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Suthep Suantai

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ON THE H–PROPERTY OF SOME BANACH SEQUENCE SPACES

SUTHEP SUANTAI

Abstract. In this paper we define a generalized Cesàro sequence space ces (p) and consider it equipped with the Luxemburg norm under which it is a Banach space, and we show that the space ces (p) posses property (H) and property (G), and it is rotund, where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$.

1. Preliminaries

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X, respectively. A point $x_0 \in S(X)$ is called

a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;

b) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x_0 (write $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$;

c) a *denting point* if for every $\epsilon > 0$, $x_0 \notin \overline{\text{conv}}\{B(X)\backslash(x_0 + \epsilon B(X))\}$.

A Banach space X is said to be *rotund* (R), if every point of $S(X)$ is an extreme point.

A Banach space X is said to posses property (H) (property (G)) provided every point of $S(X)$ is H-point (denting point).

For these geometric notions and their role in mathematics we refer to the monographs [1], [2], [6] and [13]. Some of them were studied for Orlicz spaces in [3], [7], [8], [9] and [114].

Let us denote by l^0 the space of all real sequences. For $1 \leq p < \infty$, the Cesàro sequence space (ces $_p$, for short) is defined by

$$\text{ces}_p = \left\{ x \in l^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < \infty \right\}$$

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equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^{p}\right)^{\frac{1}{p}}$$

This space was introduced by J. S. Shue [16]. It is useful in the theory of matrix operator and others (see [10] and [12]). Some geometric properties of the Cesàro sequence space ces_p were studied by many mathematicians. It is known that ces_p is LUR and posses property (H) (see [12]). Y. A. Cui and H. Hudzik [14] proved that ces_p has the Banach-Saks of type p if $p > 1$, and it was shown in [5] that ces_p has property (β).

Now, let $p = (p_k)$ be a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Nakano sequence space $l(p)$ is defined by

$$l(p) = \{x \in l^0 : \sigma(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\},$$

where $\sigma(x) = \sum_{i=1}^{\infty}|x(i)|^{p_i}$. We consider the space $l(p)$ equipped with the norm

$$\|x\| = \inf\left\{\lambda > 0 : \sigma\Big(\frac{x}{\lambda}\Big) \leq 1\right\},$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$l(p) = \left\{x \in l^0 : \sum_{i=1}^{\infty}|x(i)|^{p_i} < \infty\right\}.$$

Several geometric properties of $l(p)$ were studied in [1] and [4].

The Cesàro sequence space $\operatorname{ces}(p)$ is defined by

$$\operatorname{ces}(p) = \{x \in l^0 : \varrho(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\},$$

where $\varrho(x) = \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n}$. We consider the space $\operatorname{ces}(p)$ equipped with the so-called Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 : \rho\Big(\frac{x}{\lambda}\Big) \leq 1\right\}$$

under which it is a Banach space. If $p = (p_k)$ is bounded, then we have

$$\operatorname{ces}(p) = \left\{x = x(i) : \sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^{p_n} < \infty\right\}.$$

W. Sanhan [15] proved that $\operatorname{ces}(p)$ is nonsquare when $p_k > 1$ for all $k \in \mathbb{N}$. In this paper, we show that the Cesàro sequence space $\operatorname{ces}(p)$ equipped with the Luxemburg norm is rotund (R) and posses property (H) and property (G) when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

Throughout this paper we assume that $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$, and $M = \sup_k p_k$.

2. Main Results

We begin with giving some basic properties of modular on the space $\operatorname{ces}(p)$.

Proposition 2.1. *The functional* ϱ *on the Cesàro sequence space* $\operatorname{ces}(p)$ *is a convex modular.*

Proof. It is obvious that $\varrho(x) = 0 \Leftrightarrow x = 0$ and $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$. If $x, y \in \operatorname{ces}(p)$ and $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$, by the convexity of the function $t \to |t|^{p_k}$ for every $k \in \mathbb{N}$, we have

$$\begin{aligned}
\varrho(\alpha x + \beta y) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |\alpha x(i) + \beta y(i)| \right)^{p_k} \\
&\leq \sum_{k=1}^{\infty} \left(\alpha \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right) + \beta \left(\frac{1}{k} \sum_{i=1}^{k} |y(i)| \right) \right)^{p_k} \\
&\leq \alpha \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \beta \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |y(i)| \right)^{p_k} \\
&= \alpha \varrho(x) + \beta \varrho(y) \, .
\end{aligned}$$

Proposition 2.2. *For* $x \in \operatorname{ces}(p)$, *the modular* ϱ *on* $\operatorname{ces}(p)$ *satisfies the following properties:*

(i) *if* $0 < a < 1$, *then* $a^M \varrho(\frac{x}{a}) \leq \varrho(x)$ *and* $\varrho(ax) \leq a\varrho(x)$,

(ii) *if* $a \geq 1$, *then* $\varrho(x) \leq a^M \varrho(\frac{x}{a})$,

(iii) *if* $a \geq 1$, *then* $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.

Proof. It is obvious that (iii) is satisfied by the convexity of ϱ. It remains to prove (i) and (ii).

For $0 < a < 1$, we have

$$\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} = \sum_{k=1}^{\infty} \left(\frac{a}{k} \sum_{i=1}^{k} \left| \frac{x(i)}{a} \right| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} a^{p_k} \left(\frac{1}{k} \sum_{i=1}^{k} \left| \frac{x(i)}{a} \right| \right)^{p_k} \geq \sum_{k=1}^{\infty} a^M \left(\frac{1}{k} \sum_{i=1}^{k} \left| \frac{x(i)}{a} \right| \right)^{p_k} \\
&= a^M \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} \left| \frac{x(i)}{a} \right| \right)^{p_k} = a^M \varrho\Big(\frac{x}{a}\Big) \, ,
\end{aligned}$$

and it implies by the convexity of ϱ that $\varrho(ax) \leq a\varrho(x)$, hence (i) is satisfied.

Now, suppose that $a \geq 1$. Then we have

$$\varrho(x) = \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} = \sum_{k=1}^{\infty} a^{p_k}\left(\frac{1}{k}\sum_{i=1}^{k}\left|\frac{x(i)}{a}\right|\right)^{p_k}$$
$$\leq a^M \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\left|\frac{x(i)}{a}\right|\right)^{p_k} = a^M \varrho\Big(\frac{x}{a}\Big).$$

So (ii) is obtained. □

Next, we give some relationships between the modular ϱ and the Luxemburg norm on $\operatorname{ces}(p)$.

Proposition 2.3. *For any $x \in \operatorname{ces}(p)$, we have*
(i) *if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$,*
(ii) *if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$,*
(iii) *$\|x\| = 1$ if and only if $\varrho(x) = 1$,*
(iv) *$\|x\| < 1$ if and only if $\varrho(x) < 1$,*
(v) *$\|x\| > 1$ if and only if $\varrho(x) > 1$,*
(vi) *if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$, and*
(vii) *if $a \geq 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.*

Proof. (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|.\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ and $\varrho(\frac{x}{\lambda}) \leq 1$. From Proposition 2.2 (i) and (iii), we have

$$\varrho(x) \leq \varrho\left(\frac{(\|x\| + \epsilon)}{\lambda}x\right) = \varrho\left((\|x\| + \epsilon)\frac{x}{\lambda}\right)$$
$$\leq (\|x\| + \epsilon)\,\varrho\Big(\frac{x}{\lambda}\Big) \leq \|x\| + \epsilon\,,$$

which implies that $\varrho(x) \leq \|x\|$, so (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \frac{\|x\|-1}{\|x\|}$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By definition of $\|.\|$ and by Proposition 2.2 (i), we have

$$1 < \varrho\left(\frac{x}{(1-\epsilon)\|x\|}\right) \leq \frac{1}{(1-\epsilon)\|x\|}\varrho(x)\,,$$

so $(1-\epsilon)\|x\| < \varrho(x)$ for all $\epsilon \in \big(0, \frac{\|x\|-1}{\|x\|}\big)$. This implies that $\|x\| \leq \varrho(x)$, hence (ii) is obtained.

(iii) Assume that $\|x\| = 1$. By definition of $\|x\|$, we have that for $\epsilon > 0$, there exists $\lambda > 0$ such that $1 + \epsilon > \lambda > \|x\|$ and $\varrho\big(\frac{x}{\lambda}\big) \leq 1$. From Proposition 2.2 (ii), we have $\varrho(x) \leq \lambda^M \varrho\big(\frac{x}{\lambda}\big) \leq \lambda^M < (1+\epsilon)^M$, so $(\varrho(x))^{\frac{1}{M}} < 1 + \epsilon$ for all $\epsilon > 0$, which implies $\varrho(x) \leq 1$. If $\varrho(x) < 1$, then we can choose $a \in (0,1)$ such that

$\varrho(x) < a^M < 1$. From Proposition 2.2 (i), we have $\varrho\left(\frac{x}{a}\right) \le \frac{1}{a^M}\varrho(x) < 1$, hence $\|x\| \le a < 1$, which is a contradiction. Therefore $\varrho(x) = 1$.

On the other hand, assume that $\varrho(x) = 1$. Then $\|x\| \le 1$. If $\|x\| < 1$, we have by (i) that $\varrho(x) \le \|x\| < 1$, which contradicts our assumption. Therefore $\|x\| = 1$.

(iv) follows directly from (i) and (iii).

(v) follows from (iii) and (iv).

(vi) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\left\|\frac{x}{a}\right\| > 1$. By (v), we have $\varrho\left(\frac{x}{a}\right) > 1$. Hence, by Proposition 2.2 (i), we obtain that $\varrho(x) \ge a^M \varrho(\frac{x}{a}) > a^M$.

(vii) Suppose $a \ge 1$ and $\|x\| < a$. Then $\left\|\frac{x}{a}\right\| < 1$. By (iv), we have $\varrho\left(\frac{x}{a}\right) < 1$. If $a = 1$, it is obvious that $\varrho(x) < 1 = a^M$. If $a > 1$, then, by Proposition 2.2 (ii), we obtain that $\varrho(x) \le a^M \varrho(\frac{x}{a}) < a^M$. □

Proposition 2.4. *Let* (x_n) *be a sequence in* $\operatorname{ces}(p)$.

(i) *If* $\|x_n\| \to 1$ *as* $n \to \infty$, *then* $\varrho(x_n) \to 1$ *as* $n \to \infty$.

(ii) *If* $\varrho(x_n) \to 0$ *as* $n \to \infty$, *then* $\|x_n\| \to 0$ *as* $n \to \infty$.

Proof. (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0, 1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < 1 + \epsilon$ for all $n \ge N$. By Proposition 2.3 (vi) and (vii), we have $(1-\epsilon)^M < \varrho(x_n) < (1+\epsilon)^M$ for all $n \ge N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) Suppose $\|x_n\| \not\to 0$ as $n \to \infty$. Then there is an $\epsilon \in (0, 1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proposition 2.3 (vi), we have $\varrho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho(x_n) \not\to 0$ as $n \to \infty$. □

Next, we shall show that $\operatorname{ces}(p)$ has the property (H). To do this, we need a lemma.

Lemma 2.5. *Let* $x \in \operatorname{ces}(p)$ *and* $(x_n) \subseteq \operatorname{ces}(p)$. *If* $\varrho(x_n) \to \rho(x)$ *as* $n \to \infty$ *and* $x_n(i) \to x(i)$ *as* $n \to \infty$ *for all* $i \in \mathbb{N}$, *then* $x_n \to x$ *as* $n \to \infty$.

Proof. Let $\epsilon > 0$ be given. Since $\rho(x) = \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} < \infty$, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} < \frac{\epsilon}{3}\frac{1}{2^{M+1}}. \tag{2.1}$$

Since $\rho(x_n) - \sum_{k=1}^{k_0}(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|)^{p_k} \to \rho(x) - \sum_{k=1}^{k_0}(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)^{p_k}$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that

$$\varrho(x_n) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right)^{p_k} < \varrho(x) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} \tag{2.2}$$

for all $n \ge n_0$, and

$$\sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} < \frac{\epsilon}{3}. \tag{2.3}$$

for all $n \geq n_0$.
It follows from (2.1), (2.2) and (2.3) that for $n \geq n_0$,

$$
\begin{aligned}
\varrho(x_n - x) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} \\
&= \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} \\
&< \frac{\epsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(\varrho(x_n) - \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&< \frac{\epsilon}{3} + 2^M \left(\varrho(x) - \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(2 \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} \right) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

This show that $\varrho(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 2.4 (ii), we have $\|x_n - x\| \to 0$ as $n \to \infty$.

Theorem 2.6. *The space* $\operatorname{ces}(p)$ *has the property* (H).

Proof. Let $x \in S(\operatorname{ces}(p))$ and $(x_n) \subseteq \operatorname{ces}(p)$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. From Proposition 2.3 (iii), we have $\varrho(x) = 1$, so it follows from Proposition 2.4 (i) that $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$. Since the mapping $p_i : \operatorname{ces}(p) \to \mathbb{R}$, defined by $p_i(y) = y(i)$, is a continuous linear functional on $\operatorname{ces}(p)$, it follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus, we obtain by Lemma 2.5 that $x_n \to x$ as $n \to \infty$. □

Theorem 2.7. *The space* $\operatorname{ces}(p)$ *is rotund.*

Proof. Let $x \in S\big(\operatorname{ces}(p)\big)$ and $y, z \in B\big(\operatorname{ces}(p)\big)$ with $x = \frac{y+z}{2}$. By Proposition 2.3 and the convexity of ϱ we have

$$1 = \varrho(x) \leq \frac{1}{2}\big(\varrho(y) + \varrho(z)\big) \leq \frac{1}{2}(1+1) = 1\,,$$

so that $\varrho(x) = \frac{1}{2}(\varrho(y) + \varrho(z)) = 1$. This implies that

$$\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{y(i)+z(i)}{2}|\right)^{p_k} = \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)^{p_k} + \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}|z(i)|\right)^{p_k} \tag{2.4}$$

for all $k \in \mathbb{N}$.

We shall show that $y(i) = z(i)$ for all $i \in \mathbb{N}$.

From (2.4), we have

$$|x(1)|^{p_1} = \left|\frac{y(1)+z(1)}{2}\right|^{p_1} = \frac{1}{2}\big(|y(1)|^{p_1} + |z(1)|^{p_1}\big). \tag{2.5}$$

Since the mapping $t \to |t|^{p_1}$ is strictly convex, it implies by (2.5) that $y(1) = z(1)$.

Now assume that $y(i) = z(i)$ for all $i = 1, 2, 3, ..., k-1$. Then $y(i) = z(i) = x(i)$ for all $i = 1, 2, 3, \ldots, k-1$. From (2.4), we have

$$\begin{aligned}\left(\frac{1}{k}\sum_{i=1}^{k}\left|\frac{y(i)+z(i)}{2}\right|\right)^{p_k} &= \left(\frac{\frac{1}{k}\sum_{i=1}^{k}|y(i)| + \frac{1}{k}\sum_{i=1}^{k}|z(i)|}{2}\right)^{p_k} \\ &= \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)^{p_k} + \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}|z(i)|\right)^{p_k}\end{aligned} \tag{2.6}$$

By convexity of the mapping $t \to |t|^{p_k}$, it implies that $\frac{1}{k}\sum_{i=1}^{k}|y(i)| = \frac{1}{k}\sum_{i=1}^{k}|z(i)|$. Since $y(i) = z(i)$ for all $i = 1, 2, 3, \ldots, k-1$, we get that

$$|y(k)| = |z(k)|\,. \tag{2.7}$$

If $y(k) = 0$, then we have $z(k) = y(k) = 0$. Suppose that $y(k) \neq 0$. Then $z(k) \neq 0$. If $y(k)z(k) < 0$, it follows from (2.7) that $y(k) + z(k) = 0$. This implies by (2.6) and (2.7) that

$$\left(\frac{1}{k}\sum_{i=1}^{k-1}|x(i)|\right)^{p_k} = \left(\frac{1}{k}\left(\sum_{i=1}^{k-1}|x(i)| + |y(k)|\right)\right)^{p_k},$$

which is a contradiction. Thus, we have $y(k)z(k) > 0$. This implies by (2.5) that $y(k) = z(k)$. Thus, we have by induction that $y(i) = z(i)$ for all $i \in \mathbb{N}$, so $y = z$. □

Bor-Luh Lin, Pei-Kee Lin and S. L. Troyanski proved (cf. Theorem iii [11]) that element x in a bounded closed convex set K of a Banach space is a denting point of K iff x is an H-point of K and x is an extreme point of K. Combining this result with our results (Theorem 2.6 and Theorem 2.7), we obtain the following result.

Corollary 2.8. *The space* ces(p) *has the property* (G).

For $1 < r < \infty$, let $p = (p_k)$ with $p_k = r$ for all $k \in \mathbb{N}$. We have that $\text{ces}_r = \text{ces}(p)$, so the following results are obtained directly from Theorem 2.6, Theorem 2.7 and Corollary 2.8, respectively.

Corollary 2.9. *For* $1 < r < \infty$, *the Cesàro sequence space* ces_r *has the property* (H).

Corollary 2.10. *For* $1 < r < \infty$, *the Cesàro sequence space* ces_r *is rotund.*

Corollary 2.11. *For* $1 < r < \infty$, *the Cesàro sequence space* ces_r *has the property* (G).

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Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
E-mail: scmti005@chiangmai.ac.th