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On an effective criterion of solvability of boundary value problems for ordinary differential equation of n-th order

ON AN EFFECTIVE CRITERION OF SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATION OF n-TH ORDER

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ABSTRACT. New sufficient conditions for the existence of a solution of the boundary value problem for an ordinary differential equation of n-th order with certain functional boundary conditions are constructed by a method of a priori estimates.

INTRODUCTION

In this paper we give new sufficient conditions for the existence of a solution of the ordinary differential equation

$$(1) \qquad u^{(n)}(t) = f\Big(t, u(t), \dots, u^{(n-1)}(t)\Big)$$

with the boundary conditions

$$(2) \qquad \Phi_{0i}\Big(u^{(i-1)}\Big) = \varphi_i(u) \qquad i = 1, \dots, n,$$

resp.

$$(3_1) \qquad l_i\Big(u, u', \dots, u^{(k_0-1)}\Big) = 0 \qquad i = 1, \dots, k_0,$$

$$(3_2) \qquad \Phi_{0i}\Big(u^{(i-1)}\Big) = \varphi_i\Big(u^{(k_0)}\Big) \qquad i = k_0 + 1, \dots, n,$$

where $f : [a, b] \times R^n \to R$ satisfies the local Carathéodory conditions, $n \geq 2$, and $1 \leq k_0 \leq n - 2$.

For each index i, the functional Φ_{0i} in the conditions (2), resp. (3_2), is supposed to be linear, nondecreasing, nontrivial, continuous on $C([a, b])$, and concentrated on $[a_i, b_i] \subseteq [a, b]$ (i.e., the value of functional Φ_{0i} depends only on a function restricted to $[a_i, b_i]$ and this segment can be degenerated to a point). In general $\Phi_{0i}(1) \in R$, without loss of generality we can suppose that $\Phi_{0i}(1) = 1$, which simplifies the notation.

In the condition (3_1), the functionals $l_i : [C([a,b])]^{k_0} \to R$ $(i = 1, \ldots, k_0)$ are linear and continuous.

For each index i $(i = 1, \ldots, n)$, the functional $\varphi_i : C^{n-1}([a,b]) \to R$ in the conditions (2) is continuous and satisfies

$$(4_1) \qquad \xi_i(\rho) = \frac{1}{\rho} \sup \left\{ |\varphi_i(\rho v)| \; : \; \|v\|_{C^{n-1}_{([a,b])}} \leq 1 \right\} \to 0 \quad \text{as} \quad \rho \to +\infty \, .$$

For each index i $(i = k_0 + 1, \ldots, n)$, the functional $\varphi_i : C^{n-1-k_0}([a,b]) \to R$ in the conditions (3_2) is continuous and satisfies

$$(4_2) \qquad \delta_i(\rho) = \frac{1}{\rho} \sup \left\{ |\varphi_i(\rho v)| \; : \; \|v\|_{C^{n-1-k_0}_{([a,b])}} \leq 1 \right\} \to 0 \quad \text{as} \quad \rho \to +\infty \, .$$

The special cases of boundary conditions (2) are

$$(5_1) \qquad\qquad u^{(i-1)}(t_i) = \varphi_i(u) \qquad\qquad i = 1, \ldots, n \, ,$$

where $a \leq a_i \leq t_i \leq b_i \leq b$ $(i = 1, \ldots, n)$ or

$$(5_2) \qquad\qquad \int_{a_i}^{b_i} u^{(i-1)}(t) \, d\sigma_i(t) = \varphi_i(u) \qquad i = 1, \ldots, n \, .$$

The integral is understood in the Lebesgue–Stieltjes sense, where σ_i is nondecreasing in $[a_i, b_i]$ and $\sigma(b_i) - \sigma(a_i) > 0$ $(i = 1, \ldots, n)$. We know that the problem (1), (5_1) was studied by B. Půža in the paper [4], so in this paper we will receive more general results than in [4].

Problem (1), (3) was studied by Nguyen Anh Tuan in the paper [5] and by Gegelia G. T. in the paper [1]. In this paper, however, we will give new sufficient conditions for the existence of a solution of the problem (1), (3).

Main results

We adopt the following notation:

$[a, b]$ – a segment, $-\infty < a \leq a_i \leq b_i \leq b < +\infty$ $(i = 1, \ldots, n)$.

R^n – n-dimensional real space with elements $x = (x_i)_{i=1}^n$ normed by $\|x\| = \sum_{i=1}^{n} |x_i|$.

$R_+^n = \{x \in R^n \; : \; x_i \geq 0, \, i = 1, \ldots, n\}$, $(0, +\infty) = R_+ - \{0\}$.

$C^{n-1}([a,b])$ – the space of functions continuous together with their derivatives up to the order $(n-1)$ on $[a,b]$ with the norm

$$\|u\|_{C^{n-1}_{([a,b])}} = \max \left\{ \sum_{i=1}^{n} |u^{(i-1)}(t)| \; : \; a \leq t \leq b \right\} .$$

$AC^{n-1}([a,b])$ – the set of all functions absolutely continuous together with their derivatives up to the order $(n-1)$ on $[a,b]$.

$L^p([a,b])$ – the space of functions Lebesgue integrable on $[a,b]$ in the p-th power with the norm

$$\|u\|_{L^p_{([a,b])}} = \begin{cases} \left(\int\limits_a^b |u(t)|^p \, dt \right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < +\infty \,, \\ \operatorname{ess\,sup} \{ |u(t)| \; : \; a \le t \le b \} & \text{if} \quad p = +\infty \,. \end{cases}$$

$L^p([a,b], R_+) = \{ u \in L^p([a,b]) \; : \; u(t) \ge 0 \text{ for a. a. } a \le t \le b \}$.

Let $x = (x_i(t))_{i=1}^n, y = (y_i(t))_{i=1}^n \in [C([a,b])]^n$. We will say that $x \le y$ if $x_i(t) \le y_i(t)$ for all $t \in [a,b]$ and $i = 1, \dots, n$.

A functional $\Phi : [C([a,b])]^n \to R$ is said to be nondecreasing if $\Phi(x) \le \Phi(y)$ for all $x, y \in [C([a,b])]^n$, $x \le y$, and positively homogeneous if $\Phi(\lambda x) = \lambda \Phi(x)$ for all $\lambda \in (0, +\infty)$ and $x \in [C([a,b])]^n$.

Let us consider the problems (1), (2) and (1), (3). Under a solution of the problem (1), (2), resp. (1), (3), we understand a function $u \in AC^{n-1}([a,b])$ which satisfies the equation (1) almost everywhere on $[a,b]$ and fulfils the boundary conditions (2), resp. (3).

Theorem 1. *Let the inequalities*

(6_1)
$$f(t, x_1, x_2, \dots, x_n) \operatorname{sign} x_n \le \omega(|x_n|) \sum_{i=1}^{n-1} \sum_{j=1}^m g_{ij}(t) h_{ij}(x_i) |x_{i+1}|^{\frac{1}{q_{ij}}}$$
$$\text{for} \quad t \in [a_n, b], (x_i)_{i=1}^n \in R^n$$

(6_2)
$$f(t, x_1, x_2, \dots, x_n) \operatorname{sign} x_n \ge -\omega(|x_n|) \sum_{i=1}^{n-1} \sum_{j=1}^m g_{ij}(t) h_{ij}(x_i) |x_{i+1}|^{\frac{1}{q_{ij}}}$$
$$\text{for} \quad t \in [a, b_n], (x_i)_{i=1}^n \in R^n$$

hold, where $g_{ij} \in L^{p_{ij}}([a,b], R_+)$, $p_{ij}, q_{ij} \ge 1$, $1/p_{ij} + 1/q_{ij} = 1$ ($i = 1, \dots, n-1; j = 1, \dots, m$), $\omega : R_+ \to (0, +\infty)$ and $h_{ij} : R \to R_+$ ($i = 1, \dots, n-1; j = 1, \dots m$) are continuous nondecreasing functions satisfying

(7)
$$\Omega(\rho) = \int\limits_0^\rho \frac{ds}{\omega(s)} \to +\infty \quad as \quad \rho \to +\infty$$

and

(8)
$$\lim_{\rho \to +\infty} \frac{\Omega(\rho \xi_n(\rho))}{\Omega(\rho)} = 0 = \lim_{\rho \to +\infty} \frac{\|h_{ij}\|_{L^{q_{ij}}_{([-\rho,\rho])}}}{\Omega(\rho)}$$
$$i = 1, \dots, n-1; \; j = 1, \dots, m \,.$$

Then the problem (1), (2) has at least one solution.

To prove Theorem 1 we need the following

Lemma 1. *Let the functions ω, Ω, g_{ij}, h_{ij} and the numbers p_{ij}, q_{ij} ($i = 1, \dots, n-1; j = 1, \dots, m$) be given as in Theorem 1, and let $\eta_i : R_+ \to R_+$ ($i = 1, \dots, n$) be*

nondecreasing functions satisfying

(9) $$\lim_{\rho \to +\infty} \frac{\Omega(\eta_n(\rho))}{\Omega(\rho)} = 0 = \lim_{\rho \to +\infty} \frac{\eta_i(\rho)}{\rho} \qquad i = 1, \ldots, n.$$

Then there exists a constant $\rho_0 > 0$ such that the estimate

(10) $$\|u\|_{C^{n-1}_{([a,b])}} \leq \rho_0$$

holds for each solution $u \in AC^{n-1}([a,b])$ of the differential inequalities

(11$_1$) $u^{(n)}(t)\operatorname{sign} u^{(n-1)}(t)$

$$\leq \omega(|u^{(n-1)}(t)|) \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}\big(u^{(i-1)}(t)\big)|u^{(i)}(t)|^{\frac{1}{q_{ij}}}$$

$$for \quad t \in [a_n, b]$$

(11$_2$) $u^{(n)}(t)\operatorname{sign} u^{(n-1)}(t)$

$$\geq -\omega\big(|u^{(n-1)}(t)|\big) \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}\big(u^{(i-1)}(t)\big)|u^{(i)}(t)|^{\frac{1}{q_{ij}}}$$

$$for \quad t \in [a, b_n]$$

with the boundary condition

(12) $$\min\big\{|u^{(i-1)}(t)| \ : \ a_i \leq t \leq b_i\big\} \leq \eta_i\big(\|u\|_{C^{n-1}_{([a,b])}}\big) \qquad i = 1, \ldots, n.$$

Proof. Put

$$\mu = \sum_{i=1}^{n}(b-a)^{n-i} \quad \text{and} \quad \varepsilon = [2\mu(n-1)]^{-1}.$$

Then according to (9) there exists a number $r_0 > 0$ such that

(13) $$\eta_i(\rho) \leq \varepsilon\rho \quad \text{for} \quad \rho > r_0 \qquad i = 1, \ldots, n.$$

We suppose that the estimate (10) does not hold. Then for arbitrary $\rho_1 \geq r_0$ there exists a solution u of the problem (11), (12) such that

(14) $$\|u\|_{C^{n-1}_{([a,b])}} > \rho_1.$$

We put

(15) $$\rho = \max\big\{|u^{(n-1)}(t)| \ : \ a \leq t \leq b\big\}$$

and choose $\tau_i \in [a_i, b_i]$ $(i = 1, \ldots, n)$ such that

$$|u^{(i-1)}(\tau_i)| = \min\big\{|u^{(i-1)}(t)| \ : \ a_i \leq t \leq b_i\big\}.$$

Then from (12) we have

(16) $$|u^{(i-1)}(\tau_i)| \leq \eta_i\big(\|u\|_{C^{n-1}_{([a,b])}}\big) \qquad i = 1, \ldots, n.$$

Using (15), (16) we have

(17)
$$|u^{(n-2)}(t)| \le \Big| \int_{\tau_{n-1}}^{t} |u^{(n-1)}(\tau)| \, d\tau \Big| + |u^{(n-2)}(\tau_{n-1})|$$
$$\le (b-a)\rho + \eta_{n-1}\big(\|u\|_{C^{n-1}_{([a,b])}}\big) \quad \text{for} \quad t \in [a,b].$$

Integrating $u^{(n-2)}$ from τ_{n-2} to t and using (16) and (17) again we get

$$|u^{(n-3)}(t)| \le \Big| \int_{\tau_{n-2}}^{t} |u^{(n-2)}(\tau)| \, d\tau \Big| + |u^{(n-3)}(\tau_{n-2})|$$
$$\le (b-a)^2\rho + (b-a)\eta_{n-1}\big(\|u\|_{C^{n-1}_{([a,b])}}\big) + \eta_{n-2}\big(\|u\|_{C^{n-1}_{([a,b])}}\big)$$

for $t \in [a,b]$. Applying this procedure $(n-1)$-times we obtain

$$\|u\|_{C^{n-1}_{([a,b])}} \le \mu\Big(\rho + \sum_{i=1}^{n-1}\eta_i\big(\|u\|_{C^{n-1}_{([a,b])}}\big)\Big).$$

Using (13) and (14) we get

$$\|u\|_{C^{n-1}_{([a,b])}} \le \mu\Big(\rho + (n-1)\varepsilon\|u\|_{C^{n-1}_{([a,b])}}\Big) = \mu\rho + \frac{1}{2}\|u\|_{C^{n-1}_{([a,b])}}.$$

Therefore we have

(18)
$$\|u\|_{C^{n-1}_{([a,b])}} \le 2\mu\rho.$$

We choose a point $\tau^* \in [a,b]$ such that $\tau^* \ne \tau_n$ and

$$|u^{(n-1)}(\tau^*)| = \max\big\{|u^{(n-1)}(t)| \; : \; a \le t \le b\big\}.$$

Then either $\tau_n < \tau^*$ or $\tau^* < \tau_n$.

If $\tau_n < \tau^*$, then the integration of (11_1) from τ_n to τ^*, in view of (18) and using Hölder's inequality, we get

(19)
$$\int_{\tau_n}^{\tau^*} \frac{u^{(n)}(t)\,\operatorname{sign} u^{(n-1)}(t)\,dt}{\omega\big(|u^{(n-1)}(t)|\big)} \le \int_{\tau_n}^{\tau^*} \sum_{i=1}^{n-1}\sum_{j=1}^{m} g_{ij}(t)h_{ij}\big(u^{(i-1)}(t)\big)|u^{(i)}(t)|^{\frac{1}{q_{ij}}}\,dt$$
$$\le \sum_{i=1}^{n-1}\sum_{j=1}^{m} \|g_{ij}\|_{L^{p_{ij}}_{([a,b])}} \|h_{ij}\|_{L^{q_{ij}}_{([-2\mu\rho,2\mu\rho])}}.$$

Applying (15), (16), (18), and the definition of Ω in (19), we get

$$\Omega(\rho) \le \Omega(\eta_n(2\mu\rho)) + \sum_{i=1}^{n-1}\sum_{j=1}^{m} \|g_{ij}\|_{L^{p_{ij}}_{([a,b])}} \|h_{ij}\|_{L^{q_{ij}}_{([-2\mu\rho,2\mu\rho])}}.$$

Now, in view of (8), (9), (14), and (18), since ρ_1 was chosen arbitrarily, we get

$$\lim_{\rho \to +\infty} \frac{\Omega(\rho)}{\Omega(2\mu\rho)} = 0.$$

On the other hand, in view of (7) and the facts that $2\mu > 1$ and ω is a nonde-creasing function, we have

$$\liminf_{\rho \to +\infty} \frac{\Omega(\rho)}{\Omega(2\mu\rho)} > 0\,,$$

a contradiction.

If $\tau^* < \tau_n$, then the integration of (11_2) from τ^* to τ_n yields the same contra-diction in analogous way. $\qquad\square$

Proof of Theorem 1. Let ρ_0 be the constant from Lemma 1. Put

$$\chi(s) = \begin{cases} 1 & \text{if} \quad |s| \leq \rho_0 \\ 2 - \dfrac{|s|}{\rho_0} & \text{if} \quad \rho_0 < |s| < 2\rho_0, \\ 0 & \text{if} \quad |s| \geq 2\rho_0 \end{cases}$$

(20)
$$\widetilde{f}(t, x_1, \ldots, x_n) = \chi\left(\|x\|\right) f(t, x_1, \ldots, x_n) \quad \text{for} \quad a \leq t \leq b, \quad (x_i)_{i=1}^n \in R^n\,,$$
$$\widetilde{\varphi}_i(u) = \chi\left(\|u\|_{C_{([a,b])}^{n-1}}\right) \varphi_i(u) \quad \text{for} \quad u \in C^{n-1}([a,b]) \quad i = 1, \ldots, n$$

and consider the problem

(21) $$u^{(n)}(t) = \widetilde{f}\left(t, u(t), \ldots, u^{(n-1)}(t)\right),$$

(22) $$\Phi_{0i}\left(u^{(i-1)}\right) = \widetilde{\varphi}_i(u) \qquad i = 1, \ldots, n\,.$$

From (20) it immediately follows that $\widetilde{f} : [a,b] \times R^n \to R$ satisfies the local Carathéodory conditions, $\widetilde{\varphi}_i : C^{n-1}([a,b]) \to R$ $(i = 1, \ldots, n)$ are continuous functionals and

(23_1) $$\sup\left\{|\widetilde{f}(\cdot, x_1, \ldots, x_n)| \ : \ (x_i)_{i=1}^n \in R^n\right\} \in L([a,b])\,,$$

(23_2) $$\sup\left\{|\widetilde{\varphi}_i(u)| \ : \ u \in C^{n-1}([a,b])\right\} < +\infty \qquad i = 1, \ldots, n\,.$$

Now we will show that the homogeneous problem

(21_0) $$v^{(n)}(t) = 0\,,$$

(22_0) $$\Phi_{0i}\left(v^{(i-1)}\right) = 0 \qquad i = 1, \ldots, n$$

has only the trivial solution.

Let v be an arbitrary solution of this problem. Integrating (21_0) we get

$$v^{(n-1)}(t) = \text{const} \quad \text{for} \quad a \leq t \leq b\,.$$

According to (22_0) we have

$$v^{(n-1)}(a)\Phi_{0n}(1) = 0\,.$$

However, since $\Phi_{0n}(1) = 1$, we have $v^{(n-1)}(t) = 0$ for $a \leq t \leq b$. Referring to (22_0) and $\Phi_{0i}(1) = 1$ $(i = 1, \ldots, n-1)$, we come to the conclusion that $v(t) \equiv 0$. Using Theorem 2.1 from [3], in view of (23) and the uniqueness of the trivial solution of the problem (21_0), (22_0), we get the existence of a solution of the problem (21), (22).

Let u be a solution of the problem (21), (22). Then, using (6), we get

$$u^{(n)}(t) \operatorname{sign} u^{(n-1)}(t) = \widetilde{f}(t, u(t), \ldots, u^{(n-1)}(t)) \operatorname{sign} u^{(n-1)}(t)$$

$$= \chi\Big(\sum_{j=1}^{n} |u^{(j-1)}(t)|\Big) f(t, u(t), \ldots, u^{(n-1)}(t)) \operatorname{sign} u^{(n-1)}(t)$$

$$\leq \omega\big(|u^{(n-1)}(t)|\big) \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(u^{(i-1)}(t)) |u^{(i)}(t)|^{\frac{1}{q_{ij}}}$$

for $t \in [a_n, b]$, and

$$u^{(n)}(t) \operatorname{sign} u^{(n-1)}(t) = \widetilde{f}(t, u(t), \ldots, u^{(n-1)}(t)) \operatorname{sign} u^{(n-1)}(t)$$

$$= \chi\Big(\sum_{j=1}^{n} |u^{(j-1)}(t)|\Big) f(t, u(t), \ldots, u^{(n-1)}(t)) \operatorname{sign} u^{(n-1)}(t)$$

$$\geq -\omega\big(|u^{(n-1)}(t)|\big) \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(u^{(i-1)}(t)) |u^{(i)}(t)|^{\frac{1}{q_{ij}}}$$

for $t \in [a, b_n]$. Put

$$\eta_i(\rho) = \sup\big\{|\widetilde{\varphi}_i(v)| \ : \ \|v\|_{C^{n-1}_{([a,b])}} \leq \rho\big\} \qquad i = 1, \ldots, n.$$

From (4_1) and (8), it immediately follows that the functions η_i $(i = 1, \ldots, n)$ satisfy (9) and

$$\min\big\{|u^{(i-1)}(t)| \ : \ a_i \leq t \leq b_i\big\} = \Phi_{0i}\big(\min\big\{|u^{(i-1)}(t)| \ : \ a_i \leq t \leq b_i\big\}\big)$$

$$\leq |\Phi_{0i}(u^{(i-1)})| = |\widetilde{\varphi}_i(u)| \leq \eta_i\big(\|u\|_{C^{n-1}_{([a,b])}}\big)$$

$$i = 1, \ldots, n.$$

Therefore, by Lemma 1 we get

$$\|u\|_{C^{n-1}_{([a,b])}} \leq \rho_0.$$

Consequently,

$$\chi\Big(\sum_{i=1}^{n} |u^{(i-1)}(t)|\Big) = 1 \qquad \text{for} \ a \leq t \leq b$$

and

$$\chi\big(\|u\|_{C^{n-1}_{([a,b])}}\big) = 1.$$

Using these equalities in (20), we obtain that u is a solution of the problem (1), (2). $\qquad \square$

Remark 1. If $\Phi_{0i}(u^{(i-1)}) = u^{(i-1)}(t_i)$, $a \leq a_i \leq t_i \leq b_i \leq b$ $(i = 1, \ldots, n)$, then Theorem 1 is Theorem in [4].

Now we give new sufficient conditions guaranteeing the existence of a solution of the problem (1), (3) provided that the equation

$$(24) \qquad\qquad\qquad u^{(k_0)} = 0$$

with the boundary conditions (3_1) has only the trivial solution.

Theorem 2. *Let the problem* (24), (3_1) *have only the trivial solution and let the inequalities*

$$(25_1) \qquad f(t, x_1, \ldots, x_n) \operatorname{sign} x_n \le \omega\left(|x_n|\right) \sum_{i=k_0+1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(x_i) |x_{i+1}|^{\frac{1}{q_{ij}}}$$

$$for \quad t \in [a_n, b], (x_i)_{i=1}^n \in R^n$$

$$(25_2) \qquad f(t, x_1, \ldots, x_n) \operatorname{sign} x_n \ge -\omega\left(|x_n|\right) \sum_{i=k_0+1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(x_i) |x_{i+1}|^{\frac{1}{q_{ij}}}$$

$$for \quad t \in [a, b_n], (x_i)_{i=1}^n \in R^n$$

hold, where $g_{ij} \in L^{p_{ij}}([a, b], R_+)$, $p_{ij}, q_{ij} \ge 1$, $1/p_{ij} + 1/q_{ij} = 1$ ($i = k_0 + 1, \ldots, n - 1; j = 1, \ldots, m$), $\omega : R_+ \to (0, +\infty)$ *and* $h_{ij} : R \to R_+$ ($i = k_0 + 1, \ldots, n - 1; j = 1, \ldots, m$) *are continuous nondecreasing functions satisfying* (7) *and*

$$(26) \qquad \begin{aligned} &\lim_{\rho \to +\infty} \frac{\Omega(\rho \delta_n(\rho))}{\Omega(\rho)} = 0 \\ &\lim_{\rho \to +\infty} \frac{\|h_{ij}\|_{L^{q_{ij}}_{([-\rho, \rho])}}}{\Omega(\rho)} = 0 \qquad i = k_0 + 1, \ldots, n - 1; j = 1, \ldots, m\,. \end{aligned}$$

Then the problem (1), (3) *has at least one solution.*

To prove Theorem 2 we need the following

Lemma 2. *Let the problem* (24), (3_1) *have only the trivial solution and let the functions* ω, Ω, g_{ij}, h_{ij} *and the numbers* p_{ij}, q_{ij} ($i = k_0 + 1, \ldots, n - 1$; $j = 1, \ldots, m$) *be given as in Theorem 2, and let* $\eta_i : R_+ \to R_+$ ($i = k_0 + 1, \ldots, n$) *be nondecreasing functions satisfying*

$$\lim_{\rho \to +\infty} \frac{\Omega(\eta_n(\rho))}{\Omega(\rho)} = 0 = \lim_{\rho \to +\infty} \frac{\eta_i(\rho)}{\rho} \qquad i = k_0 + 1, \ldots, n\,.$$

Then there exists a constant $\rho_0 > 0$ *such that the estimate* (10) *holds for each solution* $u \in AC^{n-1}([a, b])$ *of the differential inequalities*

$$u^{(n)}(t) \operatorname{sign} u^{(n-1)}(t) \le \omega\left(|u^{(n-1)}(t)|\right) \sum_{i=k_0+1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}\left(u^{(i-1)}(t)\right) |u^{(i)}(t)|^{\frac{1}{q_{ij}}}$$

$$(27_1) \qquad\qquad\qquad for \quad t \in [a_n, b]$$

$$u^{(n)}(t)\,\operatorname{sign} u^{(n-1)}(t) \geq -\omega\big(|u^{(n-1)}(t)|\big) \sum_{i=k_0+1}^{n-1} \sum_{j=1}^{m} g_{ij}(t)h_{ij}\big(u^{(i-1)}(t)\big)\Big|u^{(i)}(t)\Big|^{\frac{1}{q_{ij}}}$$

$$(27_2) \qquad\qquad for \quad t \in [a, b_n]$$

with the boundary conditions (3_1) and

$$(28) \quad \min\big\{|u^{(i-1)}(t)| : a_i \leq t \leq b_i\big\} \leq \eta_i\big(\|u^{(k_0)}\|_{C^{n-k_0-1}_{([a,b])}}\big) \qquad i = k_0+1, \ldots, n\,.$$

Proof. Let u be an arbitrary solution of the problem (27), (3_1), (28). Put

$$(29) \qquad\qquad v(t) = u^{(k_0)}(t)\,.$$

Then the formulas (27) and (28) imply that

$$v^{(n-k_0)}(t)\,\operatorname{sign} v^{(n-k_0-1)}(t) \leq \omega\big(|v^{(n-k_0-1)}(t)|\big)$$

$$\times \sum_{i=1}^{n-k_0-1} \sum_{j=1}^{m} g_{ij}(t)h_{ij}\big(v^{(i-1)}(t)\big)|v^{(i)}(t)|^{\frac{1}{q_{ij}}}$$

$$for \quad t \in [a_n, b]\,,$$

$$v^{(n-k_0)}(t)\,\operatorname{sign} v^{(n-k_0-1)}(t) \geq -\omega\big(|v^{(n-k_0-1)}(t)|\big)$$

$$\times \sum_{i=1}^{n-k_0-1} \sum_{j=1}^{m} g_{ij}(t)h_{ij}\big(v^{(i-1)}(t)\big)|v^{(i)}(t)|^{\frac{1}{q_{ij}}}$$

$$for \quad t \in [a, b_n]\,,$$

and

$$\min\big\{|v^{(i-1)}(t)| : a_i \leq t \leq b_i\big\} \leq \eta_i\big(\|v\|_{C^{n-k_0-1}_{([a,b])}}\big) \qquad i = 1, \ldots, n-k_0\,.$$

Consequently, according to Lemma 1 there exists $\rho_1 > 0$ such that

$$(30) \qquad\qquad \|v\|_{C^{n-k_0-1}_{([a,b])}} \leq \rho_1\,.$$

By virtue of the assumption that the problem (24), (3_1) has only the trivial solution, there exists a Green function $G(t, s)$ such that

$$(31) \qquad u^{(i-1)}(t) = \int_a^b \frac{\partial^{i-1} G(t,s)}{\partial t^{i-1}} v(s)\,ds \quad \text{for} \quad t \in [a, b] \qquad i = 1, \ldots, k_0$$

(see e.g., [2]).

Put

$$\rho_2 = \max_{a \leq t \leq b} \int_a^b \sum_{i=1}^{k_0} \Big|\frac{\partial^{i-1} G(t,s)}{\partial t^{i-1}}\Big|\,ds\,.$$

According to (30) and (31) we have

$$\|u\|_{C^{k_0-1}_{([a,b])}} \leq \rho_1 \rho_2\,.$$

Therefore we obtain (10), where $\rho_0 = \rho_1 + \rho_2 \rho_1$. $\qquad\qquad\qquad\qquad\square$

Theorem 2 can be proved analogously to Theorem 1 using Lemma 2.

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