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Universality of separoids

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UNIVERSALITY OF SEPAROIDS

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Abstract. A *separoid* is a symmetric relation $\dagger \subset \binom{2^S}{2}$ defined on disjoint pairs of subsets of a given set S such that it is closed as a filter in the canonical partial order induced by the inclusion (i.e., $A \dagger B \preceq A' \dagger B' \iff A \subseteq A'$ and $B \subseteq B'$). We introduce the notion of *homomorphism* as a map which preserve the so-called "minimal Radon partitions" and show that separoids, endowed with these maps, admits an embedding from the category of all finite graphs. This proves that separoids constitute a *countable universal partial order*. Furthermore, by embedding also all hypergraphs (all set systems) into such a category, we prove a "stronger" universality property.

We further study some structural aspects of the category of separoids. We completely solve the *density* problem for (all) separoids as well as for separoids of points. We also generalise the classic Radon's theorem in a categorical setting as well as Hedetniemi's product conjecture (which can be proved for oriented matroids).

1. Preliminaries

In order to be self-contained, we start with some basic definitions and examples.

1.1. **The objects.** A *separoid* [1, 3, 8, 9, 10, 14, 15, 16, 17] is a (finite) set S endowed with a symmetric relation $\dagger \subset \binom{2^S}{2}$ defined on its family of subsets with the following properties: if $A, B \subseteq S$ then

- • $$A \dagger B \implies A \cap B = \emptyset\,,$$
- •• $$A \dagger B \quad \text{and} \quad B \subset B' \subseteq S \setminus A \implies A \dagger B'\,.$$

A related pair $A \dagger B$ is called a *Radon partition* (and we often say "A is not separated from B"). Each part, A and B, is known as a (*Radon*) *component* and their union $A \cup B$ is called the *support* of the partition. The *order* of the separoid is the cardinal $|S|$, and its *size* the cardinal $|\dagger|$, the number of Radon partitions. Due to the second condition, if the separoid is finite, then *minimal* Radon partitions determines the separoid. The separoid is sometimes denoted as the pair $(S, \dagger)$.

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A pair of disjoint sets $\alpha, \beta \subseteq S$ that are not a Radon partition, are said to be *separated* and denoted $\alpha \mid \beta$. The separation relation is a symmetric, *quasi-antireflexive* and *ideal* relation; that is, it satisfies for $\alpha, \beta \subseteq S$:

$$\circ \qquad \alpha \mid \alpha \implies \alpha = \emptyset\,,$$
$$\circ\circ \quad \alpha \mid \beta \quad \text{and} \quad \beta' \subset \beta \implies \alpha \mid \beta'\,.$$

The separoid is sometimes denoted as the pair $(S, |)$.

Clearly † and | determine each other; they are related by the following equivalence

$$A \dagger B \iff A \nmid B \quad \text{and} \quad A \cap B = \phi\,.$$

We say that the separoid S is *acyclic* if $\emptyset \mid S$.

Examples:

1. Consider a (non-empty) subset $X \subseteq \mathbb{E}^d$ of the d-dimensional Euclidean space and define the following relation

$$A \dagger B \iff \langle A \rangle \cap \langle B \rangle \neq \phi \quad \text{and} \quad A \cap B = \phi,$$

where $\langle A \rangle$ denotes de convex hull of A. The pair $P = (X, \dagger)$ is a separoid and will be called a *point separoid*. Indeed, the notion of separoid arises as an abstraction of the well-known Radon's lemma [13]: *if the set $X \subset \mathbb{E}^d$ consists of at least $d+2$ points, then there exist two disjoint subsets of it that are not separated* (*i.e., their convex hulls do intersect*). The class of all point separoids is denoted by $\mathcal{P}$.

2. Generalising the previous example, consider a family F of convex sets in $\mathbb{E}^d$ — instead of points — and define the separoid $S(F)$ as above; that is, two subsets of the family $A, B \subseteq F$ are separated $A \mid B$ if there exists a hyperplane that leaves all members of A on one side of it and those of B on the other. Clearly, $S(F) = (F, |\,)$ is a separoid. Indeed, all separoids can be represented in this way; i.e., given a separoids S, there exists a family of convex sets F such that S is isomorphic to $S(F)$; furthermore, if the separoid is acyclic, then it can be represented in the $(|S|-1)$-dimensional space (see [16, 17]). The class of all separoids is denoted by $\mathcal{S}$.

3. Consider a (simple) graph $G = (V, E)$ and define two elements $u, v \in V$ of the vertex set to be a minimal Radon partition $u \dagger v$ if and only if the pair is an edge $uv \in E$. Then $S(G) = (V, \dagger)$ is also a separoid. In fact, as we shall see on Theorem 5, this correspondence is an embedding of the category of graphs into the category of separoids, when both classes are endowed with *homomorphisms*. The class of all graphs is denoted by $\mathcal{G}$.

4. Consider an oriented matroid $M = (E, C)$ defined in terms of its circuits $C \subset \{-, 0, +\}^E$, in the usual way (see [2]). Define the following relation $\dagger \subset \binom{2^E}{2}$ on the subsets of E: $A \dagger B$ is a minimal Radon partition if and only if there exist a circuit $X \in C$ such that

$$A = X^+ := \{e \in E : X_e = +\} \quad \text{and} \quad B = X^- := \{e \in E : X_e = -\}\,.$$

Clearly $S(M) = (E, \dagger)$ is a separoid. Furthermore, the separoid is acyclic if and only if the oriented matroid is acyclic. The class of all oriented matroids is denoted by $\mathcal{M}$.

5. All acyclic separoids of order 3 come from one of the eight families of convex bodies depicted in Figure 1. Those labelled **a**, **b**, **e** and **h** are point separoids.

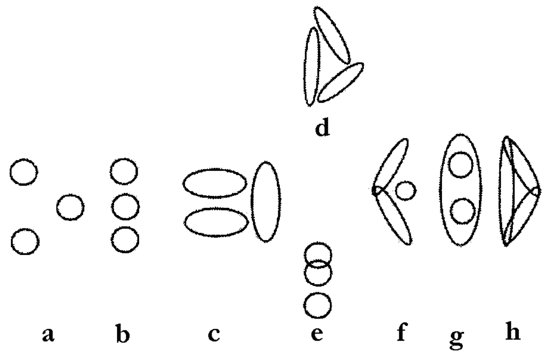


Figure 1. *The acyclic separoids of order 3.*

Example 1 suggest the following definitions. The *dimension* $\mathrm{d}(S)$ of a separoid is the minimum number d such that every subset with $d+2$ elements is the support of a Radon partition. Equivalently, the dimension of a separoid is the maximum d such that there exists a subset σ with $d+1$ elements such that every subset of it is separated from its relative complement:

$$(\star) \qquad \alpha \subset \sigma \implies \alpha \mid (\sigma \setminus \alpha)\,.$$

A separoid σ with property $(\star)$ is called a *simploid* because it can be represented with the vertex set of a simplex (Figure 1.**a** represents the simploid of dimension 2). We say that a separoid S is *in general position* if no subset with $\mathrm{d}(S)+1$ elements is the support of a Radon partition; i.e., if every subset with $\mathrm{d}(S)+1$ elements induces a simploid.

We say that a separoid is a *Radon separoid* if every minimal Radon partitions is unique in its support; that is, if $A \dagger B$ and $C \dagger D$ are minimal, then

$$A \cup B = C \cup D \implies \{A, B\} = \{C, D\}\,.$$

The class of Radon separoids is denoted by $\mathcal{R}$.

We say that the separoid S is a *Steinitz separoid* if it satisfies the Steinitz exchange axiom; namely, if $A \dagger B$ is a Radon partition whose support consists of $\mathrm{d}(S)+2$ elements, then

$$\forall x \notin A \cup B \quad \exists y \in A \cup B : (A \setminus y) \dagger (B \setminus y \cup x)\,.$$

The class of Steinitz separoids is denoted by $\mathcal{Z}$.

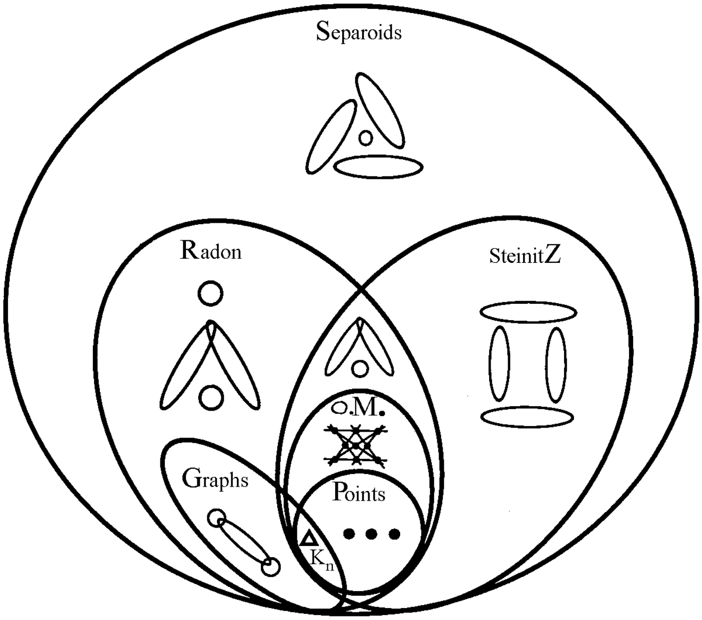


Figure 2. *Some classes of separoids.*

An *oriented matroid* is a Radon separoid whose minimal Radon partitions satisfies the week elimination axiom: if $A_i \dagger B_i$, for $i = 1, 2$, are minimal Radon partitions for which there exists an $x \in B_1 \cap A_2$, then there exists a minimal Radon partition $C \dagger D$ such that $C \subseteq \bigcup A_i$, $D \subseteq \bigcup B_i$ and $x \notin C \cup D$. As observed by Las Vergnas [7], oriented matroids are Steinitz separoids; that is, $\mathcal{M} \subset \mathcal{R} \cap \mathcal{Z}$ (see Figure 2).

Finally, we say that a separoid is a *graph* (recall Example 3), if for every minimal Radon partition $A \dagger B$ we have that $|A||B| = 1$; that is, the minimal Radon partitions are pairs of singletons. Observe that $\mathcal{G} \subset \mathcal{R}$.

1.2. **The homomorphisms.** Since it is enough to know minimal Radon partitions to reconstruct a finite separoid, we can concentrate on the study of them. In particular — when defining an operation (see below) — it is enough to define some partitions and generate the separoid as the minimal symmetric filter containing the given set as Radon partitions. That is, we can define a separoid S by defining a set of generators of the symmetric filter $(S, \dagger, \preceq)$, where $A \dagger B \preceq C \dagger D$ if $A \subseteq C$ and $B \subseteq D$.

Let S and T be two separoids. A mapping from S to T is called a *homomorphism* if the image of minimal Radon partitions are minimal Radon partitions. That is, a homomorphism is a mapping $\varphi : S \to T$ that satisfy for all $A, B \subseteq S$,

$$A \dagger B \text{ minimal } \implies \varphi(A) \dagger \varphi(B) \text{ minimal}$$

(as usual, we put $\varphi(A) = \{\varphi(x); x \in A\}$).

Two separoids are *isomorphic* $S \approx T$ if there is a bijective homomorphism between them whose inverse function is also a homomorphism. If $S \subset T$ is a subset of a separoid, the *induced* separoid $T[S]$ is the restriction to S. An *embedding* $S \hookrightarrow T$ is an injective homomorphism that is an isomorphism between the domain and the induced separoid of its image.

From now on, we will denote by $S \longrightarrow T$ the fact that there exists an homomorphism from the separoid S to the separoid T, and by $S \not\longrightarrow T$ the other case. Also, if $S \rightleftarrows T$ then we write $S \sim T$. This last defines an equivalence relation and, in its equivalence classes, a partially ordered class called the *homomorphisms order* (see [4]):

$$S \leq T \iff S \longrightarrow T\,.$$

The homomorphisms order is in fact a lattice. This is useful as separoids generalise several structures (e.g. oriented matroids) where the categorical notions are hard to define (see [6]). It is a pleasing fact that the category of separoids homomorphisms has products $\times$ and sums $+$ and they play the role of the meet (infimum) and the joint (supremum), respectively.

Given two separoids P and T, their *product* is the separoid defined on the Cartesian product $P \times T$, with projections π and τ respectively, generated by the set

$$\left\{ A \dagger B \in \binom{2^{P\times T}}{2} : \pi(A) \dagger \pi(B) \text{ and } \tau(A) \dagger \tau(B) \text{ are minimal} \right\}.$$

Given two separoids P and T, their *sum* is the separoid defined on the disjoint union $P \cup T$ generated by the set

$$\left\{ A \dagger B \in \binom{2^{P+T}}{2} : A \dagger B \text{ is minimal in } P \text{ or in } T \right\}.$$

Example 6. Consider the point separoid depicted in Figure 1.**b** and denote its elements by $P = \{-, 0, +\}$, where $\{0\} \dagger \{-+\}$. The product $P^2 = P \times P$ is a separoid of order 9 and dimension 6. P^2 is also the sum $S + T$ of two separoid: S, of order 5 and dimension 3, with the two minimal Radon partitions $\{00\} \dagger \{--, ++\}$ and $\{00\} \dagger \{-+, +-\}$; and T, of order 4 and dimension 2, with the unique Radon partition $\{0-, 0+\} \dagger \{-0, +0\}$. $P^2 = S + T$ is depicted in

Figure 3. Observe that P^2 is not a point separoid (nor an oriented matroid).

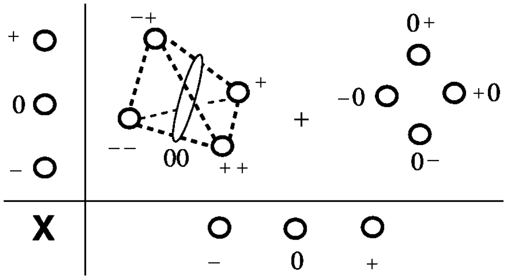


Figure 3. *An example where $P \in \mathcal{P}$ but $P^2 \notin \mathcal{P}$.*

It is easy to see that these constructions have the expected categorical properties:

Lemma 1.

1. $S \longrightarrow P \times T \iff S \longrightarrow P$ *and* $S \longrightarrow T$,
2. $P + T \longrightarrow S \iff P \longrightarrow S$ *and* $T \longrightarrow S$. □

1.3. **A comment on Radon's lemma.** One can be tempted to study maps which preserve *all* Radon partitions — not only the minimal ones. We call such maps *strong morphisms*. In the category of separoids endowed with strong morphisms, Radon's lemma can be formulated as follows (cf. Lemma 6):

Theorem 1 (Radon 1921). *S is a point separoid of order $|S| = \mathrm{d}(S) + 2$ if and only if*

$$S \not\longrightarrow K_1 \quad \text{and} \quad S \longrightarrow K_2 + \sigma\,,$$

where σ is a simploid. Furthermore, in such a case, $\sigma = \emptyset$ if and only if S is in general position.

Proof. A separoid S is a point separoid of order $\mathrm{d}(S) + 2$ if and only if it is determined by a unique minimal Radon partition $A \dagger B$. Put $C = S \setminus (A \cup B)$ and let $C = \{c_0, \ldots, c_d\}$. Now, let $K_2 = \{a, b\}$, where $a \dagger b$, and $\sigma^d = \{c'_0, \ldots, c'_d\}$. Clearly the function $\varphi\colon S \to K_2 + \sigma^d$, where

$$\varphi(s) = \begin{cases} a & \text{if} \quad s \in A\,, \\ b & \text{if} \quad s \in B\,, \\ c'_i & \text{if} \quad s = c_i\,, \end{cases}$$

is a strong morphism of separoids. Conversely, if $S \longrightarrow K_2 + \sigma$, then the preimage of K_2 determines a unique minimal Radon partitions of S. Furthermore, if this is the case, S is in general position if and only if $A \cup B = S$. □

However the category of separoids' strong morphisms seems to be less rich than that of homomorphisms and too restrictive to our purposes. In particular, such a category does not have a nice product. For this, consider the separoids $P_3 = \{0, 1, 2\}$ where $0 \dagger 12$, and $K_2 = \{a, b\}$ where $a \dagger b$. Let us denote by $P_3 \times K_2 = \{0a, 0b, 1a, 1b, 2a, 2b\}$ the elements of the product and by π and κ the two projections. If $A \dagger B$ implies that $\pi(A) \dagger \pi(B)$ and $\kappa(A) \dagger \kappa(B)$ then the natural candidates to A and B are $A = \{0a\}$, $B = \{1b, 2b\}$. However this would imply that $A \dagger B \cup \{0b\}$ but,

$$\pi(0a) \cap \pi(0b, 1b, 2b) = \{0\} \cap \{0, 1, 2\} = \{0\} \neq \phi\,.$$

Therefore $P_3 \times K_2$ should have size 0. In other words, while working with strong morphisms, the product of these separoids is equivalent to the simploid. This collapse seem to be occurring too often to make the concept of strong morphism interesting.

1.4. **Basic properties.** In this section we review some very basic facts about the homomorphism order and some of its invariants. In the sequel, we will denote by K_k the (acyclic) complete separoid; i.e., K_k is the separoid which, for all $i, j \in K_k$ we have that $i \dagger j$ — clearly, K_k is the complete graph, hence the notation. Recall also that the *d-dimensional simploid* is the separoid S of order $|S| = d + 1$ and size 0. Simploids play the roll of *independent sets* and are usually denoted by σ. A straight-forward argument shows that

Lemma 2. *The following statements are equivalent:*

1. *S is a simploid.*
2. $|S| = \mathrm{d}(S) + 1$.
3. $S \longrightarrow K_1$.
4. $\forall T \neq \emptyset : S \longrightarrow T$. □

In the study of homomorphisms, it is useful to have simple conditions which forbids them. The following is similar to the *No-Homomorphism* lemma (cf. [5]).

Lemma 3. *Let T be a separoid in general position. If $S \not\longrightarrow K_1$ and $S \longrightarrow T$ then* $\mathrm{d}(S) \geq \mathrm{d}(T)$.

Proof. For the contrary, suppose that $\mathrm{d}(S) < \mathrm{d}(T)$ (and then $\mathrm{d}(S)+2 \leq \mathrm{d}(T)+1$). Since $S \not\longrightarrow K_1$, due to Lemma 2, the order of S is at least $\mathrm{d}(S) + 2$. Then there exists a minimal Radon partition $A \dagger B$ such that $|A \cup B| \leq \mathrm{d}(S) + 2$. If $\varphi \colon S \to T$ is a function, then $|\varphi(A \cup B)| \leq \mathrm{d}(T) + 1$. Since T is in general position, $T[\varphi(A \cup B)] \sim K_1$ and $\varphi(A)$ is separated from $\varphi(B)$. Therefore φ is not an homomorphism. □

This result has to be contrasted with the No-Homomorphism lemma for graphs. There the "general position" hypothesis is replaced by the "vertex transitive" one. However, as the vertices of the regular 3-cube shows, in this broader context these hypothesis cannot be interchanged — observe that graphs, unless complete or completely disconnected, are not in general position.

The following result plays the roll of the well-known Erdős inequality $|G| \leq \chi(G)\alpha(G)$.

Lemma 4. *If $S \longrightarrow T$ then $|S| < |T|(\mathrm{d}(S)+2)$.*

Proof. If $|S| \geq |T|(\mathrm{d}(S)+2)$, by the pigeon-hole principle, in any function $\varphi\colon S \to T$ there most be $\mathrm{d}(S)+2$ elements of S mapping into the same element of T and 'colapsing' a minimal Radon partition. □

Observe that for graphs, this is an immediate consequence of Erdős inequality. For, suppose there is a homomorphism of graphs $G \longrightarrow H$, and denote by $\alpha(G) = \mathrm{d}(G)+1$ the independence number of G. Then

$$\frac{|G|}{\alpha(G)+1} < \frac{|G|}{\alpha(G)} \leq \chi(G) \leq \chi(H) \leq |H|\,.$$

In the study of any partial order, it is natural to ask how does its *chains* (induced linear orders) and *atichains* (subsets of incomparable elements) looks like. We close this section describing the antichains of $\mathcal{S}$ — we will describe the chains in Section 3. As an immediate application of the previous two lemmas, we have that

Theorem 2. *The only maximal and finite antichains in the homomorphism order of separoids are $\{K_0\}$ and $\{K_1\}$. That is, given any other finite antichain*

$$\mathcal{A} = \{S_1, \dots, S_k : i \neq j \implies S_i \not\longrightarrow S_j\}\,,$$

there exists a (point) separoid $P \notin \mathcal{A}$ such that $\mathcal{A} \cup \{P\}$ remains an antichain.

Proof. Let $\mathcal{A} = \{S_1, \dots, S_k\}$ be a finite antichain. Let d be a number such that

$$(*) \qquad d > \max \mathrm{d}(S_i),$$

and let n be a number such that

$$(**) \qquad n > (d+2)\max|S_i|\,.$$

Now, let $P \subset \mathbb{E}^d$ be a d-dimensional separoid of n points in general position. Due to $(*)$ and Lemma 3, $S_i \not\longrightarrow P$; and due to $(**)$ and Lemma 4, $P \not\longrightarrow S_i$. Therefore, $\mathcal{A} \cup \{P\}$ is an antichain. □

2. On universality

Let $\mathcal{A}_\omega$ be the *countable antichain*; that is, the order type of a countable set of incomparable elements. Theorem 2 shows that $\mathcal{A}_\omega$ can be represented by $\mathcal{P}$. Let $\mathcal{C}_\omega$ be the *countable chain*; i.e, the usual order type of the natural numbers (sometimes simply denoted by ω). It is easy to see that $\mathcal{C}_\omega$ can also be represented by $\mathcal{P}$ (e.g., consider the chain $K_1 \longrightarrow K_2 \longrightarrow \dots$). It is well known (see e.g. [5]) that these two facts implies that any finite partial order can be represented by the class $\mathcal{P}$ of all point separoids.

Theorem 3. *The homomorphisms order of $\mathcal{P}$ is universal for finite orders.* □

As suggested by Example 3, we can embed the class $\mathcal{G}$ of (finite) graphs into the class $\mathcal{S}$ of separoids. Explicitly, let us define the functor $\Psi\colon \mathcal{G} \hookrightarrow \mathcal{S}$ as follows: for each graph $G = (V,E)$, let $\Psi(G) = (V, \dagger)$ be the separoid where $i \dagger j$ is a minimal Radon partition iff $ij \in E$; and, for each homomorphism of graphs $\varphi\colon G \to H = (V', E')$, let $\Psi(\varphi) = \varphi$. It is easy to check that Ψ is an embedding — and in the sequel we often identify $\mathcal{G}$ with $\Psi(\mathcal{G})$.

It is well known that the category of graphs endowed with homomorphisms is *universal for countable categories*; i.e., every countable category can be embedded into the category of graphs. This implies that

Theorem 4. *$\mathcal{S}$ is universal for countable categories.* □

Analogously, the quasi-order $(\mathcal{G}, \leq)$, where we put $G \leq H$ whenever $G \longrightarrow H$, is *universal*. As an immediate consequence of this fact, we have that

Theorem 5. *The homomorphisms order of $\mathcal{S}$ is universal for countable orders.* □

Now, let us denote by $\mathcal{C}^{\times}$ the closure in $\mathcal{S}$ of the class $\mathcal{C}$ under products, sums and induced substructures. For example, it is well-known that $\mathcal{G} = \mathcal{K}^{\times}$ (see [5]), where $\mathcal{K}$ denotes the class of complete graphs (see also Theorem 9 and Example 7). Analogous to Theorem 4 we have the following

Theorem 6. *$\mathcal{P}^{\times}$ is universal for countable categories.*

Proof. Since $\mathcal{P}$ contains all complete graphs, thus the class $\mathcal{P}^{\times}$ contains all graphs. □

The category of separoids contains other rich (and natural) categories. We present here another example: the category $\mathcal{H}$ of *hypergraphs* (set systems). Recall that a hypergraph $H = (V, E)$ is a set V endowed with a family of subsets $E \subseteq 2^V$, called edges, and that a homomorphism of hypergraphs $H \longrightarrow H'$, where $H' = (V', E')$, is any map $\varphi\colon V \to V'$ which sends edges to edges.

Theorem 7. *There exists a functoral embedding $\Phi\colon \mathcal{H} \hookrightarrow \mathcal{S}$ from the category of hypergraphs into that of separoids. Therefore, $\mathcal{S}$ is a universal partially ordered class; that is, given a partially ordered set $(\mathcal{X}, \leq)$, there exists a monotone embedding $\iota\colon \mathcal{X} \hookrightarrow \mathcal{S}$:*

$$x \leq y \iff \iota(x) \longrightarrow \iota(y)\,.$$

Proof. Let $\Phi\colon \mathcal{H} \hookrightarrow \mathcal{S}$ be the function which assigns to each hypergraph $H = (V, E)$ (simple and without isolated points) the separoid $S = (V \cup E, \dagger)$, whose minimal Radon partitions are defined by: $U \dagger e$ minimal in S if $U \subseteq V$, $e \in E$ and $U = e$. The function Φ is injective. Let $\varphi\colon V \to V'$ be a homomorphism of hypergraphs (the image of edges are edges) that sends the hypergraph $H = (V, E)$ to the hypergraph $H' = (V', E')$. The mapping φ induces a function on the edges which will be denoted again by $\varphi\colon E \to E'$ and therefore we have also a function

$$\Phi(\varphi)\colon V \cup E \to V' \cup E'\,.$$

To see that this function $\Phi(\varphi)$ is a separoid homomorphism $\Phi(H) \longrightarrow \Phi(H')$, observe that each minimal Radon partition $U \dagger e$ is mapped to a minimal Radon partition $\varphi(U) \dagger \varphi(e)$.

Conversely, let $\psi\colon V \cup E \to V' \cup E'$ be a separoid homomorphism $\Phi(H) \longrightarrow \Phi(H')$. First observe that $\psi(V) \subseteq V'$; for, let $v \in V$ be a vertex and let $v \in U = e \in E$ be an edge that contains it. Since $U \dagger e$ then $\psi(U) \dagger \psi(e)$ and therefore $\psi(v) \in \psi(U) \subseteq V'$. That is, ψ restricts in a function from $\varphi = \psi|_V \colon V \to V'$. Now, observe that such a restriction is an homomorphism; for, let $U = e \in E$ be an edge then $U \dagger e$ and therefore $\varphi(U) \dagger \varphi(e)$. This implies that $\varphi(e) \in E'$ and therefore φ defines an homomorphism of hypergraphs. Thus we proved that $\psi = \Phi(\varphi)$. □

Remark. Theorem 4 (as Theorem 6) prove that the category of finite separoids and all their homomorphisms is universal for countable categories. But there is a fine distinction here in the infinite: Theorem 4 gives a full embedding of the category of graphs and for infinite sets the universality of $\mathcal{G}$ depends on set-theoretical axioms. On the other hand, Theorem 7 above, when extended to infinite set systems (infinite hypergraphs), gives an absolute result: the category $\mathcal{H}_\infty$ of *all* hypergraphs is universal and thus also the category $\mathcal{S}_\infty$ of *all* separoids is universal — see [12] for the background.

3. On density

In this section we describe the chains of the homomorphisms order of separoids. We will show that the homomorphisms order of separoids is *dense*; i.e., between any two separoids, there is another one.

3.1. $\mathcal{S}$ **is dense.** Let us bring some of the categorical machinery developed in [11]. We will denote by $S \to\!=\!\not\to T$ the fact that, for every separoid P holds:

$$S \longrightarrow P \iff P \not\longrightarrow T\,.$$

We say that $S \to\!=\!\not\to T$ is a *duality pair* (e.g., $K_1 \to\!=\!\not\to K_0$ is a duality pair). Also, we say that $S \longrightarrow T$ is a *gap* if for all separoids P,

$$S \longrightarrow P \longrightarrow T \implies S \sim P \quad \text{or} \quad P \sim T$$

(e.g., $K_0 \longrightarrow K_1$ is a gap).

A separoid S is *connected* if it cannot be expressed as the sum of other two separoids; that is,

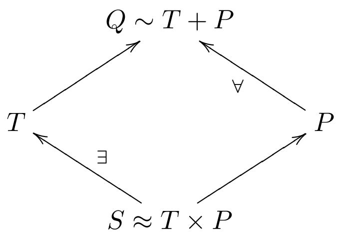
$$S \longrightarrow P + T \implies S \longrightarrow P \quad \text{or} \quad S \longrightarrow T\,.$$

Lemma 5. *Let $S \to\!=\!\not\to T$ be a duality pair. Then*

1. *T is connected, and*
2. *$S \times T \longrightarrow T$ is a gap.*

Proof. If T is not connected, then $T \approx T_0 + T_1$ and $T \not\longrightarrow T_i$. Therefore $T_i \longrightarrow S$ and then $T \longrightarrow S$ which is a contradiction. Now, suppose that $T \times S \longrightarrow P \longrightarrow T$. If $T \not\longrightarrow P$ then $P \longrightarrow S$ and $P \longrightarrow T \times S$. Therefore $P \sim T$ or $P \sim T \times S$ which concludes the proof. □

Theorem 8. *If $P \longrightarrow Q$ is a gap, and Q is not connected, then there exists a gap $S \longrightarrow T$ with T connected. Furthermore, $Q \sim T + P$ and $S \approx T \times P$:*



Proof. Let $Q = T_1 + \cdots + T_k$, where each T_i is connected. Clearly $P \longrightarrow P + T_i \longrightarrow Q$ and then $P \sim P + T_i$ or $P + T_i \sim Q$. Since $Q \not\longrightarrow P$, there exists a $T = T_i$ such that $T \not\longrightarrow P$ and therefore $P + T \not\longrightarrow P$ and $P + T \sim Q$. Finally, let R be such that $P \times T \longrightarrow R \longrightarrow T$. Since $P \longrightarrow P + R \longrightarrow T$, then $P \sim P + R$ or $P + R \sim T$. Therefore, if $T \not\longrightarrow R$, then $R \longrightarrow P$ and $R \longrightarrow T \times P$ which concludes the proof. ☐

As usual, an order $(X, \leq)$ is said to be *dense* if for all $x, y \in X$, if $x < y$ then there exists a $z \in X$ such that $x < z < y$; i.e., a dense order is an order without gaps. We now exhibit some dense classes of separoids. First observe that, due to Theorem 8, when searching for a gap it is enough to look pairs of objects $S \longrightarrow T$ for which T is connected.

Theorem 9. *$\mathcal{P}^\times$ is dense.*

Proof. Let $S \longrightarrow T \not\longrightarrow S$, with T connected, and let $n = |S|$ and $n' = |T|$. Consider a point separoid P in general position, with dimension $\mathrm{d}(P) > \mathrm{d}(T)$, and order $|P| = (\mathrm{d}(P) + 2)n^{n'}$. Clearly,

$$S \longrightarrow S + (P \times T) \longrightarrow T$$

so, it is enough to prove that the opposite arrows does not exist.

Since T is connected and $T \not\longrightarrow S$, every homomorphism $T \longrightarrow S + (P \times T)$ most be an homomorphism $T \longrightarrow P \times T$ which, followed by the projection, would lead an homomorphism $T \longrightarrow P$. However, since P is in general position and $\mathrm{d}(P) > \mathrm{d}(T)$, due to the No-Homomorphism Lemma 3, such an homomorphism does not exist.

Now, every homomorphism $S + (P \times T) \longrightarrow S$ induces to an homomorphism $\varphi\colon P \times T \longrightarrow S$. For every $p \in P$ there is a function $\varphi_p\colon T \to S$ defined as $\varphi_p(t) = \varphi(p, t)$ — such functions does not have to be homomorphisms. Since there are at most $|S^T| = n^{n'}$ different functions, there exists a subset $P' \subseteq P$ of order $|P'| = \mathrm{d}(P) + 2$ such that for every $p, p' \in P'$ we have that $\varphi_p = \varphi_{p'}$. Let $A \dagger B$ be a minimal Radon partition in $P[P']$.

Since $T \not\longrightarrow S$, there exists a minimal Radon partition $\alpha \dagger \beta$ in T such that

$$\varphi_{p'}(\alpha) \mid \varphi_{p'}(\beta) \qquad (\text{or} \quad \varphi_{p'}(\alpha) \cap \varphi_{p'}(\beta) \neq \phi)\,.$$

But $\varphi_{p'}(\alpha) = \varphi(p' \times \alpha) = \varphi(A \times \alpha)$ and $\varphi_{p'}(\beta) = \varphi(B \times \beta)$, therefore we have also that

$$\varphi_{p'}(\alpha) \dagger \varphi_{p'}(\beta)\,,$$

an obvious contradiction. Hence the homomorphism φ does not exists and we are done. □

Example 7. Notice that $\mathcal{P}^{\times} \neq \mathcal{S}$. It is also easy to see that $\mathcal{R}^{\times} = \mathcal{R}$. It is natural to ask if there is a "nice" proper subclass which generates all separoids. Or even more concrete, a nice proper subclass which generates all Radon separoids. However, it seems that such a nice subclass may not exist. For, consider the separoid depicted in Figure 2 as an example of a separoid in the class $\mathcal{R} \cap \mathcal{Z} \setminus \mathcal{M}$. Such a separoid is indeed a *prime* separoid; i.e., it cannot be expressed as a product of other two separoids. It is part of an infinite family of prime separoids, each of the form: $S = \{1, \ldots, n\}$ and

$$1 \dagger 2, \quad 12 \dagger 3, \quad 123 \dagger 4, \quad \ldots, \quad 1 \ldots (n-1) \dagger n\,.$$

Observe that the previous family contains an element in each dimension. On the other extreme, we have that the family of all separoids of 5 or more points in the plane in *convex position* (i.e., where each element is separated from its complement) are prime separoids. We do not know whether there is a "good" characterisation of prime separoids.

Theorem 9 implies that

Corollary 10. *Let $\mathcal{C}$ be any class of non-empty separoids. If $\mathcal{P} \subseteq \mathcal{C}$ then $\mathcal{C}^{\times}$ is dense.* □

In particular, the class of all non-empty separoids is dense.

At first glance it seems that $\mathcal{P}$ is responsible for the density of $\mathcal{S}$; however we can find dense subclasses of separoids which does not arise from point separoids (see the following Remark).

Theorem 11. *$\mathcal{S} \setminus K_0$ is dense.*

Proof. Let $S \longrightarrow T \not\longrightarrow S$ with T connected. Let P be the separoid of order $|P| = 2|T||S|^{|T|}$ and Radon partitions as follows: for every $A, B \subseteq P$

$$A \dagger B \iff |T| \leq \min\{|A|, |B|\} \quad \text{and} \quad A \cap B = \phi\,.$$

Clearly, $\mathrm{d}(P) = 2|T| - 2$ and P is in general position. We can now prove, analogously as in Theorem 9, that $S < S + P \times T < T$. □

Remark. If P is the separoid described in the previous proof, then $P \in \mathcal{Z} \setminus \mathcal{R}$. Therefore P is neither a point separoid nor an oriented matroid nor a graph.

Corollary 12.

1. *The only gap of $\mathcal{S}$ is $K_0 \longrightarrow K_1$,*
2. *The only duality pair of $\mathcal{S}$ is $K_1 \rightarrow\!=\!\not\rightarrow K_0$.*

Proof. For the first, due to Theorem 8, there is a gap only if there is a *connected* gap. Therefore, since there are no other connected gap (Theorem 11), there are no other gap at all.

For the second, due to Lemma 5, there is a duality pair only if there is a gap. Since there are no other gaps, there are no other duality pairs. □

3.2. $\mathcal{P}$ **is not dense.** We now analyse in more detail the density of $\mathcal{P}$. We first describe the *principal ideal* in $\mathcal{P}$ generated by K_2; i.e., we analyse the class

$$\longrightarrow K_2 := \{P \in \mathcal{P} : P \longrightarrow K_2\}$$

(see Figure 4). For, let us denote by χ^i_j the separoid of order $i + j$ with a unique Radon partition of the form $\{1, \ldots, i\} \dagger \{i + 1, \ldots, j\}$. In particular, $\chi^1_1 \approx K_2$ and χ^1_2 is depicted in Figure 1.**b**.

It is easy to see that, P is a point separoid of order $|P| = \mathrm{d}(P) + 2$ if and only if P is defined by a unique minimal Radon partition, say $A \dagger B$ (see e.g. [10]); therefore, P is isomorphic to the separoid $\chi^{|A|}_{|B|} + \sigma$, for some simploid σ.

Lemma 6. *Let P be a point separoid.*

$$P \longrightarrow K_2 \implies P \approx \sum_{(i,j)\in I} \chi^i_j + \sigma\,,$$

for some simploid σ and some set of indices $I \subset \mathbb{N}^2$.

Proof. For, it is enough to prove that, if $P \longrightarrow K_2$ and P is connected, then $|P| = \mathrm{d}(P) + 2$. So, let us suppose that $|P| \geq \mathrm{d}(P) + 3$. Let us identify K_2 with the set $\{1, 2\}$, where $1 \dagger 2$, and let $\varphi\colon P \to K_2$ be an homomorphism. Let $A \dagger B$ be a minimal Radon partition of P, then $|A \cup B| \leq \mathrm{d}(S) + 2$. Furthermore, we may suppose that $|A|$ is minimal over all minimal Radon partitions. Let $c \notin A \cup B$ be another element. Since P is a Steinitz separoid, there exists a $d \in A \cup B$ such that $(A \setminus d) \dagger (B \setminus d \cup c)$; furthermore, by the minimality of A, we may suppose that $d \in B$ and then $A \dagger (B \setminus d \cup c)$. Now, with out loose of generality, suppose that $\varphi(A) = 1$ and $\varphi(B \setminus d \cup c) = 2$; in particular, $\varphi(c) = 2$. But then, by the same Steinitz argument, there exists an $e \in A \cup B$ such that $(A \setminus e \cup c) \dagger (B \setminus e)$ which implies that $\varphi(A \setminus e \cup c) \cap \varphi(B \setminus e) \neq \emptyset$. This contradicts the fact that φ is a homomorphism. □

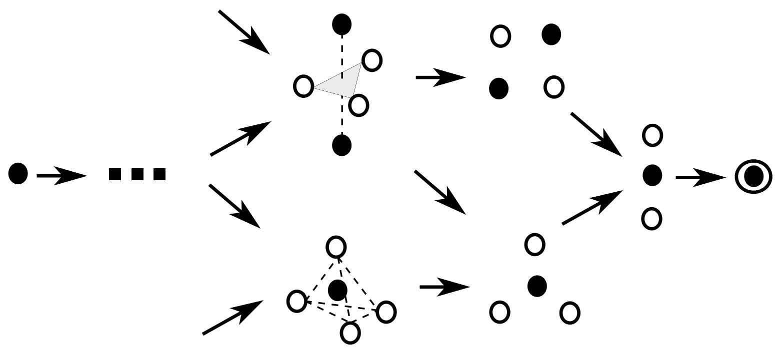


Figure 4. *The interval* $K_1 \longrightarrow K_2$ *in* $\mathcal{P}$.

It follows immediately that,

Corollary 13. $\chi^i_{j+1} + \chi^{i+1}_j \longrightarrow \chi^i_j$ *is a gap in* $\mathcal{P}$.

Proof. Simply observe that if $i \leq j$ and $\ell \leq m$ then $\chi^\ell_m \longrightarrow \chi^i_j$ if and only if $i \leq \ell$ and $j \leq m$. The result follows from Lemma 6. □

More generally, let us denote by $\|\cdot\|$ the number of minimal Radon partitions.

Theorem 14. *Let* Z *be a Steinitz separoid in general position, and* R *a Radon separoid. Then*

$$Z \longrightarrow R \implies \|Z\| \leq \|R\| .$$

Proof. Let $\varphi\colon Z \to R$ be a homomorphism. If $\|Z\| > \|R\|$ then there are two minimal Radon partitions of Z, say $A_i \dagger B_i$, for $i = 1, 2$, mapping into the same minimal Radon partition of R $\alpha \dagger \beta$, say $\varphi(A_i) = \alpha$ and $\varphi(B_i) = \beta$. With out loose of generality, we may suppose that there exists an $x \in A_2 \setminus A_1$ (observe that $x \notin B_1$). Since Z is in general position and it is Steinitz, $|A_1 \cup B_1| = \mathrm{d}(Z) + 2$ and there exists a $y \in A_1 \cup B_1$ such that $(A_1 \setminus y) \dagger (B_1 \setminus y \cup x)$ is a minimal Radon partition of Z.

Suppose that $y \in B_1$. Then $\varphi(A_1 \setminus y) = \varphi(A_1) = \alpha$ and $\varphi(x) \in \varphi(B_1 \setminus y \cup x) \cap \alpha$. These contradict the fact that $\varphi(A_1 \setminus y) \dagger \varphi(B_1 \setminus y \cup x)$ is a minimal Radon partition of R.

So, $y \in A_1$. Since $\varphi(A_1 \setminus y) = \alpha$ leads to same contradiction as before, we may suppose that $\varphi(A_1 \setminus y) = \alpha \setminus \varphi(y)$. Also, $\varphi(B_1 \cup x) = \beta \cup \varphi(x)$. From here it follows that

$$(\alpha \setminus \varphi(y)) \cap (\beta \cup \varphi(x)) \neq \emptyset \quad \text{or} \quad \varphi(x) = \varphi(y) .$$

The first case contradicts that φ is a homomorphism; the second that R is a Radon separoid. □

Remark. It is easy to see that if P is a point separoid in general position then $\|P\| = \binom{n}{d+2}$, where $n = |P|$ and $d = \mathrm{d}(P)$. Since $\mathcal{P} \subset \mathcal{R} \cap \mathcal{Z}$, it follows immediately that if $P \longrightarrow K_k$ then $\binom{n}{d+2} \leq \binom{k}{2}$. Thus, the point separoids rarely maps into complete separoids.

This has to be contrasted with graphs which always maps into complete ones. Furthermore, complete graphs are those graphs K with the property that, if $K \longrightarrow G$ then such a homomorphism has to be an embedding. This motivates the following definition. We say that a separoid F is *epifinal* if every homomorphism $F \longrightarrow T$ is an embedding (e.g., the vertices of the pentagon form an epifinal point separoid). It is easy to see that epifinal separoids are prime.

Let P^u the point separoid obtained from P by *blowing* the point $u \in P$; that is, P^u is the separoid defined in $P \setminus \{u\} \cup \{u', u''\}$ with minimal Radon partitions of the form (see Figure 5):

$$\begin{cases} A \dagger (B \setminus \{u\} \cup \{u', u''\}) & \text{if} \quad u \in B\,, \\ A \dagger B & \text{if} \quad u \notin A \cup B\,. \end{cases}$$

For example, the vertices of the n-octahedron — also known as the n-crosspolytope, which is the dual polytope of the n-cube — can be obtained by blowing all points of K_n.

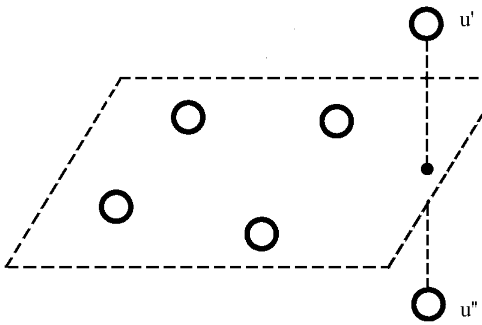


Figure 5. *Blowing a vertex of the pentagon.*

We have the following

Conjecture. *If P is epifinal in $\mathcal{P}$, then $\sum_{u \in P} P^u \longrightarrow P$ is a gap in $\mathcal{P}$.*

4. On the chromatic number

It is clear from the remark after Theorem 14 that we cannot define the chromatic number of a separoid in terms of the minimum complete graph where it can be mapped with a homomorphism. However, we can generalise this important invariant as follows. We say that a colouring $f\colon S \to \{1, \dots, k\}$ is a *proper k-colouring* if for every minimal Radon partition $A \dagger B$ it follows that $f(A) \cap f(B) = \emptyset$. The *chromatic number* of a separoid $\chi(S)$ is the minimum k such that there exists a proper k-colouring of S. A standard argument shows the following

Lemma 7.

1. $S \longrightarrow T \implies \chi(S) \leq \chi(T)$,
2. $\chi(S) = \min |T| : S \longrightarrow T$,
3. $\chi(S \times T) \leq \min\{\chi(S), \chi(T)\}$. □

The celebrated (Hedetniemi) product conjecture (see e.g. [5]) states that, in the class of graphs, the inequality in *3* is in fact an equality. This conjecture can be formulated in terms of homomorphisms as follows:

$$S \times T \longrightarrow K_k \iff S \longrightarrow K_k \quad \text{or} \quad T \longrightarrow K_k\,.$$

In this direction, we have the following

Theorem 15. *Let R be a Radon separoid, M an oriented matroid, and G a graph. Then*

$$R \times M \longrightarrow G \quad \text{and} \quad M \not\longrightarrow G \implies R \longrightarrow G\,.$$

Proof. Since $M \not\longrightarrow G$ we may suppose that there exists a Radon partition $\alpha \dagger \beta$ in M. Let $\psi\colon M \times S \longrightarrow G$ be an homomorphism. For each $s \in S$, we can define the function $\psi_s(m) = g$ whenever $\psi(m,s) = g$. If $A \dagger B$ in S, then $\psi(A \times \alpha) \dagger \psi(B \times \beta)$ in G and therefore

$$\psi_A(\alpha) \dagger \psi_B(\beta) \in G\,.$$

Now, observe the following; since G is a graph, then $|\psi_A(\alpha)| = |\psi_B(\beta)| = 1$. Say $\psi_A(\alpha) = v$ and $\psi_B(\beta) = u$, where $v \dagger u \in G$. Now, since M is an oriented matroid, then $\psi_A \equiv v$ and $\psi_B \equiv u$ are constant functions.

Let $\varphi_{AB}\colon S[A \cup B] \to G$ be defined as

$$\varphi_{AB}(s) = \begin{cases} v & s \in A \\ u & s \in B\,. \end{cases}$$

Then φ_{AB} is a homomorphism. Since R is a Radon separoid, we can extend this homomorphism to obtain a homomorphism $\varphi\colon S \longrightarrow G$ such that $\varphi|_{A\cup B} = \varphi_{AB}$ and we are done. □

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