to testing the condition $1 \in \langle P_i[\boldsymbol{x}] \mid i \in I\rangle$ for various subsets $I \subseteq L$. This can be done using the Buchberger algorithm which computes a canonical basis (for a given ordering of monomials) of the ideal $\langle P_i[\boldsymbol{x}] \mid i \in I\rangle$, a so called reduced Gröbner basis [2]. If $1 \in \langle P_i[\boldsymbol{x}] \mid i \in I\rangle$, this basis has to be $\{1\}$.

In practice for reasonable examples this algorithm may be implemented in, for example, Maple. Actually, one can save some computation and moreover obtain the explicit form of the operators $Q_J$ from (2) or $Q_{J,j}$ from (4) by using Buchberger's algorithm without seeking the reduced basis. Consider the ideal $I := \langle G\rangle \subseteq \mathbb{F}[\boldsymbol{x}]$ (for a set of polynomials $G$) such that $1 \in I$, and a Gröbner basis $G'$ of $I$. Then $\alpha \in G'$ for some scalar $\alpha \in \mathbb{F}$. The Buchberger algorithm starts with $G$ and builds $G' \supseteq G$ by adding various linear combinations (with coefficients in $\mathbb{F}[\boldsymbol{x}]$) of elements from $G$. So when $\alpha$ is added (and the algorithm stops), it has the required form which expresses 1 as a linear combination of elements of $G$ (up to a scalar multiple $\alpha$).

## Example

We shall demonstrate the previous observations on the operator

$$P[\mathcal{D}_1,\mathcal{D}_2] := \mathcal{D}_1^5\mathcal{D}_2 + \mathcal{D}_1^4\mathcal{D}_2^2 + 3\mathcal{D}_1^4\mathcal{D}_2 + \mathcal{D}^4 + 3\mathcal{D}_1^3\mathcal{D}_2^2 + 3\mathcal{D}_1^3\mathcal{D}_2 \\ + 2\mathcal{D}_1^3 + 3\mathcal{D}_1^2\mathcal{D}_2^2 + \mathcal{D}_1^2\mathcal{D}_2 + \mathcal{D}_1\mathcal{D}_2^2 - \mathcal{D}_1\,.$$

This correspond (after factoring) to the polynomial

$$P[x,y] = (x+1)(xy+y+1)x(x^2+xy+x+y-1)\,. \tag{9}$$

For example, taking $\mathcal{D}_1 = \frac{\partial}{\partial x}$ and $\mathcal{D}_2 = \frac{\partial}{\partial y}$ the differential operator $P[\frac{\partial}{\partial x},\frac{\partial}{\partial y}] : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ is a the sixth order differential operator. We apply the previous observation and Theorem 2.1 to reduce the corresponding differential equation to a lower order problem. In general, we start with the equation $P[\mathcal{D}_1,\mathcal{D}_2]u = f$ for a given $f \in \mathcal{V}$.

First we shall find the optimal (in the sense as above) dual $\beta$–decomposition, $\beta \subseteq 2^{\{0,1,2,3\}}$ and/or whether we can obtain the dual decomposition after an appropriate regrouping of the factors in (9). Many steps can be done directly in Maple. The factors

$$P_0[x,y] = x+1\,,\quad P_1[x,y] = xy+y+1\,,\quad P_2[x,y] = x\,, P_3[x,y] = x^2+xy+x+y-1$$

are irreducible; this can be verified by the command `IsPrime`. Using `gbasis` we see that

$$\begin{aligned} &1 \in \langle P_0[x,y], P_i[x,y]\rangle\,, && i \in \{1,2,3\}\,,\\ &1 \notin \langle P_i[x,y], P_j[x,y]\rangle\,, && i,j \in \{1,2,3\} \quad \text{and}\\ &1 \in \langle P_1[x,y], P_2[x,y], P_3[x,y]\rangle\,. \end{aligned} \tag{10}$$

From the first two lines in (10) and using the observation around Lemma 3.4 we conclude that $P[x,y] = P_0[x,y]\tilde{P}[x,y]$ is the decomposition for $\tilde{P}[x,y] = (xy+y+1)x(x^2+xy+x+y-1)$. Following the first line in (10), one easily computes

$$1 = -yP_0[x,y] + P_1[x,y]\,,\qquad 1 = -(x-1)P_0[x,y] + xP_2[x,y]\,,$$
$$1 = (x+y)P_0[x,y] - P_3[x,y]$$

and multiplying these three relations we obtain

$$1 = \tilde{Q}[x,y]P_0[x,y] + Q_0[x,y]\,\underbrace{P_1[x,y]P_2[x,y]P_3[x,y]}_{\tilde{P}[x,y]}$$

together with explicit form of the projectors $\tilde{Q}$ and $Q_0$. Passing to the corresponding operators on the space $\mathcal{V}$, this is the decomposition (2) of $P[\mathcal{D}_1,\mathcal{D}_2] = P_0[\mathcal{D}_1,\mathcal{D}_2]\,\tilde{P}[\mathcal{D}_1,\mathcal{D}_2]$. Now using Theorem 2.1 (and Remark 2.2) we see that every solution $u \in \mathcal{V}$ of $P[\mathcal{D}_1,\mathcal{D}_2]u = f$ can be uniquely expressed as

$$u = \tilde{Q}[\mathcal{D}_1,\mathcal{D}_2]u_0 + Q_0[\mathcal{D}_1,\mathcal{D}_2]\tilde{u}$$

where $u_0$ and $\tilde{u}$ satisfy $P_0[\mathcal{D}_1,\mathcal{D}_2]u_0 = f$ and $\tilde{P}[\mathcal{D}_1,\mathcal{D}_2]\tilde{u} = f$. So we have reduced the original problem $P[\mathcal{D}_1,\mathcal{D}_2]u = f$ to the system of latter two equations.

Using the last line in (10), we can apply Theorem 2.1 to the equation $\tilde{P}[\mathcal{D}_1,\mathcal{D}_2]\tilde{u} = f$. It is easy to compute the corresponding dual $\beta$–decomposition for the operator $\tilde{P}[\mathcal{D}_1,\mathcal{D}_2] = P_1[\mathcal{D}_1,\mathcal{D}_2]P_2[\mathcal{D}_1,\mathcal{D}_2]P_3[\mathcal{D}_1,\mathcal{D}_2]$, $\beta = \{\{1,2,3\}\}$; on the polynomial level we obtain

$$1 = \frac{1}{2}P_1[x,y] + \frac{1}{2}(x+1)P_2[x,y] - \frac{1}{2}P_3[x,y]\,.$$

This is actually also the $\alpha$–decomposition, $\alpha = \{\{1,2\},\{2,3\},\{1,3\}\}$ as $P_1[\mathcal{D}_1,\mathcal{D}_2] = P^{\{2,3\}}[\mathcal{D}_1,\mathcal{D}_2]$ etc. according to the notation in (2). Now applying Theorem 2.1 we obtain that every solution $\tilde{u} \in \mathcal{V}$ of $\tilde{P}[\mathcal{D}_1,\mathcal{D}_2]\tilde{u} = f$ has the form

$$\tilde{u} = \frac{1}{2}\tilde{u}_1 + \frac{1}{2}(\mathcal{D}_1+1)\tilde{u}_2 - \frac{1}{2}\tilde{u}_3$$

where $\tilde{u}_1$, $\tilde{u}_2$ and $\tilde{u}_3$ satisfy $P_2[\mathcal{D}_1,\mathcal{D}_2]\,P_3[\mathcal{D}_1,\mathcal{D}_2]\,\tilde{u}_1 = f$, $P_1[\mathcal{D}_1,\mathcal{D}_2]\,P_3[\mathcal{D}_1,\mathcal{D}_2]\,\tilde{u}_2 = f$ and $P_1[\mathcal{D}_1,\mathcal{D}_2]\,P_2[\mathcal{D}_1,\mathcal{D}_2]\,\tilde{u}_3 = f$. Note that such expression for $\tilde{u}$ is not generally unique.

Summarising, we have reduced the original equation $P[\mathcal{D}_1,\mathcal{D}_2]u = f$, see (9), for $\mathcal{D}_1,\mathcal{D}_2 \in \mathrm{End}(\mathcal{V})$ to the system of four equations

$$\begin{aligned} &P_i[\mathcal{D}_1,\mathcal{D}_2]P_j[\mathcal{D}_1,\mathcal{D}_2]\tilde{u}_k = f \quad \text{where} \quad \{i,j,k\} = \{1,2,3\}\,,\ i<j\,,\\ &P_0[\mathcal{D}_1,\mathcal{D}_2]u_0 = f\,. \end{aligned} \tag{11}$$

If we put $\mathcal{D}_1 := \frac{\partial}{\partial x}$ and $\mathcal{D}_2 := \frac{\partial}{\partial y}$, the original problem has order 6 and the resulting system (11) has order 3.

## 4. Systems of polynomial equations

The notion of decompositions from [7] summarised in Section 2 can be applied also to systems of equations of the form (1) with commuting $P_i$. Here we describe one possible type of such a system to demonstrate power of this machinery. We will consider only the (true) decompositions.

Let us consider a $k$–tuple of commuting linear operators $P^{(i)} \in \mathrm{End}\,\mathcal{V}$ and corresponding equations

$$P^{(i)}u = f^i\,, \quad [P^{(i)},P^{(j)}] = 0\,, \quad f^i \in \mathcal{V}\,, \quad 1 \le i,j \le k\,. \tag{12}$$

The necessary (i.e. integrability) condition for existence of a solution $u$ is obviously

$$P^{(i)}f^j = P^{(j)}f^i\,, \quad 1 \le i < j \le k\,. \tag{13}$$

If, for some $i$, $P^{(i)}$ is of the form (1) satisfying (2), we can replace $P^{(i)}u = f^{(i)}$ with several simpler equations using Theorem 2.1. But even if this is not the case, one can obtain a decomposition using an algebraic relation between the $P^{(i)}$s.

Let us consider the special case where just one equation from (12) is of the form (1) and we do not decompose the remaining ones, i.e. we have the system

$$\begin{aligned} &Pu = P_0P_1\cdots P_\ell u = f\,, \quad [P_i,P_j] = 0\,, \quad 1 \le i,j \le \ell\,,\\ &R^{(j)}u = g^j\,, \quad [P_i,R^{(j)}] = 0\,, \quad 1 \le j \le k, 1 \le i \le \ell \end{aligned} \tag{14}$$

where $P_i$ and $R^{(j)}$ satisfy the identity

$$\begin{aligned}
&\mathrm{id}_V = Q_0P^0 + \cdots + Q_\ell P^\ell + S_1R^{(1)} + \cdots + S_kR^{(k)}\,,\\
&[Q_i, P_j] = [S_p, R^{(q)}] = 0\,, \quad 1 \le i, j \le \ell\,, \quad 1 \le p, q \le k\,,\\
&[Q_i, R^{(p)}] = [S_p, P_i] = 0\,, \quad 1 \le i \le \ell,\ 1 \le p \le k
\end{aligned} \tag{15}$$

where $P^i := \Pi_{i\neq j=0}^{j=\ell} P_i, i = 0, \dots, \ell$. (Note in (14) we require not only commutativity of the left hand sides of the equations in the systems as in (12) but also commutativity of the factors $P_i$ and the left hand sides $R^{(j)}$.) The condition (13) then becomes

$$R^{(j)}f = Pg^j\,, \quad R^{(j)}g^i = R^{(i)}g^j \quad \text{for all} \quad 1 \le i, j \le k\,. \tag{16}$$

**Proposition 4.1** (Dual decomposition)**.** *Let us consider the system* (14)*. Then* (15) *is equivalent to*

$$\mathrm{id}_{\mathcal{V}} = Q_{i,j}P_i + Q_{j,i}P_j + Q^1_{i,j}R^{(1)} + \cdots + Q^k_{i,j}R^{(k)} \tag{17}$$

*where* $Q_{i,j}, Q^p_{i,j} \in \mathrm{End}(\mathcal{V})$ *satisfy* $[Q_{i,j}, P_s] = [Q_{i,j}R^{(p)}] = [Q^p_{i,j}, R^{(q)}] = [Q^p_{i,j}, P_s] = 0$ *where* $0 \le i, j, s \le \ell$, $i \neq j$ *and* $1 \le p, q \le k$.

**Proof.** This is just a straightforward modification of the proof of Proposition 2.4. □

In this setting, we obtain an analogue of Theorem 2.1 for the special case of decompositions. In this Theorem, we replaced the operator $P$ given by (1) satisfying (2) with the system (3) of $\ell+1$ simpler equations. Here we replace the system (14) with a "system of simpler systems" as follows.

**Theorem 4.2.** *Let* $\mathcal{V}$ *be a vector space over a field* $\mathbb{F}$ *and consider* $P, R^{(j)} \colon \mathcal{V} \to \mathcal{V}$ *as in* (14) *with the factorisation giving the decomposition* (15)*. Here and below we assume the range* $1 \le j \le k$*. Let us fix* $f, g^j \in \mathcal{V}$*. There is a 1-1 relationship between solutions* $u \in \mathcal{V}$ *of* (14) *and solutions* $(u_0, \dots, u_\ell) \in \oplus^{\ell+1}\mathcal{V}$ *of the problem*

$$\begin{gathered}
P_0u_0 = f, R^{(1)}u_0 = P^0g^1, \dots, R^{(k)}u_0 = P^0g^k\\
\vdots\\
P_\ell u_\ell = f, R^{(1)}u_\ell = P^\ell g^1, \dots, R^{(k)}u_\ell = P^\ell g^k\,.
\end{gathered} \tag{18}$$

*Writing* $\mathcal{V}_P^{f,\mathbf{g}}$ *for the solution space of* (14) *and (for* $i = 0, \dots, \ell$*)* $\mathcal{V}_i^{f,\mathbf{g}}$ *for the solution space of the system corresponding to the* $i$*th line in* (18)*, where* $\mathbf{g} = (g^1, \dots, g^k)^T$*. The map* $F \colon \mathcal{V}_P^{f,\mathbf{g}} \to \times_{i=0}^{\ell}\mathcal{V}_i^{f,\mathbf{g}}$ *is given by*

$$u \mapsto (P^0u, \dots, P^\ell u)\,,$$

*with inverse* $B \colon \times_{i=0}^{\ell}\mathcal{V}_i^{f,\mathbf{g}} \to \mathcal{V}_P^{f,\mathbf{g}}$ *given by*

$$(u_0, \dots, u_\ell) \mapsto \sum_{i=0}^{i=\ell} Q_iu_i + \sum_{j=1}^{j=k} S_jg^j\,.$$

*On $\mathcal{V}$ we have $B\circ F = \mathrm{id}_{\mathcal{V}_P^{f,\mathbf{g}}}$, while on the affine space $\times_{i=0}^{\ell}\mathcal{V}_i^{f,\mathbf{g}}$ we have $F\circ B = \mathrm{id}_{\times_{i=0}^{\ell}\mathcal{V}_i^{f,\mathbf{g}}}$.*

**Proof.** Suppose $u$ is a solution of (14). Then $P_iP^iu = Pu = f$ and also $R^{(j)}P^iu = P^i(R^{(j)}u) = P^ig^j$. Hence $Fu$ is a solution of (18). For the converse suppose that $(u_0,\dots,u_\ell)$ is a solution of (18) and write $u := \sum_{i=0}^{i=\ell} Q_iu_i + \sum_{j=1}^{j=k} S_jg^j$. Then

$$Pu = \sum_{i=0}^{i=\ell} PQ_iu_i + \sum_{j=1}^{j=k} PS_jg^j = \sum_{i=0}^{i=\ell} Q_iP^i(P_iu_i) + \sum_{j=1}^{j=k} S_jR^{(j)}f = f$$

where we have used $Pg^j = R^{(j)}f$ from (16) and then (15). Further

$$\begin{aligned} R^{(j)}u &= \Big(\sum_{i=0}^{i=\ell} R^{(j)}Q_iu_i\Big) + R^{(j)}S_jg^j + \Big(\sum_{j\neq p=1}^{p=k} R^{(j)}S_pg^p\Big) \\ &= \Big(\sum_{i=0}^{i=\ell} Q_iP^ig^j\Big) + R^{(j)}S_jg^j + \Big(\sum_{j\neq p=1}^{p=k} S_pR^{(p)}g^j\Big) = g^j \end{aligned}$$

where we have used $R^{(j)}u_i = P^ig^j$ from (18) and $R^{(j)}g^p = R^{(p)}g^j$ from (16) in the middle equality and (15) in the last one.

It remains to show that $F$ and $B$ are inverses. Clearly $u\in\mathcal{V}_P^{f,\mathbf{g}}$ satisfies

$$(B\circ F)u = \sum_{i=0}^{i=\ell} Q_iP^iu + \sum_{j=1}^{j=k} S_jg^j = \sum_{i=0}^{i=\ell} Q_iP^iu + \sum_{j=1}^{j=k} S_jR^{(j)}u = u$$

since $g^j = R^{(j)}u$ according to (14). Thus we obtain $B\circ F = \mathrm{id}_{\mathcal{V}_P^{f,\mathbf{g}}}$. To compute the opposite direction we need the $r^{\mathrm{th}}$ component of $FB(u_0,\dots,u_\ell)$ for $(u_0,\dots,u_\ell)\in \times_{i=0}^{\ell}\mathcal{V}_i^{f,\mathbf{g}}$. This is equal to

$$P^r\sum_{i=0}^{i=\ell} Q_iu_i + P^r\sum_{j=1}^{j=k} S_jg^j = \sum_{r\neq i=0}^{i=\ell} Q_iP^ru_i + Q_rP^ru_r + \sum_{j=1}^{j=k} S_jR^{(j)}u_r = u_r\,.$$

since $P^ru_i = P^iu_r$ for $r\neq i$ (the consistency condition given by $P_iu_i = f$ for $i=0,\dots,\ell$) and $P^rg^j = R^{(j)}u_r$ from (18). Hence $F\circ B = \mathrm{id}_{\times_{i=0}^{\ell}\mathcal{V}_i^{f,\mathbf{g}}}$. □

## 5. Higher symmetries of operators

For a vector space $\mathcal{V}$ and a linear operator $P\colon\mathcal{V}\to\mathcal{V}$, let is say that a linear operator $S\colon\mathcal{V}\to\mathcal{V}$ is a *formal symmetry* of $P$ if $PS = S'P$, for some other linear operator $S'\colon\mathcal{V}\to\mathcal{V}$. Note that $S\colon\mathcal{N}(P)\to\mathcal{N}(P)$. In [7] we called operators with the latter property "weak symmetries" and discussed the structure of the algebra of these in relation to symmetries and related maps for the component operators $P_i$. We show here that although formal symmetries are defined rather differently similar results hold using our general tools as discussed above and in [7]. For the case of $P$ a differential operator the formal symmetries agree with the "higher symmetries" considered in [3] and we thank Mike Eastwood for asking whether the ideas from [7] might be adapted to deal directly with what we are here calling formal symmetries.

Consider the case of an operator $P = P_0P_1 \cdots P_\ell$ with a decomposition

$$\mathrm{id}_{\mathcal{V}} = Q_0P^0 + \cdots + Q_\ell P^\ell \,, \tag{19}$$

i.e. (2) with $\alpha := \{J \subseteq L \mid |J| = 1\}$. Then as commented in Remark 2.2, upon restriction to $\mathcal{N}(P)$, the operators

$$Pr_i := Q_iP^i, \quad i = 0, 1, \ldots, \ell$$

are projections onto $\mathcal{N}(P_i)$. This was the crucial fact used to discuss weak symmetries and their decompositions in [7]. Here we see that it plays a similar for formal symmetries.

Now note that if $S$ is a formal symmetry of $P$ then $Pr_iSPr_i$ is a formal symmetry of $P_i$. More generally, using the assumed commutativity as in (2), we have

$$P_i(Pr_iSPr_j) = Q_i(PS)Pr_j = Q_i(S'P)Pr_j = (Q_iS'Pr_jP^j)P_j$$

so $S_{ij} := Pr_iSPr_j$ linearly maps $\mathcal{N}(P_j) \to \mathcal{N}(P_i)$. (In fact $P^iS$ would suffice (see the remark below), we use $Pr_iSPr_j$ for the link with [7].) But we may view the property $P_iS_{ij} = S'_{ij}P_j$ (with $S'_{ij}$ any linear endomorphism of $\mathcal{V}$) as a generalisation of the idea of a formal symmetry. If we have such a *generalised formal symmetry* $S_{ij}$ for all pairs $i, j \in \{0, 1, \ldots, \ell\}$ then note that for each pair $i, j$ we have

$$PS_{ij}Pr_j = P^iP_iS_{ij}Pr_j = P^iS'_{ij}P_jPr_j = (P^iS'_{ij}Q_j)P;$$

$S_{ij}Pr_j$ (and hence also $Pr_iS_{ij}Pr_j$) is a formal symmetry of $P$. Thus the decomposition of the identity (19) allows us to understand formal symmetries of $P$ in terms of the generalised formal symmetries of the component operators $P_i$, $i = 0, 1, \ldots, \ell$.

**Remark 5.1.** This result for formal symmetries follows the Theorem 4.1 in [7] where weak symmetries are treated. The decomposition of the identity (19) plays the crucial role in this theorem. Since, upon restriction to $\mathcal{N}(P)$, the $Pr_i$ are projections, the formulae above have a straightforward conceptual interpretation. However there is, in fact, an even simpler relationship between formal symmetries $S$ of $P$ and generalised formal symmetries $S_{ij}$, $i, j \in \{0, \ldots, \ell\}$. We simply put $S_{ij} := P^iS|_{\mathcal{N}(P_j)}$ for a formal symmetry $S$ and $S := S_{ij}P^j$ for a generalized formal symmetry $S_{ij}$.

For example, using a factorization from [6] (or [5]), this observation enables a treatment of the higher symmetries of the conformal Laplacian operators of [8] on conformally Einstein manifolds. In particular our approach to the higher symmetries of the Paneitz operator is an alternative to that in [4]. (In fact in [4] they consider only the square of the Laplacian on Euclidean space but by conformal invariance this may alternatively treated via the Paneitz operator on the sphere.) This will be taken up elsewhere.

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Department of Mathematics, The University of Auckland
Private Bag 92019, Auckland 1, New Zealand
*E-mail*: `gover@math.auckland.ac.nz`

Eduard Čech Center
Institut of Mathematics and Statistics, Masaryk University
Janáčkovo nám. 2a, 602 00, Brno, Czech Republic
*E-email*: `silhan@math.muni.cz`