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ON TOTALLY REAL MINIMAL SUBMANIFOLDS IN COMPLEX PROJECTIVE SPACE

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Abstract. In this paper, we obtain some pinching theorems for totally real minimal submanifolds in complex projective space.

§1. Introduction

Let $CP^n(c)$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c(c > 0)$. The pinching problem for totally real minimal submanifolds in $CP^n(c)$ has been studied by many mathematicians. Montiel, Ros and Urbano [MRU] proved a pinching result about Ricci curvature condition. Recently, Matsuyama [M1,2] has discussed the scalar curvature case which give a positive answer for Ogiue's conjecture [O]. Now, in this paper, we give a pinching condition for the norm of the second fundamental form under which the submanifolds is totally geodesic.

Throughout this paper, we use the similar notations and formulas as those used in [MRU]. Let M be an n-dimensional compact Riemannian manifold. We denote by UM the unit tangent bundle over M and by UM_p its fibre at $p \in M$. For any continuous function $f\colon UM \to R$, we have

$$\int_{UM} f dv = \int_M \int_{UM_p} f dv_p dp$$

where dp, dv_p and dv stand for the canonical measures on M, UM_p and UM respectively.

If T is a k-covariant tensor on M and ∇T is covariant derivative, then we have ([R1])

$$\int_{UM} \Big\{ \sum_{i=1}^{n} (\nabla T)(e_i, e_i, v, \cdots .v) \Big\} dv = 0 \tag{1.1}$$

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where $e_1, \dots, e_n$ is an orthonormal basis of T_pM, $p \in M$.

Suppose now that M is isometrically immersed in an $(n+p)$-dimensional Riemannian manifold $\overline{M}^{n+p}$. We denote by $\langle , \rangle$ the metric of $\overline{M}$ as well as that induced on M. Let σ be the second fundamental form of the isometrically immersion and A_ξ the Weingarten endomorphism for a normal vector ξ. If T_pM and $T_p^\perp M$ denote the tangent and normal spaces to M at p, one can define

$$L\colon T_pM \to T_pM \quad \text{and} \quad T\colon T_p^\perp M \times T_p^\perp M \to R$$

by the expressions

$$Lv = \sum_{i=1}^n A_{\sigma(v,e_i)} e_i \quad \text{and} \quad T(\xi,\eta) = \text{trace } A_\xi A_\eta$$

where $e_1, \dots, e_n$ is an orthonormal basis of T_pM. Then L is a self-adjoint linear map and T a symmetric bilinear map.

There are many submanifolds satisfying $T = k\langle , \rangle$. Obviously, hypersurfaces represent a trivial case. In $CP^{n+p}(c)$, a Kaehler submanifold of order $\{k_1, k_2\}$ for some natural numbers k_1 and k_2 is one submanifold of this type ([R3]). In this paper, we have a pinching theorem for this kind of submanifolds as following:

Theorem 3.1. *Let M^n be a totally real minimal submanifold with $T = k\langle , \rangle$ in $CP^{n+p}(c)$. If*

$$|\sigma|^2 < \frac{nc(n+2p)(n+4)}{4(n+2)(n+4) + n(n+4)^2 + 4n},$$

then M must be totally geodesic.

§2. Some Lemmas

In this section, we will prove some lemmas which will be used later. First, we give the following modified version of Simons' formula which generalizes a result from [MRU]. Now we suppose that M is a curvature-invariant submanifold of $\overline{M}$, i.e., $\overline{R}(X,Y)Z \in T_pM$ for all X, Y, $Z \in T_pM$, being $\overline{R}$ the curvature operator of $\overline{M}$.

Lemma 2.1 [LC]**.** *Let M be an n-dimensional compact curvature-invariant submanifold with parallel mean curvature vector isometrically immersed in an $(n+p)$-dimensional Riemannian manifold $\overline{M}^{n+p}$. Then we have*

$$\begin{aligned} 0 = \int_{UM} \Big\{ &\sum_{i=1}^n \big|(\nabla\sigma)(e_i,v,v)\big|^2 + \sum_{i=1}^n \langle \sigma(e_i,e_i), A_{\sigma(v,v)}v\rangle \\ &+ (n+4)\big|A_{\sigma(v,v)}v\big|^2 - 4\langle Lv, A_{\sigma(v,v)}v\rangle - 2T(\sigma(v,v),\sigma(v,v)) \\ &+ \Big[\sum_{i=1}^n \overline{R}(e_i,v,\sigma(v,e_i),\sigma(v,v)) + 2\sum_{i=1}^n \overline{R}(e_i,v,v,A_{\sigma(v,e_i)}v)\Big]\Big\}\, dv \end{aligned}$$

Remark. When the immersion is minimal, Lemma 2.1 is due to [MRU].

Remark. It's clear that submanifolds in real space forms, Kahler, and totally real submanifolds in complex space forms are curvature-invariant.

Lemma 2.2. *Let M be an n-dimensional compact submanifold isometrically immersed in a Riemannian manifold $\overline{M}^{n+p}$. Then, for $\forall p \in M$, we have:*

i)
$$\int_{UM_p} \langle Lv, A_{\sigma(v,v)}v\rangle \, dv_p = \frac{2}{n+2}\int_{UM_p} \big|Lv\big|^2 dv_p + \frac{1}{n+2}\int_{UM_p} \langle \sigma(v,v), \xi\rangle \, dv_p$$

ii)
$$\int_{UM_p} \big|\sigma(v,v)\big|^2 dv_p = \frac{2}{n+2}\int_{UM_p} \langle Lv, v\rangle \, dv_p + \frac{1}{n+2}\int_{UM_p} \sum_{i=1}^{n} \langle \sigma(v,v), \sigma(e_i,e_i)\rangle \, dv_p$$

iii)
$$\int_{UM_p} \langle Lv, v\rangle \, dv_p = \frac{1}{n}\int_{UM_p} \big|\sigma\big|^2 dv_p$$

iv)
$$\int_{UM_p} \langle \sigma(v,v), \eta\rangle dv_p = \frac{1}{n}\int_{UM_p} \sum_{i=1}^{n} \langle \sigma(e_i,e_i), \eta\rangle \, dv_p$$

Where $\xi = \sum_{i=1}^{n} \sigma(e_i, Le_i)$ and η is a fixed vector in normal bundle.

Proof. Let α^1 be the 1-form on UM_p defined by

$$\alpha^1(e) = \langle Lv, A_{\sigma(v,v)}e\rangle, \quad v \in UM_p, \quad e \in T_vUM_p$$

For any $v \in UM_p$, let $e_1, \dots, e_{n-1}, e_n = v$ be an orthonormal basis of T_pM. Then

$$(\delta\alpha^1)(v) = -(n+2)\langle Lv, A_{\sigma(v,v)}v\rangle + 2\big|Lv\big|^2 + \langle \sigma(v,v), \xi\rangle\,.$$

Integrating it over UM_p, we obtain i).

ii), iii) and iv) are obtained by using the same technique for the 1-forms α^2, α^3 and α^4 on UM_p defined by

$$\begin{aligned} \alpha_v^2(e) &= \langle \sigma(v,v), \sigma(v,e)\rangle \\ \alpha_v^3(e) &= \langle Lv, e\rangle \\ \alpha_v^4(e) &= \langle \sigma(v,e), \eta\rangle \end{aligned}$$

□

Lemma 2.3. *Let M be an n-dimensional compact submanifold isometrically immersed in a Riemannian manifold $\overline{M}^{n+p}$. Then we have*

$$\int_{UM_p} \big|A_{\sigma(v,v)}v\big|^2 dv_p \geq \frac{2}{n+2}\int_{UM_p} \langle Lv, A_{\sigma(v,v)}v\rangle \, dv_p + \frac{1}{n+2}\int_{UM_p} \langle A_{\sigma(e_i,e_i)}v, A_{\sigma(v,v)}v\rangle \, dv_p$$

Proof. Let $\triangle$ denote the Laplace operator on S^{n-1}. Then, for the function $f : UM_p \to T_pM$ defined by $f(v) = A_{\sigma(v,v)}v$, we have

$$(\triangle f)(v) = -3(n+1)A_{\sigma(v,v)}v + 4Lv + 2A_{\sigma(e_i,e_i)}v\,.$$

Since UM_p is a $(n-1)$-dimensional sphere, the first eigenvalue of $-\triangle = \nabla_{\nabla_{e_i}e_i} - \nabla_{e_i}\nabla_{e_i}$ is $n-1$. Then

$$-\int_{UM_p} \langle \triangle f, f\rangle\, dv_p \ge (n-1)\int_{UM_p} |f|^2 dv_p$$

and the lemma follows. □

Let α be a 1-form on UM_p defined by

$$\alpha_v(e) = \langle A_{\sigma(v,v)}e, A_{\sigma(v,v)}v\rangle$$

where $v \in UM_p$, and $e \in T_vUM_p$. If $e_1, \dots, e_{n-1}$ is an orthnormal basis of T_vUM_p, then the codifferential of α is

$$\begin{aligned}(\delta\alpha) &= \sum_{i=1}^{n} e_i \cdot \alpha_v(e_i)\\ &= -(n+4)\big|A_{\sigma(v,v)}v\big|^2 + 2\langle Lv, A_{\sigma(v,v)}v\rangle\\ &\quad + T(\sigma(v,v),\sigma(v,v)) + 2\sum_{i=1}^{n}\langle A_{\sigma(v,v)}e_i, A_{\sigma(v,e_i)}v\rangle\,,\end{aligned}$$

where $e_1, \dots, e_{n-1}, e_n = v$ is an orthonormal basis of T_pM. Now integrating the above equality over UM_p and using divergence theorem, we have

$$\begin{aligned}&2\int_{UM_p}\Big\{\sum_{i=1}^{n}\langle A_{\sigma(v,v)}e_i, A_{\sigma(v,e_i)}v\rangle\Big\}\, dv_p\\ &\quad = (n+4)\int_{UM_p}\big|A_{\sigma(v,v)}v\big|^2 dv_p - 2\int_{UM_p}\langle Lv, A_{\sigma(v,v)}v\rangle\, dv_p\\ &\qquad - \int_{UM_p} T(\sigma(v,v),\sigma(v,v))\, dv_p \end{aligned}\tag{2.1}$$

In a similar way, for the 1-form α defined by

$$\alpha_v(e) = \langle A_{\sigma(v,e)}v, A_{\sigma(v,v)}v\rangle\,,$$

we have

$$\begin{aligned}(\delta\alpha)(v) &= \sum_{i=1}^{n}\{2\big|A_{\sigma(v,e_i)}v\big|^2 + \langle A_{\sigma(v,e_i)}v, A_{\sigma(v,v)}e_i\rangle\\ &\quad + \langle A_{\sigma(e_i,e_i)}v, A_{\sigma(v,v)}v\rangle\} - (n+4)\big|f(v)\big|^2 + \langle Lv, f(v)\rangle\,.\end{aligned}$$

Integrating this and using (2.1), we get

$$\begin{aligned} 2\int_{UM_p}\sum_{i=1}^{n}\left|A_{\sigma(v,e_i)}v\right|^2 dv_p = \int_{UM_p} & \Big\{\frac{n+4}{2}\left|f(v)\right|^2 - \langle A_{nH}v, f(v)\rangle \\ & + \frac{1}{2}T(\sigma(v,v),\sigma(v,v))\Big\}\, dv_p \end{aligned} \tag{2.2}$$

By (2.1),(2.2) and

$$\begin{aligned} 2\sum_{i=1}^{n}\langle A_{\sigma(v,e_i)}v, A_{\sigma(v,v)}e_i\rangle &\le a\sum_{i=1}^{n}\left|A_{\sigma(v,e_i)}v\right|^2 + \frac{1}{a}\sum_{i=1}^{n}\left|A_{\sigma(v,v)}e_i\right|^2 \\ &= a\sum_{i=1}^{n}\left|A_{\sigma(v,e_i)}v\right|^2 \frac{1}{a}T(\sigma(v,v),\sigma(v,v))\,, \end{aligned} \tag{2.3}$$

By (2.1),(2.2) and (2.3), we have, for $\forall\ b > 0$,

$$\begin{aligned} \int_{UM_p} & \Big\{\Big(n+4-\frac{b(n+4)}{4}\Big)\left|f(v)\right|^2 - 2\langle Lv, f(v)\rangle \\ & - \Big(1+\frac{b}{4}+\frac{1}{b}\Big)T(\sigma(v,v),\sigma(v,v))\Big\}\, dv \le 0\,. \end{aligned} \tag{2.4}$$

Now, we can prove the following lemma:

Lemma 2.4. *Let* $M^n \to \overline{M}^{n+p}$ *be a compact Riemannian immersion. Then we have*

$$\begin{aligned} & \int_{UM_p}(n+2)\langle A_H v, f(v)\rangle\, dv_p \\ & \quad = \int_{UM_p}\Big\{2\sum_{i=1}^{n}\langle A_H e_i, A_{\sigma(v,e_i)}v\rangle + T(H,\sigma(v,v))\Big\}\, dv_p \end{aligned} \tag{1}$$

$$\int_{UM_p}\langle A_H v, Lv\rangle\, dv_p = \int_{UM_p}\sum_{i=1}^{n}\langle A_H e_i, A_{\sigma(v,e_i)}v\rangle\, dv_p \tag{2}$$

$$\begin{aligned} \int_{UM_p}\langle A_H v, Lv\rangle\, dv_p &= \frac{1}{n}\int_{UM_p}\sum_{i=1}^{n}\langle A_H e_i, Le_i\rangle\, dv_p \\ &= \frac{1}{n}\int_{UM_p}\langle H\cdot\xi\rangle\, dv_p \end{aligned} \tag{3}$$

$$\int_{UM_p}T(H<\sigma(v,v))dv_p = \int_{UM_p}T(H,H)\, dv_p \tag{4}$$

$$\begin{aligned}(5)\qquad &\int_{UM_p}(n+2)T(\sigma(v,v),\sigma(v,v))\,dv_p\\ &\quad=\int_{UM_p}\{nT(H,\sigma(v,v))+2\sum_{i=1}^{n}T(\sigma(v,e_i),\sigma(v,e_i))\}\,dv_p\end{aligned}$$

$$(6)\qquad \int_{UM_p}\sum_{i=1}^{n}T(\sigma(v,e_i),\sigma(v,e_i))\,dv_p=\frac{1}{n}\int_{UM_p}\sum_{i,j=1}^{n}T(\sigma(e_i,e_j),\sigma(e_i,e_j))\,dv_p$$

$$(7)\qquad \int_{UM_p}\langle A_Hv,f(v)\rangle\,dv_p=\int_{UM_p}\Big\{\frac{1}{n+2}T(H,H)+\frac{2}{n(n+2)}\langle H,\xi\rangle\Big\}\,dv_p$$

$$\begin{aligned}(8)\qquad \int_{UM_p}T(\sigma(v,v),\sigma(v,v))\,dv_p=\int_{UM_p}&\Big\{\frac{n}{n+2}T(H,H)\\ &+\frac{2}{n(n+2)}\sum_{i,j=1}^{n}T(\sigma(e_i,e_j),\sigma(e_i,e_j))\Big\}\,dv_p\end{aligned}$$

$$\begin{aligned}(9)\qquad &\int_{UM_p}(2-\frac{b(n+4)}{4})\big|f(v)\big|^2dv_p\\ &\le\int_{UM_p}\Big\{\Big(1+\frac{b}{4}+\frac{1}{b}\Big)T(\sigma(v,v),\sigma(v,v))-\Big(1+\frac{b}{2}\Big)n\langle A_Hv,f(v)\rangle\Big\}\,dv_p\,,\end{aligned}$$

for each b.

Proof. By taking some proper 1-form on UM_p respectively as above, we can obtain (1) $\sim$ (6) and then (7) and (8) as their corollaries. Using Lemma 2.3, (2.4) implies (9). □

Remark. When $b(>0)$ is small, (9) gives a estimation of the upper bound of $\big|f(v)\big|^2$.

§3. TOTALLY REAL SUBMANIFOLDS WITH $T = k\langle,\rangle$ IN COMPLEX PROJECTIVE SPACES

There are many submanifolds satisfying $T = k\langle,\rangle$. Obviously, hypersurfaces represent a trivial case. In $CP^{n+p}(c)$, a Kaehler submanifold of order $\{k_1,k_2\}$ for some natural numbers k_1 and k_2 is one submanifold of this type ([R3]). Let M^n be a totally real minimal submanifold with $T = k\langle , \rangle$ immersed in $CP^{n+p}(c)$. Then

$$\begin{aligned} P(\overline{R}) =& \sum_{i=1}^{n} \overline{R}(e_i, v, \sigma(v, e_i), \sigma(v, v)) + 2\sum_{i=1}^{n} \overline{R}(e_i, v, v, A_{\sigma(v,e_i)}v) \\ =& \frac{c}{2}\langle Lv, v\rangle - \frac{c}{2}\left|\sigma(v, v)\right|^2 + \frac{c}{4}\sum_{i=1}^{n}\langle \sigma(v, v), Je_i\rangle^2 \\ & - \frac{c}{4}\sum_{i=1}^{n}\langle Jv, \sigma(e_i, e_i)\rangle\langle Jv, \sigma(v, v)\rangle \, . \end{aligned} \tag{3.1}$$

Now, we define a map $g^1 : UM_p \to T_pM$ by

$$g^1(v) = A_{\sigma(v,v)}v - Lv \, .$$

By a direct computation, we have

$$(-\triangle g^1)(v) = 3(n+1)f(v) - (n+3)Lv - 2nA_Hv \, .$$

Here $\triangle$ is the Laplacian of UM_p. Since $\int_{UM_p} g^1(v)dv_p = 0$, we get

$$\int_{UM_p}\langle(-\triangle g^1)(v), g^1(v)\rangle \geq (n-1)\int_{UM_p}\left|g^1(v)\right|^2 .$$

Then, the above relation gives

$$\begin{aligned} \int_{UM_p} \{&(2n+4)\left|f(v)\right|^2 - (2n+8)\langle Lv, f(v)\rangle \\ & - 2n\langle f(v), A_Hv\rangle + 4\left|Lv\right|^2 + 2n\langle Lv, A_Hv\rangle\}\, dv_p \geq 0 \, . \end{aligned} \tag{3.2}$$

In a similar way, for the 1-form $g^2(v) = f(v) + Lv$, we have

$$\begin{aligned} \int_{UM_p} \{&(2n+4)\left|f(v)\right|^2 - 2n\langle Lv, f(v)\rangle \\ & - 2n\langle f(v), A_Hv\rangle - 4\left|Lv\right|^2 - 2n\langle Lv, A_Hv\rangle\}\, dv_p \geq 0 \, . \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we get

$$\begin{aligned} \int_{UM_p} \{&(2n+4)\left|f(v)\right|^2 - (2kn+4k+4)\langle Lv, f(v)\rangle \\ & - 2n\langle f(v), A_Hv\rangle + 4k\left|Lv\right|^2 - 2nk\langle Lv, A_Hv\rangle\}\, dv_p \geq 0 \, . \end{aligned} \tag{3.4}$$

Since M is minimal, by (3.4) with $k=-\frac{2}{n+2}$, we have

$$\int_{UM_p}\left|f(v)\right|^2 dv_p \geq \frac{4}{(n+2)^2}\int_{UM_p}\left|Lv\right|^2 dv_p\,.$$

From this and Lemma 2.2 i) we get

$$\int_{UM_p}\left|f(v)\right|^2 dv_p \geq \frac{2}{n+2}\int_{UM_p}\langle Lv, f(v)\rangle\, dv_p\,. \tag{3.5}$$

From (3.1),(3.5) and Lemma 2.1 we have

$$\begin{aligned} 0 = \int_{UM} \Big\{ &\sum_{i=1}^{n}\left|(\nabla\sigma)(e_i,v,v)\right|^2 + (n+4)\left|f(v)\right|^2 \\ &- 4\langle Lv, f(v)\rangle - 2T(\sigma(v,v),\sigma(v,v)) \\ &+ \Big[\frac{c}{2}\langle Lv, v\rangle - \frac{c}{2}\left|\sigma(v,v)\right|^2 + \frac{c}{4}\sum_{i=1}^{n}\langle\sigma(v,v), Je_i\rangle^2\Big]\Big\}\, dv \\ \geq \int_{UM} \Big\{ &\sum_{i=1}^{n}\left|(\nabla\sigma)(e_i,v,v)\right|^2 + \frac{nc}{4}\left|\sigma(v,v)\right|^2 \\ &- n\left|f(v)\right|^2 - 2T(\sigma(v,v),\sigma(v,v))\Big\}\, dv\,. \end{aligned} \tag{3.6}$$

Assuming now that M is minimal, and putting $b=\frac{4}{n+4}$ in formula (9) of Lemma 2.4,we obtain

$$\int_{UM_p}\left|f(v)\right|^2 dv_p \leq \Big(1+\frac{1}{n+4}+\frac{n+4}{4}\Big)\int_{UM_p} T(\sigma(v,v),\sigma(v,v))\, dv_p\,. \tag{3.7}$$

By (3.6), (3.7) and the fact that $T=\frac{\left|\sigma\right|^2}{2p+n}g$ we get

$$\begin{aligned} 0 \geq \int_{UM} \Big\{ &\sum_{i=1}^{n}\left|(\nabla\sigma)(e_i,v,v)\right|^2 \\ &+ \Big[\frac{nc}{4} - \frac{n(1+\frac{1}{n+4}+\frac{n+4}{4})+2}{2p+n}\left|\sigma\right|^2\Big]\cdot\left|\sigma(v,v)\right|^2\Big\}\, dv\,. \end{aligned}$$

From this we immediately have

Theorem 3.1. *Let M^n be a totally real minimal submanifold with $T=k\langle,\rangle$ in $CP^{n+p}(c)$. If*

$$\left|\sigma\right|^2 < \frac{nc(n+2p)(n+4)}{4(n+2)(n+4)+n(n+4)^2+4n}\,, \tag{3.8}$$

then M must be totally geodesic.

Remark. Xia [X] gave a pinching constant $\frac{nc}{6}$ without the assumption: $T = k\langle , \rangle$. When $p > \frac{n(n+4)}{12} + \frac{2}{3} + \frac{n}{3(n+4)} - \frac{n}{6}$, our pinching constant is larger than Xia's.

Remark. When the target manifold is the quaternionic space form $QP^{n+p}(c)$, we have also a corresponding result, i.e., changing the factor $n + 2p$ in (3.8) to $3n + 4p$. So our result is better than that of [Sh1] in case when p is large enough.

Remark. B. Y. Chen and K. Ogiue ([CO]) had proved that, for a submanifold M of nonflat complex space form, M is curvature-invariant if and only if M is holomorphic or totally real submanifold. So we can use Lemma 2.1 in the proof of Theorem 3.1.

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REFERENCES

[CO] Chen, B. Y. and Ogiue, K., *On totally real submanifolds*, Trans. Amer. Math. Soc. **193** (1994), 257–266.

[LC] Li, Y. W and Chao, X. L., *A modified version of Simons formula*, preprint (2001).

[M1] Matsuyama, Y., *Curvature pinching for totally real submanifolds of complex projective space*, J. Math. Soc. Japan **52** (2000), 51–64.

[M2] Matsuyama, Y., *On totally real submanifolds of a complex projective space*, Nihonkai Math. J. **13** (2002), 153–157.

[MRU] Montiel, S., Ros, A. and Urbano, F., *Curvature pinching and eigenvalue rigidity for minimal submanifolds*, Math. Z. **191** (1986), 537-548.

[O] Ogiue, K., *Recent topics of differential geometry*, Mathematics: the publication of the Mathematical Society of Japan **39** (1987), 51–64.

[R1] Ros, A., *A characterization of seven compact Kahler submanifolds by holomorphic pinching*, Ann. Math. **121** (1985), 377–382.

[R2] Ros A., *Eigenvalue inequalities for minimal submanifolds and P-manifolds*, Math. Z. **187** (1984), 393–404.

[R3] Ros, A., *On spectral geometry of Kaehlar submanifolds*, J. Math. Soc. Japan **36** (1984), 433–448.

[Si] Simons, J., *Minimal verieties in Riemannian manifolds*, Ann. Math. **88** (1968), 62–65.

[Sh1] Shen, Y. B., *Totally real minimal submanifolds in quaternionic projective space*, Chin. Ann. Math. **14B** (1993), 297–306.

[Sh2] Shen, Y. B., *On scalar curvature of totally real minimal submanifolds*, Chin. Ann. Math. **12A** (1991), 573–577.

[Sh3] Shen, Y. B., *On curvature pinching for minimal and Kahler submanifolds with isotropic second fundamental form*, Chin. Ann. Math. **12B** (1991), 454–463.

[Ub] Urbano F., *Totally real submanifolds*, Geometry and Topology of Submanifolds, Proceedings (1987), 198–208.

[X] Xia, C. Y., *On the minimal submanifolds in* $CP^m(c)$ *and* $S^n(1)$, Kodai Math. J. **15** (1992), 141–153.

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