Archivum Mathematicum

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Archivum Mathematicum, Vol. 42 (2006), No. 1, 31--42

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), 31 – 42

ON INTEGERS WITH A SPECIAL DIVISIBILITY PROPERTY

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Abstract. In this note, we study those positive integers n which are divisible by $\sum_{d|n}\lambda(d)$, where $\lambda(\cdot)$ is the Carmichael function.

1. Introduction

Let $\varphi(\cdot)$ denote the *Euler function*, whose value at the positive integer n is given by

$$\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times = \prod_{p^\nu \,\|\, n} p^{\nu-1}(p-1).$$

Let $\lambda(\cdot)$ denote the *Carmichael function*, whose value $\lambda(n)$ at the positive integer n is defined to be the largest order of any element in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$. More explicitly, for a prime power p^ν, one has

$$\lambda(p^\nu) = \begin{cases} p^{\nu-1}(p-1) & \text{if} \quad p \ge 3 \quad \text{or} \quad \nu \le 2\,, \\ 2^{\nu-2} & \text{if} \quad p = 2 \quad \text{and} \quad \nu \ge 3\,, \end{cases}$$

and for an arbitrary integer $n \ge 2$ with prime factorization $n = p_1^{\nu_1}\dots p_k^{\nu_k}$, one has

$$\lambda(n) = \mathrm{lcm}\big[\lambda(p_1^{\nu_1}),\,\dots,\,\lambda(p_k^{\nu_k})\big]\,,$$

Note that $\lambda(1) = 1$.

Since $\lambda(d) \le \varphi(d)$ for all $d \ge 1$, it follows that

$$\sum_{d|n}\lambda(d) \le \sum_{d|n}\varphi(d) = n$$

for every positive integer n, and it is clear that the equality

$$\sum_{d|n}\lambda(d) = n \tag{1}$$

cannot hold unless $\lambda(n) = \varphi(n)$. The latter condition is equivalent to the statement that $(\mathbb{Z}/n\mathbb{Z})^\times$ is a *cyclic* group, and by a well known result of Gauss, this happens

2000 *Mathematics Subject Classification.* 11N37.
Key words and phrases. Euler function, Carmichael function.
Received August 23, 2004.

only if $n = 1,\ 2,\ 4,\ p^\nu$ or $2p^\nu$ for some odd prime p and integer exponent $\nu \ge 1$. For such n, $\lambda(d) = \varphi(d)$ for every divisor d of n, hence we see that the equality (1) is in fact *equivalent* to the statement that $\lambda(n) = \varphi(n)$.

When $\lambda(n) < \varphi(n)$, the equality (1) is not possible. However, it may happen that the sum appearing on the left side of (1) is a *proper* divisor of n. Indeed, one can easily find many examples of this phenomenon:

$$n = 140, 189, 378, 1375, 2750, 2775, 2997, 4524, 5550, 5661, 5994, \ldots.$$

These positive integers n are the subject of the present paper.

Throughout the paper, the letters p, q and r are always used to denote prime numbers. For a positive integer n, we write $P(n)$ for the largest prime factor of n, $\omega(n)$ for the number of distinct prime divisors of n, and $\tau(n)$ for the total number of positive integer divisors of n. For a positive real number x and a positive integer k, we write $\log_k x$ for the function recursively defined by $\log_1 x = \max\{\log x, 1\}$ and $\log_k x = \log_1(\log_{k-1} x)$, where $\log(\cdot)$ denotes the natural logarithm. We also use the Vinogradov symbols $\gg$ and $\ll$, as well as the Landau symbols O and o, with their usual meanings.

Acknowledgements. This work was done during a visit by the second author to the University of Missouri, Columbia; the hospitality and support of this institution are gratefully acknowledged. During the preparation of this paper, W. B. was supported in part by NSF grant DMS-0070628, and F. L. was supported in part by grants SEP-CONACYT 37259-E and 37260-E.

2. Main Results

Let $b(\cdot)$ be the arithmetical function whose value at the positive integer n is given by

$$b(n) = \sum_{d|n} \lambda(d)\,.$$

Our aim is to investigate the set $\mathcal{B}$ defined as follows:

$$\mathcal{B} = \{n : b(n) \text{ is a proper divisor of } n\}\,.$$

For a positive real number x, let $\mathcal{B}(x) = \mathcal{B} \cap [1, x]$. Our first result provides a nontrivial upper bound on $\#\mathcal{B}(x)$ as $x \to \infty$:

Theorem 1. *The following inequality hold as $x \to \infty$:*

$$\#\mathcal{B}(x) \le x \exp\left(-2^{-1/2}(1 + o(1))\sqrt{\log x \log_2 x}\right)\,.$$

Proof. Our proof closely follows that of Theorem 1 in [2]. Let x be a large real number, and let

$$y = y(x) = \exp\left(2^{-1/2}\sqrt{\log x \log_2 x}\right)\,.$$

Also, put

$$u = u(x) = \frac{\log x}{\log y} = 2^{1/2}\sqrt{\frac{\log x}{\log_2 x}}\,. \tag{2}$$

Finally, we recall that a number m is said to be *powerful* if $p^2|m$ for every prime factor p of m.

Let us consider the following sets:

$$\begin{aligned}
\mathcal{B}_1(x) &= \{n \in \mathcal{B}(x) : P(n) \le y\}\,,\\
\mathcal{B}_2(x) &= \{n \in \mathcal{B}(x) : \omega(n) \ge u\}\,,\\
\mathcal{B}_3(x) &= \{n \in \mathcal{B}(x) : m|n \text{ for some powerful number } m > y^2\}\,,\\
\mathcal{B}_4(x) &= \{n \in \mathcal{B}(x) : \tau(\varphi(n)) > y\}\,,\\
\mathcal{B}_5(x) &= \mathcal{B}(x) \setminus (\mathcal{B}_1(x) \cup \mathcal{B}_2(x) \cup \mathcal{B}_3(x) \cup \mathcal{B}_4(x))\ .
\end{aligned}$$

Since $\mathcal{B}(x)$ is the union of the sets $\mathcal{B}_j(x)$, $j = 1, \ldots, 5$, it suffices to find an appropriate bound on the cardinality of each set $\mathcal{B}_j(x)$.

By the well known estimate (see, for instance, Tenenbaum [7]):

$$\Psi(x,y) = \#\{n \le x : P(n) \le y\} = x \exp\{-(1+o(1))u \log u\}\,,$$

which is valid for u satisfying (2), we derive that

$$\#\mathcal{B}_1(x) \le x \exp\left(-2^{-1/2}(1+o(1))\sqrt{\log x \log_2 x}\right)\ . \tag{3}$$

Next, using Stirling's formula together with the estimate

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1)\,,$$

we obtain that

$$\begin{aligned}
\#\{n \le x : \omega(n) \ge u\} &\le \sum_{p_1 \cdots p_{\lfloor u \rfloor} \le x} \frac{x}{p_1 \ldots p_{\lfloor u \rfloor}} \le \frac{x}{\lfloor u \rfloor !}\Big(\sum_{p \le x} \frac{1}{p}\Big)^{\lfloor u \rfloor}\\
&\le x\Big(\frac{e \log\log x + O(1)}{\lfloor u \rfloor}\Big)^{\lfloor u \rfloor}\\
&\le x \exp\left(-(1+o(1))u \log u\right)\ ,
\end{aligned}$$

therefore

$$\#\mathcal{B}_2(x) \le x \exp\Big(-2^{-1/2}(1+o(1))\sqrt{\log x \log_2 x}\Big)\,. \tag{4}$$

We also have

$$\#\mathcal{B}_3(x) \le \sum_{\substack{m > y^2 \\ m \text{ powerful}}} \frac{x}{m} \ll \frac{x}{y} = x \exp\Big(-2^{-1/2}\sqrt{\log x \log_2 x}\Big)\,, \tag{5}$$

where the second inequality follows by partial summation from the well known estimate:

$$\#\{m \le x : m \text{ powerful}\} \ll \sqrt{x}\,.$$

(see, for example, Theorem 14.4 in [5]).

By a result from [6], it is known that

$$\sum_{n \le x} \tau(\varphi(n)) \le x \exp\left(O\left(\sqrt{\frac{\log x}{\log_2 x}}\right)\right)\,. \tag{6}$$

Therefore,

$$\begin{aligned}\#\mathcal{B}_4(x) &\le \sum_{\substack{n\le x\\ \tau(\varphi(n))>y}} 1 < \frac{1}{y}\sum_{n\le x}\tau(\varphi(n)) \le \frac{x}{y}\exp(O(u))\\ &\le x\exp\Big(-2^{-1/2}(1+o(1))\sqrt{\log x\log_2 x}\Big). \qquad (7)\end{aligned}$$

In view of the estimates (3), (4), (5) and (7), to complete the proof it suffices to show that

$$\#\mathcal{B}_5(x) \le x\exp\Big(-2^{-1/2}(1+o(1))\sqrt{\log x\log_2 x}\Big). \qquad (8)$$

We first make some comments about the integers in the set $\mathcal{B}_5(x)$. For each $n \in \mathcal{B}_5(x)$, write $n = n_1n_2$, where $\gcd(n_1n_2) = 1$, n_1 is powerful, and n_2 is squarefree. Since $n_1 \le y^2$ (as $n \notin \mathcal{B}_3(x)$) and $P(n) > y$ (as $n \notin \mathcal{B}_1(x)$), it follows that $P(n)|n_2$; in particular, $P(n)\|n$. By the multiplicativity of $\tau(\cdot)$, we also have

$$\tau(n) = \tau(n_1)\tau(n_2).$$

Since $n \notin \mathcal{B}_2(x)$,

$$\tau(n_2) \le 2^{\omega(n)} < 2^u = \exp\big(O(u)\big),$$

Also,

$$\tau(n_1) \le \exp\Big(O\Big(\frac{\log n_1}{\log\log n_1}\Big)\Big) \le \exp\Big(O\Big(\frac{\log y}{\log\log y}\Big)\Big) = \exp\big(O(u)\big).$$

In particular,

$$\tau(n) \le \exp\big(O(u)\big). \qquad (9)$$

Now let $n \in B_5(x)$, and write $n = Pm$, where $P = P(n)$ and m is a positive integer with $m \le x/y$. Put

$$D_1 = \gcd(P-1,\lambda(m)) \qquad \text{and} \qquad D_2 = \gcd\big(m, b(n)\big). \qquad (10)$$

Since $b(n)$ is a (proper) divisor of $n = Pm$, it follows that $b(n) = D_2P^\delta$, where $\delta = 0$ or 1. Since $P\|n$ and $P \ne 2$, we also have

$$\begin{aligned} b(n) &= \sum_{d|n}\lambda(d) = \sum_{d|m}\lambda(d) + \sum_{d|m}\operatorname{lcm}[P-1,\lambda(d)]\\ &= b(m) + \sum_{d|m}\frac{(P-1)\lambda(d)}{\gcd(D_1,\lambda(d))} = b(m) + (P-1)b(D_1,m),\end{aligned}$$

where

$$b(D_1,m) = \sum_{d|m}\frac{\lambda(d)}{\gcd\big(D_1,\lambda(d)\big)}.$$

Consequently,

$$b(m) + (P-1)b(D_1,m) = D_2P^\delta,$$

and thus

$$P = \begin{cases} 1 + \dfrac{D_2 - b(m)}{b(D_1, m)} & \text{if} \quad \delta = 0\,, \\[2ex] \dfrac{b(m) - b(D_1, m)}{D_2 - b(D_1, m)} & \text{if} \quad \delta = 1\,. \end{cases} \tag{11}$$

We remark that $D_2 \neq b(D_1, m)$ in the second case. Indeed, noting that $m > 2$ (since n is neither prime nor twice a prime), it follows that D_1 is even; in particular, $D_1 \geq 2$. Thus,

$$1 = \frac{\lambda(1)}{\gcd(D_1, \lambda(1))} \leq b(D_1, m) \leq \sum_{\substack{d|m \\ d<m}} \lambda(d) + \frac{\lambda(m)}{D_1} < b(m)\,,$$

which shows that $b(m) - b(D_1, m) > 0$, and therefore D_2 cannot be equal to $b(D_1, m)$ in view of (11). Hence, from (11), we conclude that for all fixed choices of m, an even divisor D_1 of $\lambda(m)$, and a divisor D_2 of m, there are at most two possible primes P satisfying (10) and such that $Pm \in \mathcal{B}_5(x)$. Using (6) and (9), and recalling that $m \leq x/y$, we derive that

$$\begin{aligned} \#B_5(x) &\ll \sum_{m \leq x/y} \tau(m)\tau\big(\lambda(m)\big) \leq \exp(O(u)) \sum_{m \leq x/y} \tau\big(\varphi(m)\big) \\ &\ll \frac{x}{y} \exp\big(O(u)\big)\,. \end{aligned}$$

The estimate (8) now follows from our choice of y, and this completes the proof. □

Our next result provides a complete characterization of those odd integers $n \in \mathcal{B}$ with $\omega(n) = 2$.

Theorem 2. *Suppose that $n = p^a q^b$, where p and q are odd primes with $p < q$, and a, b are positive integers. If $n \neq 2997$, then $n \in \mathcal{B}$ if and only if $b = 1$ and there exists a positive integer k such that*

$$q = 2p^{(p^k - 1)/(p-1)} + 1 \qquad \text{and} \qquad a = k + 2(p^k - 1)/(p - 1)\,.$$

Proof. Let c be the largest nonnegative integer such that $p^c | (q - 1)$.

First, suppose that $p \nmid (q - 1)$ (that is, $c = 0$). We must show that $n \notin \mathcal{B}$. Indeed, let $t = \gcd(p - 1, q - 1)$; then

$$\begin{aligned} b(n) &= 1 + \sum_{j=1}^{a} \lambda(p^j) + \sum_{k=1}^{b} \lambda(q^k) + \sum_{j=1}^{a}\sum_{k=1}^{b} \lambda(p^j q^k) \\ &= 1 + \sum_{j=1}^{a} \varphi(p^j) + \sum_{k=1}^{b} \varphi(q^k) + \sum_{j=1}^{a}\sum_{k=1}^{b} \frac{\varphi(p^j q^k)}{t} \\ &= 1 + (p^a - 1) + (q^b - 1) + t^{-1}(p^a q^b - p^a - q^b + 1). \end{aligned}$$

If $n \in \mathcal{B}$, $b(n) = p^e q^f$ for some integers e, f with $0 \le e \le a$ and $0 \le f \le b$. Thus,

$$tp^e q^f = (t-1)(p^a + q^b - 1) + p^a q^b \tag{12}$$

If $e \le a - 1$, then since $t \le p - 1$, it follows that

$$tp^e q^f < p^{e+1} q^f \le p^a q^b,$$

which contradicts (12); therefore, $e = a$. A similar argument shows that $f = b$. But then $b(n) = p^a q^b = n$, which is not possible since $b(n)$ is a *proper* divisor of n. This contradiction establishes our claim that $n \notin \mathcal{B}$.

If $c \ge 1$, we have

$$\begin{aligned} b(n) &= 1 + \sum_{j=1}^{a} \lambda(p^j) + \sum_{k=1}^{b} \lambda(q^k) + \sum_{\substack{1 \le j \le a \\ j \le c}} \sum_{k=1}^{b} \lambda(p^j q^k) \\ &\quad + \sum_{\substack{1 \le j \le a \\ j \ge c+1}} \sum_{k=1}^{b} \lambda(p^j q^k) \\ &= 1 + \sum_{j=1}^{a} \varphi(p^j) + \sum_{k=1}^{b} \varphi(q^k) + \sum_{\substack{1 \le j \le a \\ j \le c}} \sum_{k=1}^{b} \frac{\varphi(pq^k)}{t} \\ &\quad + \sum_{\substack{1 \le j \le a \\ j \ge c+1}} \sum_{k=1}^{b} \frac{\varphi(p^{j-c} q^k)}{t}. \end{aligned}$$

For any integer $r \ge 1$, we have the identity:

$$\sum_{k=1}^{b} \varphi(p^r q^k) = \varphi(p^r) \sum_{k=1}^{b} \varphi(q^k) = (p^r - p^{r-1})(q^b - 1).$$

Hence, it follows that

$$b(n) = p^a + q^b - 1 + \frac{(q^b - 1)}{t} \Big((p-1) \min\{a, c\} + p^{\max\{a-c,\, 0\}} - 1 \Big). \tag{13}$$

Assuming that $n \in \mathcal{B}$, write $b(n) = p^e q^f$ as before.

We claim that $c < a$. Indeed, if $c \ge a$, then reducing (13) modulo p^c (and recalling that $q \equiv 1 \pmod{p^c}$), we obtain that

$$p^e \equiv p^e q^f = b(n) \equiv p^a \pmod{p^c},$$

which implies that $e = a$. Then

$$p^a q^f = b(n) = p^a + q^b - 1 + \frac{(q^b - 1)(p-1)a}{t},$$

which in turn gives

$$tp^a(q^f - 1) = (q^b - 1)(1 + (p - 1)a)\,. \tag{14}$$

The following result can be easily deduced from [1].

Lemma 3. *For every odd prime q and integer $b \geq 2$, then there exists a prime P such that $P|(q^b - 1)$, but $P \nmid (q^f - 1)$ for any positive integer $f < b$, except in the case that $b = 2$ and q is a Mersenne prime.*

If $f < b$ and the prime P of Lemma 3 exists, the equality (14) is not possible as P divides only the right-hand side. Thus, if (14) holds and $f < b$, it must be the case that $b = 2$, $f = 1$, and $q = 2^r - 1$ for some prime r. But this leads to the equality

$$tp^a = 2^r(1 + (p - 1)a)\,,$$

and since t divides $(q - 1) \equiv 2 \pmod 4$, we obtain a contradiction after reducing everything modulo 4. Therefore, $f = b$, and we again have that $b(n) = p^a q^b = n$, contradicting the fact that $n \in \mathcal{B}$. This establishes our claim that $c < a$.

From now on, we can assume that $c < a$; then (13) takes the form:

$$p^e q^f = b(n) = p^a + q^b - 1 + \frac{(q^b - 1)}{t}\left((p - 1)c + p^{a-c} - 1\right)\,.$$

Reducing this equation modulo p^c, we immediately deduce that $e \geq c$. Thus,

$$\Big(\frac{q^b - 1}{q - 1}\Big)\Big(\frac{q - 1}{p^c}\Big)\Big(1 + \frac{(p - 1)c + p^{a-c} - 1}{t}\Big) = (p^{e-c}q^f - p^{a-c})\,, \tag{15}$$

where each term enclosed by parentheses is an integer. Using the trivial estimates

$$\frac{q^b - 1}{q - 1} \geq q^{b-1}\,, \qquad \frac{q - 1}{p^c} \geq t\,,$$

and

$$1 + \frac{(p - 1)c + p^{a-c} - 1}{t} > \frac{p^{a-c}}{t}\,,$$

we obtain that

$$p^{a-c}(q^{b-1} + 1) < p^{e-c}q^f\,, \tag{16}$$

which clearly forces $f = b$.

Now put $D = (q^b - 1)/(q - 1)$; then $D|(q^b - 1)$ and $D|(p^{e-c}q^b - p^{a-c})$ (since $f = b$); thus,

$$p^{e-c} \equiv p^{a-c} \pmod D\,. \tag{17}$$

Write $D = p^d D_0$, where $p \nmid D_0$. From the definition of D, it easy to see that d is also the largest nonnegative integer such that $p^d|b$; therefore,

$$d \leq \frac{\log b}{\log p}\,. \tag{18}$$

On the other hand, from (17), it follows that $d \le e - c$; hence,

$$p^{e-c-d} \equiv p^{a-c-d} \pmod{D_0},$$

which implies that $D_0 | (p^{a-e} - 1)$. Consequently,

$$p^{a-e} > p^{a-e} - 1 \ge D_0 = p^{-d} D \ge p^{-d} q^{b-1} > p^{-d} (p^{a-e})^{b-1},$$

where in the last step we have used the bound $q > p^{a-e}$, which follows from (16) (with $f = b$). Thus,

$$d > (a-e)(b-2). \tag{19}$$

Combining the estimates (18) and (19), and using the fact that $a - e \ge 1$, we see that $b \le 2$. Moreover, if $b = 2$, then since $p^d | b$ and p is odd, it follows that $d = 0$, which is impossible in view of (19). Hence, $b = 1$.

At this point, (15) takes the form

$$\left(\frac{q-1}{p^c}\right)\left(1 + \frac{(p-1)c + p^{a-c} - 1}{t}\right) = p^{e-c} q - p^{a-c}. \tag{20}$$

Since $t \le p - 1$, we have

$$p^{e-c} q > \Big(\frac{q-1}{p^c}\Big)\Big(\frac{p^{a-c}}{p-1}\Big) = p^{a-2c}\Big(\frac{q-1}{p-1}\Big) > p^{a-2c}\Big(\frac{q}{p}\Big) = p^{a-2c-1} q,$$

thus $a \le e + c$.

We now write $q - 1 = p^c t \mu$ for some positive integer μ. Then from (20), it follows that

$$p^{a-c}(\mu + 1) - p^e t \mu = p^{e-c} + \mu - t\mu - (p-1)c\mu. \tag{21}$$

First, let us distinguish a few special cases. If $t = 2$ and $\mu = 1$, we have

$$2p^{a-c} - 2p^e = p^{e-c} - 1 - (p-1)c.$$

If $a \le e + c - 1$, we see that

$$p^{e-c} - 1 - (p-1)c \le 2p^{e-1} - 2p^e;$$

hence,

$$2p^{e-1}(p-1) \le c(p-1) + 1 - p^{e-c} \le e(p-1),$$

which is not possible for any $e \ge 1$. Thus, $a = e + c$, and it follows that

$$c = \frac{p^{e-c} - 1}{p - 1}.$$

Taking $k = e - c$ (which is positive since c is an integer), we have

$$q = 2p^c + 1 = 2p^{(p^k - 1)/(p-1)} + 1,$$

and

$$a = e + c = k + 2c = k + 2(p^k - 1)/(p - 1);$$

hence, our integer $n = p^a q$ has the form stated in the theorem.

Next, we claim that $e \neq 1$. Indeed, if $e = 1$, then $c = 1$; as $c < a \leq e + c$, it follows that $a = 2$. Substituting into (21), we obtain that

$$p(\mu + 1) - pt\mu = 1 + \mu - t\mu - (p - 1)\mu\,,$$

or

$$p(1 + 2\mu - t\mu) = 1 + 2\mu - t\mu\,.$$

This last equality implies that $1 + 2\mu - t\mu = 0$, therefore $\mu = 1$ and $t = 3$, which is not possible since t is an even integer.

For convenience, let S denote the value on either side of the equality (21). We note that the relation (20) implies that $p^{e-c} | (t + (p - 1)c - 1)$; thus,

$$S \leq t + (p - 1)c - 1 + \mu - t\mu - (p - 1)c\mu = (1 - \mu)\big(t + (p - 1)c - 1\big)\,.$$

In the case that $S \geq 0$, we immediately deduce that $\mu = 1$, which implies that $S = 0$. Then $2p^{a-c} = p^e t$, and we conclude that $t = 2$ (and $a = e + c$), which is a case we have already considered.

Suppose now that $S < 0$. From (21) we derive that

$$\frac{-S}{p^{e-c}\mu} = p^c t - p^{a-e}\Big(1 + \frac{1}{\mu}\Big) = \frac{t + (p - 1)c}{p^{e-c}} - \frac{1}{\mu} - \frac{1}{p^{e-c}}\,,$$

and since we already know that $a \leq e + c$, $t \leq p - 1$ and $c \leq e$, it follows that

$$p^c\Big(t - 1 - \frac{1}{\mu}\Big) < \frac{t + (p - 1)c}{p^{e-c}} \leq \frac{(p - 1)(c + 1)}{p^{e-c}} \leq \frac{(p - 1)(e + 1)}{p^{e-c}}\,.$$

If $t \neq 2$ or $\mu \neq 1$ (which have already been considered), then $(t - 1 - 1/\mu) \geq 1/2$, and therefore

$$e + 1 > \frac{p^e}{2(p - 1)}\,.$$

This implies that $e \leq 2$ for $p = 3$, and $e = 1$ for $p \geq 5$. Since we have already ruled out the possibility $e = 1$, this leaves only the case where $p = 3$ and $e = 2$. To handle this, we observe that $(t - 1 - 1/\mu) \geq 2/3$ if $\mu \geq 3$, and we obtain the bound

$$e + 1 > \frac{2p^e}{3(p - 1)}\,,$$

which is not possible for $p = 3$ and $e = 2$. Thus, we left only with the case $p = 3$ and $e = t = \mu = 2$. Since $c \leq e$, $c < a \leq e + c$, and $q = 4 \cdot 3^c + 1$, it follows that $n \in \{117,\ 351,\ 999,\ 2997\}$. It may be checked that, of these four integers, only 2997 lies in the set $\mathcal{B}$.

To complete the proof, it remains only to show that if

$$q = 2p^{(p^k-1)/(p-1)} + 1 \qquad \text{and} \qquad a = k + 2(p^k - 1)/(p - 1)$$

for some positive integer k, then $n = p^a q$ lies in the set $\mathcal{B}$. For such primes p, q, we have $t = 2$, $c = (p^k - 1)/(p - 1)$, $q = 2p^c + 1$, and $a = k + 2c$; taking $e = a - c = k + (p^k - 1)/(p - 1)$, we immediately verify (20). Noting that $e < a$, it follows that $b(n)$ is a proper divisor of n. □

As a complement to Theorem 2, we have:

Theorem 4. *If n is even and $\omega(n) = 2$, then $n \notin \mathcal{B}$.*

Proof. Write $n = 2^a q^b$, where q is an odd prime and a, b are positive integers, and suppose first that $a \ge 3$. For any divisor $d = 2^e q^f$ of n, the congruence $\lambda(d) \equiv 0 \pmod 4$ holds whenever $e \ge 4$. On the other hand, if $e \le 3$, then $\lambda(d) = \lambda(q^f)$ since $2|(q-1)$. Reducing $b(n)$ modulo 4, we have

$$b(n) \equiv \sum_{j=0}^{3} \lambda(2^j) + \sum_{j=0}^{3}\sum_{k=1}^{b} \lambda(2^j q^k) = 6 + 4\sum_{k=1}^{b} \lambda(q^k) \equiv 2 \pmod 4,$$

which implies that $2 \| b(n)$. If $n \in \mathcal{B}$, then $b(n)$ is a divisor of n, thus $b(n) \le 2q^b$. On the other hand,

$$b(n) \ge 6 + 4\sum_{k=1}^{b} \lambda(q^k) = 2 + 4\sum_{k=0}^{b} \varphi(q^k) = 2 + 4q^b,$$

which contradicts the preceding estimate. This shows that $n \notin \mathcal{B}$.

If $a = 1$, then n is twice a prime power, thus $n \notin \mathcal{B}$.

Finally, suppose that $a = 2$. Then

$$\begin{aligned} b(n) = \sum_{j=0}^{2} \lambda(2^j) + \sum_{j=0}^{2}\sum_{k=1}^{b} \lambda(2^j q^k) &= 4 + 3\sum_{k=1}^{b} \lambda(q^k) \\ &= 1 + 3\sum_{k=0}^{b} \varphi(q^k) = 1 + 3q^b, \end{aligned}$$

which clearly cannot divide $n = 4q^b$. □

3. Comments

In Theorem 2, the condition $k = 1$ is equivalent to $a = 3$ and $q = 2p+1$; that is, q is a *Sophie Germain prime*. Under the classical Hardy-Littlewood conjectures (see [3, 4]), the number of such primes $q \le y$ should be asymptotic to $y/(\log y)^2$ as $y \to \infty$; thus, we expect $\mathcal{B}$ to contain roughly $x^{1/4}/(\log x)^2$ odd integers n of the form $n = p^3 q$. When $k \ge 2$, then

$$\frac{1}{\log q} \ll \frac{1}{p^{k-1}\log p},$$

and since the series

$$\sum_{\substack{p \ge 3 \\ k \ge 2}} \frac{1}{p^{k-1}\log p}$$

converges, classical heuristics suggest that there should be only finitely many numbers $n \in \mathcal{B}$ with $\omega(n) = 2$ and $k > 1$. Unconditionally, we can only say that the number of such odd integers $n \in \mathcal{B}$ with $n \le x$ is $O((\log x)/(\log_2 x))$.

We do not have any conjecture about the correct order of magnitude of $\#\mathcal{B}(x)$ as $x \to \infty$. In fact, we cannot even show that $\mathcal{B}$ is an infinite set, although computer searches produce an abundance of examples.

Let $p_1, p_2, \dots, p_k$ be distinct primes such that $(p_1-1)|(p_2-1)|\dots|(p_k-1)$. Taking $n = p_1 \dots p_k$, we see that

$$b(n) = \sum_{d|n} \lambda(d) = 1 + (p_1-1) + 2(p_2-1) + \cdots + 2^{k-1}(p_k-1)\,. \tag{22}$$

Indeed, this formula is clear if $k=1$. For $k>1$, put $m = p_1 \dots p_{k-1}$, and note that the divisibility conditions among the primes imply that $\lambda(m)|(p_k-1)$. Therefore,

$$\begin{aligned} b(n) = \sum_{d|n} \lambda(d) &= \sum_{d|m} \lambda(d) + \sum_{d|m} \mathrm{lcm}[p_k-1, \lambda(d)] \\ &= \sum_{d|m} \lambda(d) + (p_k-1)\tau(m) = b(m) + 2^{k-1}(p_k-1)\,, \end{aligned}$$

and an immediate induction completes the proof of formula (22). If $p>5$ is a prime congruent to 1 modulo 4 such that $q=2p-1$ is also prime, then $p_1=5$, $p_2=p$ and $p_3=q$ fulfill the stated divisibility conditions; thus, with $n=5pq$, we have

$$b(n) = \sum_{d|n} \lambda(d) = 1 + (5-1) + 2(p-1) + 4(q-1) = 10p - 5 = 5q\,,$$

which is a divisor of n. The Hardy-Littlewood conjectures also predict that if x is sufficiently large, there exist roughly $x^{1/2}/(\log x)^2$ of such positive integers $n \le x$, which suggests that the inequality $\#\mathcal{B}(x) \gg x^{1/2}/(\log x)^2$ holds.

Finally, we note that $b(2n) = 2b(n)$ whenever n is odd, therefore $2n \in \mathcal{B}$ whenever n is an odd element of $\mathcal{B}$.

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