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**ON THE DEGENERATION OF HARMONIC SEQUENCES
FROM SURFACES INTO COMPLEX GRASSMANN
MANIFOLDS**

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ABSTRACT. Let $f : M \rightarrow G(m, n)$ be a harmonic map from surface into complex Grassmann manifold. In this paper, some sufficient conditions for the harmonic sequence generated by f to have degenerate ∂' -transform or ∂'' -transform are given.

1. INTRODUCTION

Let $G(m, n)$ be the Grassmann manifold of all m -dimensional subspaces C^m in complex space C^n , M be a connected Riemannian surface. Given a harmonic map $f : M \rightarrow G(m, n)$, Chern-Wolfson obtain the following sequence of harmonic maps by using the ∂' -transforms and ∂'' -transforms:

$$(1.1) \quad \begin{aligned} f &= f_0 \xrightarrow{\partial'} f_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} f_\alpha \xrightarrow{\partial'} \cdots, \\ f &= f_0 \xrightarrow{\partial''} f_{-1} \xrightarrow{\partial''} \cdots \xrightarrow{\partial''} f_{-\alpha} \xrightarrow{\partial''} \cdots, \end{aligned}$$

(1.1) is called the harmonic sequence generated by $f = f_0$. It is important to ask when the harmonic sequence (1.1) includes degenerate ∂' -transform or ∂'' -transform. If $m = 1$, then the degeneration of ∂' -transform or ∂'' -transform is equivalent to the isotropy of f . We know that the harmonic sequence (1.1) must have degenerate ∂' -transform or ∂'' -transform for an arbitrary harmonic map $f : M \rightarrow G(m, n)$ if one of the following conditions holds:

- (i) $g = 0$, i.e., M is homeomorphic to the 2-sphere S^2 [2];
- (ii) $g = 1$ and $\deg(f) \neq 0$ [2];
- (iii) $m = 1$, $|\deg(f)| > (n - 1)(g - 1)$ [3, 4];
- (iv) $m = 1$, $r(\partial'_0) + r(\partial''_0) > 2n(g - 1)$ [4, 5]; where g denotes the genus of M , $\deg(f)$ is the degree of the map and $r(\partial'_0)$ and $r(\partial''_0)$ are the ramification indices of ∂'_0 and ∂''_0 respectively.

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So far have we known above sufficient conditions to guarantee the existence of degenerate ∂' -transform or ∂'' -transform, but when $m > 1$ and $g > 1$ or when M is non-compact, it seems that there aren't any results about it. The main purpose of the present paper is to find some sufficient conditions to ensure the existence of degenerate ∂' -transform or ∂'' -transform. In order to do this, we establish the generalized Frenet formulae for harmonic maps and then use it to obtain some relative results.

2. HARMONIC SEQUENCES

We equip C^n with the standard Hermitian inner product $\langle \cdot, \cdot \rangle$, so that, for any two column vectors $X, Y \in C^n$, $\langle X, Y \rangle = Y^* X$, where Y^* denotes the conjugate and transpose of Y .

Let M be a connected Riemannian surface with Riemannian metric $ds_M^2 = \varphi \bar{\varphi}$, where φ is a complex-valued 1-form defined up to a factor of norm one. The structure equations of M are

$$(2.1) \quad d\varphi = -\sqrt{-1}\rho \wedge \varphi, \quad d\rho = -\frac{\sqrt{-1}}{2}K\varphi \wedge \bar{\varphi},$$

where ρ is the real connection 1-form of M , and K the Gaussian curvature of M . Let $f : M \rightarrow G(m, n)$ be a harmonic map. Choose a local unitary frame Z_1, \dots, Z_n along f suitably such that Z_1, \dots, Z_m span f . We write

$$(2.2) \quad dZ_i = X_i\varphi + Y_i\bar{\varphi} \pmod{f},$$

where $X_i, Y_i \in f^\perp$, $i = 1, \dots, m$. It follows from [1] that except at isolated points the ranks of $\text{span}\{X_1, \dots, X_m\}$ and $\text{span}\{Y_1, \dots, Y_m\}$ are constant, and they define two harmonic maps $f_1 = \partial'f : M \rightarrow G(m_1, n)$ and $f_{-1} : M \rightarrow G(m_{-1}, n)$, where m_1 and m_{-1} are the ranks of $\text{span}\{X_1, \dots, X_m\}$ and $\text{span}\{Y_1, \dots, Y_m\}$ respectively. Repeating in this way, we can get the harmonic sequence (1.1), where $f_{\pm\alpha} : M \rightarrow G(m_{\pm\alpha}, n)$. If $m_\alpha > m_{\alpha+1}$, then the ∂' -transform of f_α is called degenerate. Similarly, when $m_{-\alpha} > m_{-\alpha-1}$, then the ∂'' -transform of $f_{-\alpha}$ is called degenerate. In order to avoid confusion, sometimes we denote the ∂' -transform and ∂'' -transform of f_k by ∂'_k and ∂''_k respectively, $k = 0, \pm 1, \dots$. If $f_{-1} \perp f_1$ then $f = f_0$ is called strongly conformal [6]. If for any $\alpha > 0$, $f_\alpha \neq 0$, then the number

$$(2.3) \quad r = \max\{j : f_0 \perp f_j, \forall 1 \leq i \leq j\}$$

must be finite, and it is called the isotropy order of f [6]. It is known that when $\partial'_k \neq 0$ or $\partial''_k \neq 0$, then ∂'_k or ∂''_k has only isolated zeros, $k = 0, \pm 1, \dots$. Hence, when M is compact, the number of zeros of ∂'_k or ∂''_k , counted according to multiplicity, is finite which is called the ramification index of ∂'_k or ∂''_k , and will be denoted by $r(\partial'_k)$ or $r(\partial''_k)$.

From [6] we know that there is a one-to-one correspondence between smooth map $f : M \rightarrow G(m, n)$ and the subbundle \underline{f} of the trivial bundle $M \times C^n$ of rank m which has fiber at $x \in M$ given by $\underline{f}_x = f(x)$. Therefore, we can identify f with the Hermitian orthogonal projection from $M \times C^n$ onto \underline{f} . From this point of view, we have

Lemma 2.1 ([7]). *Let $f : M \rightarrow G(m, n) \subset U(n)$ be a smooth map. Then f is harmonic if and only if $d * A = 0$, where $A = \frac{1}{2}s^{-1}ds$, $s = f - f^\perp$.*

For the harmonic sequence (1.1), let $Z_1^{(k)}, \dots, Z_{m_k}^{(k)}$ be the local unitary frame for $\underline{f}_k, k = 0, \pm 1, \dots$. Then the Hermitian orthogonal projection f_k can be expressed by

$$(2.4) \quad f_k = W_k W_k^*,$$

where $W_k = (Z_1^{(k)}, \dots, Z_{m_k}^{(k)})$ is the $(n \times m_k)$ -matrix.

3. THE GENERALIZED FRENET FORMULAE

Let $f = f_0 : M \rightarrow G(m, n)$ be a harmonic map which generates the harmonic sequence (1.1). The exterior derivative d has the decomposition $d = \partial + \bar{\partial}$. From [6] we see that $\partial'(\partial'' \underline{f}_k) \subset \underline{f}_k$ and $\partial''(\partial' \underline{f}_k) \subset \underline{f}_k, k = 0, \pm 1, \dots$. Hence, locally there exist $(m_k \times m_k)$ -matrices $\underline{B}_k, \underline{D}_k, (m_{k+1} \times m_k)$ -matrix A_k and $(m_{k-1} \times m_k)$ -matrix C_k so that

$$\begin{aligned} \partial W_k &= (W_{k+1} A_k + W_k B_k) \varphi, \\ \bar{\partial} W_k &= (W_{k-1} C_k + W_k D_k) \bar{\varphi}. \end{aligned}$$

By the construction of the harmonic sequence (1.1), we have $\underline{f}_k \perp \underline{f}_{k+1}$ and so $W_k W_k^* = I_{m_k}$ and $W_k W_{k+1}^* = 0$. Operating $\bar{\partial}$ on them we obtain $B_k + D_k^* = 0$ and $A_k^* + C_{k+1} = 0$. Consequently we get the following generalized Frenet formulae:

$$(3.1) \quad \begin{aligned} \partial W_k &= (W_{k+1} A_k + W_k B_k) \varphi, \\ \bar{\partial} W_k &= -(W_{k-1} A_{k-1}^* + W_k B_k^*) \bar{\varphi}. \end{aligned}$$

Since $\text{rank}(\underline{f}_k)$ is constant for each k except at isolated points, A_k is of full rank except at these isolated points. When f is an isometric immersion, we have [8]

$$(3.2) \quad |A_0|^2 + |A_{-1}|^2 = 1, \quad \cos \alpha = |A_0|^2 - |A_{-1}|^2,$$

here the norm $|Q|$ of a matrix Q is defined by $|Q|^2 = \text{tr}(QQ^*)$ in a standard manner, and α is the Kaehler angle of f .

Lemma 3.1. *In the generalized Frenet formulae (3.1), we have*

$$(3.3) \quad \begin{aligned} dA_k + (A_k B_k^* - B_{k+1}^* A_k) \bar{\varphi} - \sqrt{-1} \rho A_k &\equiv 0 \pmod{\varphi}, \\ d(B_k^* \bar{\varphi}) - d(B_k \varphi) &= (A_k^* A_k - A_{k-1} A_{k-1}^* + B_k^* B_k - B_k B_k^*) \varphi \wedge \bar{\varphi}. \end{aligned}$$

Proof. Set $s_k = f_k - f_k^\perp, A^{(k)} = \frac{1}{2}s_k^{-1}ds_k$. Thus $A^{(k)}$ is one half of the pull-back of Maurer- Cartan form of $U(n)$ by f_k , and it satisfies

$$(3.4) \quad dA^{(k)} + 2A^{(k)} \wedge A^{(k)} = 0.$$

From Lemma 2.1. we see that

$$(3.5) \quad d * A^{(k)} = 0.$$

On the other hand, by virtue of (2.4), (3.1) and the definition of $A^{(k)}$ we get

$$(3.6) \quad \begin{aligned} A^{(k)} = & (-W_{k+1}A_kW_k^* - W_kA_{k-1}W_{k-1}^*)\varphi \\ & + (W_kA_k^*W_{k+1}^* + W_{k-1}A_{k-1}^*W_k^*)\overline{\varphi}. \end{aligned}$$

Combining (3.4)–(3.6) one can obtain (3.3)₁. Substituting (3.1) into $W_k^*d^2W_k = 0$ yields (3.3)₂. □

Lemma 3.2. *If $m_k = m_{k+1}$, then at points where $\det A_k \neq 0$, we have*

$$(3.7) \quad \Delta \log |\det A_k| = m_k K + 2(|A_{k-1}|^2 - 2|A_k|^2 + |A_{k+1}|^2).$$

Moreover, if M is a compact surface with genus g , then

$$(3.8) \quad r(\det A_k) = 2m_k(g - 1) + \deg(f_k) - \deg(f_{k+1}),$$

where $r(\det A_k)$ denotes the number of zeros of $\det A_k$ counted according to multiplicity.

Proof. Note that $d \log(\det A_k) = \text{tr}(A_k^{-1}dA_k)$, by (3.3)₁ we get

$$(3.9) \quad d \log(\det A_k) + \text{tr}(B_k^* - B_{k+1}^*)\overline{\varphi} - \sqrt{-1}m_k\rho \equiv 0 \pmod{\varphi}.$$

A standard computation together with (3.3)₂ and (3.9) yields (3.7). Integrating (3.7) on M and using Lemma 4.1 of [9], the Gauss-Bonnet theorem together with the definition of $\deg(f_k)$ and $\deg(f_{k+1})$ [2] we finally get (3.8). □

4. THE MAIN RESULTS

In this section we shall study the sufficient conditions to ensure the existence of degenerate ∂' -transform or ∂'' -transform in harmonic sequence (1.1). First we have

Theorem 4.1. *Let M be a connected and complete Riemannian surface with non-negative Gaussian curvature K and $f : M \rightarrow G(m, n)$ be a harmonic isometric immersion. If there exists a positive number $\varepsilon > 0$ so that $|\cos \alpha| \geq \varepsilon$, where α is the Kaehler angle of the immersion f , then at least one of the ∂' -transforms or ∂'' -transforms in harmonic sequence (1.1) generated by f is degenerate.*

Proof. Suppose that under the conditions of the theorem, none of the ∂' -transforms and ∂'' -transforms in (1.1) generated by f is degenerate, that is to say, $m = m_0 = m_{\pm 1} = \dots$. Equivalently speaking, square matrices $A_0, A_{\pm 1}, \dots$ are all non-singular. Without loss of generality, we may assume that $\cos \alpha \geq \varepsilon > 0$. Then, for any positive integer p , a direct computation together with (3.2) and (3.7) yields (c.f. [8])

$$(4.1) \quad \begin{aligned} \Delta \log \prod_{k=-1}^{-p} \prod_{j=-1}^k |\det A_j| \\ = \frac{1}{2}mp(p+1)K + (2p+1)\cos \alpha - 1 + 2|A_{-p-1}|^2. \end{aligned}$$

Since $\cos \alpha \geq \varepsilon > 0$, we can choose p such that $(2p + 1) \cos \alpha - 1 > 0$. Thus from (4.1) we conclude that

$$(4.2) \quad \Delta \log \prod_{k=-1}^{-p} \prod_{j=-1}^k |\det A_j| > 0,$$

from which it follows that the function

$$\prod_{k=-1}^{-p} \prod_{j=-1}^k |\det A_j|$$

is a subharmonic function on a complete surface M with non-negative Gaussian curvature, and it must be a constant. But this is in contradiction with (4.2). So the theorem is proved. \square

Corollary 4.2. *Let $f : M \rightarrow CP^n$ be an isometric minimal immersion of a complete and connected surface M with non-negative Gaussian curvature into CP^n . If the Kaehler angle α of the immersion f satisfies $|\cos \alpha| \geq \varepsilon$, where ε is a positive number, then f must be isotropy.*

Proposition 4.3. *Let $f : M \rightarrow G(m, (p + 1)m)$ be a harmonic map of a compact surface with genus g into $G(m, (p + 1)m)$. If none of the ∂' -transforms and ∂'' -transforms in harmonic sequence (1.1) generated by f is degenerate, and the isotropy order of f is p . Then $|\deg(f)| \leq mp(g - 1)$.*

Proof. It is easy to see that under the assumption of the proposition, $\underline{f}_0, \underline{f}_1, \dots, \underline{f}_p$ are mutually orthogonal, and that $\underline{f}_k = \underline{f}_{k+p+1}$, $k = 0, \pm 1, \dots$. Therefore, from the definition of $\deg(\cdot)$ it is clear that

$$(4.3) \quad \sum_{k=0}^p \deg(f_k) = 0,$$

from which together with (3.8) yields

$$(4.4) \quad \begin{aligned} \sum_{k=0}^p r(\det A_k) &= 2m(g - 1)(p + 1), \\ \sum_{k=0}^p (p - k)r(\det A_k) &= (p + 1) \deg(f_0) + mp(p + 1)(g - 1), \\ \sum_{k=0}^p kr(\det A_k) &= -(p + 1) \deg(f_0) + mp(p - 1)(g - 1). \end{aligned}$$

From (4.4) we get

$$\begin{aligned} |\deg(f)| &= |\deg(f_0)| = \frac{1}{2} \left| \sum_{k=0}^p \left(\frac{p - 2k}{p + 1} \right) r(\det A_k) \right| \\ &\leq \frac{1}{2} \sum_{k=0}^p \left(\left| \frac{p - 2k}{p + 1} \right| r(\det A_k) \right) \leq \frac{p}{2(p + 1)} \sum_{k=0}^p r(\det A_k) = mp(g - 1). \end{aligned}$$

Thus the proposition is proved. □

The following two theorems are the direct consequences of Proposition 4.3.

Theorem 4.4. *Let $f : M \rightarrow G(m, 2m)$ be a harmonic map of a compact surface M with genus g into $G(m, 2m)$. If $|\deg(f)| > m(g - 1)$, then at least one of the ∂' -transforms or ∂'' -transforms in (1.1) is degenerate.*

Theorem 4.5. *Let $f : M \rightarrow G(m, 3m)$ be a strongly conformal harmonic map of a compact surface M with genus g into $G(m, 3m)$. If $|\deg(f)| > 2m(g - 1)$, then at least one of the ∂' -transforms or ∂'' -transforms in (1.1) is degenerate.*

Remark. Theorem 4.5 generalizes Proposition 7.8 of [10].

Now let M be a compact surface with genus g and $f : M \rightarrow G(m, n)$ be a harmonic map which generates (1.1) with non-degenerate ∂' -transforms and ∂'' -transforms. This implies that $m = m_0 = m_{\pm 1} = \dots$. Suppose that the isotropy order of f is p so that

$$(4.5) \quad \begin{aligned} W_{k+i+1}^* W_k &= 0, & 0 \leq i \leq p - 1, \\ W_{k+p+1}^* W_k &\neq 0, \end{aligned}$$

here $k = 0, \pm 1, \dots$. Set $P_k = W_{k+p+1}^* W_k$, then from (3.1) and (4.5) we get

$$(4.6) \quad A_{k+p}^* P_k = P_{k-1} A_{k-1}^*, \quad \partial P_k = (P_k B_k - B_{k+p+1} P_k) \varphi.$$

It follows from (4.6) that $\text{rank}(P_0) = \text{rank}(P_{\pm 1}) = \dots$ except at isolated points. Thus we can choose the local frame $W_k, k = 0, 1, \dots, 2p + 1$ suitably such that

$$(4.7) \quad P_k = \begin{pmatrix} Q_k & 0 \\ 0 & 0 \end{pmatrix}, \quad k = 0, 1, \dots, p,$$

where Q_k 's are non-singular $(t \times t)$ -matrices except at isolated points, and $t = \text{rank}(P_0) = \text{rank}(P_1) = \dots$. Assume the corresponding blocks of the matrices A_k and B_k are

$$(4.8) \quad A_k = \begin{pmatrix} A_{k11} & A_{k12} \\ A_{k21} & A_{k22} \end{pmatrix}, \quad B_k = \begin{pmatrix} B_{k11} & B_{k12} \\ B_{k21} & B_{k22} \end{pmatrix}.$$

Combining (4.6)–(4.8) it follows that

$$(4.9) \quad \begin{aligned} A_{k21} &= 0, & 0 \leq k \leq p - 1, \\ B_{k12} &= 0, & 0 \leq k \leq p, \\ B_{k21} &= 0, & p + 1 \leq k \leq 2p + 1, \end{aligned}$$

from which together with (3.3) and (4.6) yields

$$(4.10) \quad \begin{aligned} dA_{k11} + (A_{k11} B_{k11}^* - B_{k+1,11}^* A_{k11}) \bar{\varphi} - \sqrt{-1} \rho A_{k11} &\equiv 0 \pmod{\varphi}, & 0 \leq k \leq p, \\ \partial Q_0 = (Q_0 B_{011} - B_{p+1,11} Q_0) \varphi. \end{aligned}$$

From (4.10) we can calculate out that

$$(4.11) \quad d \log (\det(A_{011} \dots A_{p11} Q_0^*)) - \sqrt{-1} t (p + 1) \rho \equiv 0 \pmod{\varphi},$$

and consequently,

$$(4.12) \quad \Delta \log |\det(A_{011} \dots A_{p11} Q_0^*)| = t(p+1)K.$$

Integrating (4.12) on M and making use of Gauss-Bonnet theorem, Lemma 4.1 of [9] and Noticing the fact that

$$(4.13) \quad r(\partial'_k) = r(|A_k|) \leq r(|A_{k11}|) \leq \frac{1}{t} r(\det A_{k11}),$$

we get

$$(4.14) \quad \sum_{k=0}^p r(\partial'_k) \leq 2(p+1)(g-1).$$

Similarly we can prove

$$(4.15) \quad \sum_{k=-1}^{p-1} r(\partial'_k) \leq 2(p+1)(g-1).$$

Note that $r(\partial'_{-1}) = r(|A_{-1}|) = r(\partial''_0)$, from (4.15) we get

$$(4.16) \quad r(\partial'_0) + r(\partial''_0) \leq 2(p+1)(g-1) \leq 2\frac{n}{m}(g-1).$$

By (4.16) we can easily obtain the following theorem.

Theorem 4.6. *Let $f : M \rightarrow G(m, n)$ be a harmonic map of a compact surface M with genus g into $G(m, n)$ which generates the harmonic sequence (1.1). If*

$$r(\partial'_0) + r(\partial''_0) > 2\frac{n}{m}(g-1),$$

then at least one of the ∂' -transforms or ∂'' -transforms in (1.1) is degenerate.

Remark. Theorem 4.6 generalizes the corresponding results in [4,5].

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