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EXISTENCE OF TWO SOLUTIONS FOR QUASILINEAR PERIODIC DIFFERENTIAL EQUATIONS WITH DISCONTINUITIES

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Abstract. In this paper we examine a quasilinear periodic problem driven by the one- dimensional p-Laplacian and with discontinuous forcing term f. By filling in the gaps at the discontinuity points of f we pass to a multivalued periodic problem. For this second order nonlinear periodic differential inclusion, using variational arguments, techniques from the theory of nonlinear operators of monotone type and the method of upper and lower solutions, we prove the existence of at least two non trivial solutions, one positive, the other negative.

1. Introduction

The purpose of this paper is to study the following quasilinear periodic problem

$$\begin{aligned} -(|x'(t)|^{p-2}x'(t))' = f(t,x(t))\,, \quad & \text{a.e. on} \quad T=[0,b] \\ x(0)=x(b)\,,\ x'(0)=x'(b)\,, \quad & \quad 2\le p<\infty\,. \end{aligned} \tag{1}$$

where $f : T\times\mathbb{R}\to\mathbb{R}$ is a function. We do not assume that the right hand side function $f(t,x)$ is continuous in the variable $x\in\mathbb{R}$. So problem (1) need not have a solution. In order to be able to develop a satisfactory existence theory, we need to pass to a multivalued problem by, roughly speaking, filling in the gaps at the discontinuity points of $f(t,\cdot)$. More precisely, we introduce the following two functions:

$$f_1(t,x)=\liminf_{x'\to x} f(t,x') \quad \text{and} \quad f_2(t,x)=\limsup_{x'\to x} f(t,x')$$

and then we define the multifunction $\widehat{f}(t,x)=[f_1(t,x),f_2(t,x)]$. Evidently, if $f(t,\cdot)$ is continuous at x, then $\widehat{f}(t,x)=\{f(t,x)\}$. Instead of (1) we examine the

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following multivalued variant of it:

$$(2) \qquad \begin{aligned} &-(|x'(t)|^{p-2}x'(t))' \in \widehat{f}(t,x(t))\,, \qquad \text{a.e. on} \qquad T=[0,b] \\ &x(0)=x(b)\,,\ x'(0)=x'(b)\,, \qquad\qquad 2 \le p < \infty\,. \end{aligned}$$

Our goal is to establish the existence of at least two nontrivial solutions for problem (2). Earlier works on the periodic problem deal with semilinear equations (i.e. $p=2$) which have a continuous right hand side f and they prove the existence of only one nontrivial solution. We refer to the papers of S. Ahmad-A. Lazer [1], of A. Fonda-D. Lupo [9] and of J.-P. Gossez-P. Omari [11]. Multiplicity results were proved by J. Mawhin-M. Willem [17] for the forced pendulum equation and by C. Fabry-J. Mawhin-M. N. Nkashama [8], where f depends also in x' but it is continuous in all three variables. They prove an Ambrosetti-Prodi type theorem for the equation they examine.

The quasilinear problem, driven by the one-dimensional p-Laplacian, was studied only recently. We refer to the works of L. Boccardo-P. Drábek-D. Giacchetti-M. Kučera [2], H. Dang-S. F. Oppenheimer [3], M. Del Pino-M. Elgueta-R. Manasevich [4], P. Drábek [6], C. Fabry-D. Fayyad [7], Z. Guo [12] and R. Manasevich-J. Mawhin [16]. Of the above papers L. Boccardo-P. Drábek-D. Giacchetti-M. Kučera [2], M. Del Pino-M. Elgueta-R. Manasevich [4] and P. Drábek [6] deal with the Dirichlet problem, R. Manasevich-J. Mawhin [16] study the periodic problem, Z. Guo [12] studies both the periodic and the Neumann problems and finally H. Dang-S. F. Oppenheimer [3] study all three problems (i.e. the Dirichelet, periodic and Neumann problems). It should be mentioned that H. Dang-S. F. Oppenheimer [3] and R. Manasevich-J. Mawhin [16] have a general p-Laplacian-like operator which is not necessarily homogeneous and has no growth restrictions. Moreover, R. Manasevich-J. Mawhin [16] consider systems. The multiplicity question for quasilinear equations was addressed by M. Del Pino-R.Manasevich-A. Murua [5] who assumed that f is continuous on both variables and used conditions on the interaction of the vector field f with the Fučík spectrum of the p-Laplacian differential operator.

In this paper we prove the existence of at least two nontrivial solutions for problem (2). One solution is positive and the other is negative. Our approach uses variational arguments, the theory of operators of monotone type and the method of upper and lower solutions.

2. Mathematical preliminaries

Let X be a reflexive Banach space and X^* its topological dual. A map $A : D \subset X \to 2^{X^*}$ is said to be "monotone" if, for all $x, y \in D$ and for all $x^* \in A(x), y^* \in A(y)$, we have $(x^* - y^*, x - y) \ge 0$ (here by $(\cdot,\cdot)$ we denote the duality brackets for the pair (X^*, X)). If $(x^* - y^*, x - y) = 0$ implies that $x = y$, we say that A is "strictly monotone ". The map A is said to be "maximal monotone", if the graph of A, Gr $A = \{[x, x^*] \in X \times X^* : x^* \in A(x)\}$, is maximal with respect to inclusion among the graphs of the all monotone maps. It is easy to see that the graph of a maximal monotone map A is sequentially closed in $X \times X^*_w$ and in $X_w \times X^*$ (here by

X_w and X_w^* we denote the spaces X and X^* furnished with their respective weak topologies). A map $A : X \to X^*$ which is single-valued and everywhere defined (i.e. $D = X$) is said to be "demicontinuous", if for any sequence $\{x_n\}_{n\ge1} \subset X$ such that $x_n \to x$ in X it follows that $Ax_n \rightharpoonup Ax$ weakly in X^*. A monotone, demicontinuous map is maximal monotone. A map $A : D \subset X \to 2^{X^*}$ is said to be "coercive", if D is bounded or D is unbounded and $\frac{\inf[\|x^*\| : x^* \in A(x)]}{\|x\|} \to \infty$ as $\|x\| \to \infty$. A maximal monotone, coercive map is surjective.

A map $A : D \subset X \to 2^{X^*}$ is said to be "pseudomonotone", if

(a) for every $x \in X$, $A(x)$ is nonempty, weakly compact and convex in X^*;

(b) A is upper semicontinuous as a multifunction from X into X_w^*; i.e. for every $C \subset X^*$ nonempty and weakly closed, the set $A^-(C) = \{x \in X : A(x) \cap C \neq \emptyset\}$ is closed;

(c) if $\{x_n\}_n \subset X$ is a sequence such that $x_n \rightharpoonup x$ weakly in X and if $x_n^* \in A(x_n), n \ge 1$ is such that $\limsup\limits_{n\to\infty}(x_n^*, x_n - x) \le 0$ then, for every $y \in X$, there exists $x^*(y) \in A(x)$ such that $(x^*(y), x - y) \le \liminf\limits_{n\to\infty}(x_n^*, x_n - y)$.

If a map A is bounded (i.e. it maps bounded sets into bounded sets) and satisfies condition (c), then it satisfies also condition (b). A map $A : X \to 2^{X^*}$ is said to be "generalized pseudomonotone" if for every sequence $\{x_n\}_n \subset X, x_n \rightharpoonup x$ weakly in X and for every sequence $\{x_n^*\}_n, x_n^* \in A(x_n), n \ge 1$ such that $x_n^* \rightharpoonup x^*$ weakly in X^* and $\limsup\limits_{n\to\infty}(x_n^*, x_n - x) \le 0$ then we have $x^* \in A(x)$ and $(x_n^*, x_n) \to (x^*, x)$ as $n \to \infty$. Every maximal monotone operator is a generalized pseudomonotone operator. Also a pseudomonotone operator is always generalized pseudomonotone. The converse is true if the operator is also bounded. A pseudomonotone coercive operator is surjective. For further details on these and related issues we refer to the books of S. Hu-N. S. Papageorgiou [13] and E. Zeidler [18].

As we already mentioned earlier our approach also uses the method of upper and lower solution. So let us define these two notions. We denote with $C_{\text{per}}(T)$ and $C^1_{\text{per}}(T)$ respectively the sets $C_{\text{per}}(T) = \{x \in C(T) : x(0) = x(b)\}$ and $C^1_{\text{per}}(T) = \{x \in C^1(T) : x(0) = x(b), x'(0) = x'(b)\}$. A function $\varphi \in C_{\text{per}}(T)$ with $|\varphi'|^{p-2}\varphi' \in W^{1,q}(T)(\frac{1}{p} + \frac{1}{q} = 1)$ is said to be an "upper solution" of problem (2) if

$$\begin{gathered} -(|\varphi'(t)|^{p-2}\varphi'(t))' \ge f_2(t, \varphi(t))\,, \quad \text{a.e. on} \quad T \\ \varphi(0) = \varphi(b)\,; \end{gathered}$$

while a function $\psi \in C_{\text{per}}(T)$ with $|\psi'|^{p-2}\psi' \in W^{1,q}(T)$ is said to be a "lower solution" of problem (2) if

$$\begin{gathered} -(|\psi'(t)|^{p-2}\psi'(t))' \le f_1(t, \psi(t))\,, \quad \text{a.e. on} \quad T \\ \psi(0) = \psi(b)\,. \end{gathered}$$

Finally by a solution of problem (2) we mean a function $x \in C^1_{\text{per}}(T)$ such that $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(T)$ and there exists $g \in L^q(T)$ with $f_1(t, x(t)) \le g(t) \le f_2(t, x(t))$ a.e. on T such that $-(|x'(t)|^{p-2}x'(t))' = g(t)$ a.e. on T.

3. Eigenvalues of the ordinary p-Laplacian

In this section we develop some basic facts about the spectrum of the one-dimensional periodic p-Laplacian, i.e. of $(-\triangle_p, W^{1,p}_{\rm per}(T))$ where $\triangle_p x = (|x'|^{p-2}x')'$ and $W^{1,p}_{\rm per}(T) = \{x \in W^{1,p}(T) : x(0) = x(b)\}$. In $W^{1,p}_{\rm per}(T)$ we consider the norm endowed by $W^{1,p}(T)$. Note that $W^{1,p}(T)$ (with $(1 < p < \infty)$) is embedded continuously (in fact compactly) in $C(T)$ and so the evaluation at $t = 0$ and $t = b$ makes sense. We identify the part of the spectrum related to the well known Poincaré-Wirtinger inequality (see Hu-Papageorgiou [14], p. 866 or Kesavan [15], p. 88) which is crucial in the study of periodic problems. So our results are of independent interest.

It is well-known that $W^{1,p}_{\rm per}(T) = \mathbb{R} \bigoplus \mathbb{V}$ where V is the subspace of $W^{1,p}_{\rm per}(T)$ defined by $V = \{x \in W^{1,p}_{\rm per}(T) : \int_0^b x(t)\,dt = 0\}$. We introduce the following quantity

$$\widehat{\lambda}_1 = \inf\left[\frac{\|v'\|_p^p}{\|v\|_p^p} : v \in V, v \neq 0\right],$$

where $\|\cdot\|_p$ denotes the norm in the space $L^p(T)$.

We will show that $\widehat{\lambda}_1$ is a nonzero eigenvalue of the periodic one-dimensional p-Laplacian. In other words $\widehat{\lambda}_1$ is a nonzero number for which the problem

$$(3)\qquad \begin{aligned} &-(|x'(t)|^{p-2}x'(t))' = \lambda|x(t)|^{p-2}x(t)\,, \qquad \text{a.e. on} \quad T\\ &x(0) = x(b)\,,\ x'(0) = x'(b)\,, \end{aligned}$$

has a nontrivial solution. Note that $\lambda_0 = 0$ is an eigenvalue for the nonlinear problem (3). The corresponding to $\lambda_0 = 0$ eigenspace is the one-dimensional subspace of constant functions, i.e. $\mathbb{R}$. Clearly problem (3) can not have a negative eigenvalue.

Proposition 1. $\widehat{\lambda}_1$ *is positive.*

Proof. Suppose that $\widehat{\lambda}_1 = 0$. Then we can find a sequence $\{v_n\}_n \subset V$ such that $\|v_n\|_p = 1, \forall n \in \mathbb{N}$ and $\|v_n'\|_p \to 0$ as $n \to \infty$. Hence the sequence $\{v_n'\}_n$ converges to 0 in $L^p(T)$ and so, using Poincaré-Wirtinger inequality (cf. [14], p. 866 or [15], p. 88) we have that $\{v_n\}_n$ converges to 0 in $W^{1,p}(T)$ which is a contradiction with the fact that $\|v_n\|_p = 1, \forall n \in \mathbb{N}$. □

Proposition 2. *There exists $v \in V$ such that $\|v\|_p = 1$ and $\|v'\|_p^p = \widehat{\lambda}_1$.*

Proof. Let $\{v_n\}_n \subset V$ a sequence such that $\|v_n\|_p = 1, \forall n \in \mathbb{N}$ and $\|v_n'\|_p^p \to \widehat{\lambda}_1$ as $n \to \infty$. So by virtue of the Poincaré-Wirtinger inequality (cf. [14], p. 866 or [15], p. 88) we have that $\{v_n\}_n$ is bounded in $W^{1,p}_{\rm per}(T)$, hence by passing to a subsequence if necessary we may assume that $v_n \rightharpoonup v$ weakly in $W^{1,p}_{\rm per}(T)$. Since $W^{1,p}_{\rm per}(T)$ is embedded compactly in $L^p(T)$, we also have that there exists a subsequence of $\{v_n\}_n$, denoted again by $\{v_n\}_n$, which converges to v in $L^p(T)$. Therefore we obtain that $\|v\|_p = 1$. Now from the weak lower semicontinuity of the norm functional we have

$$\|v'\|_p^p \le \liminf_{n\to\infty} \|v_n'\|_p^p = \widehat{\lambda}_1\,,$$

which implies, recalling the definition of $\widehat{\lambda}_1$, that $\|v'\|_p^p = \widehat{\lambda}_1$. $\square$

Remark 1. By virtue of the strict convexity of the Lebesgue space $L^p(T)$, we see that the element $v \in L^p(T), \|v\|_p = 1$ obtained in Proposition 2 is unique.

Proposition 3. *If $v \in V, \|v\|_p = 1$ is as in Proposition 2, then $v \in C^1_{\mathrm{per}}(T)$, $|v'|^{p-2}v' \in W^{1,q}(T)$ and v is a solution of the problem* (3).

Proof. Let $L : V \to \mathbb{R}$ be defined by

$$L(w) = \frac{1}{p}\|w'\|_p^p - \frac{\widehat{\lambda}_1}{p}\|w\|_p^p\,, \quad \forall w \in V\,.$$

From Proposition 2, we know that

$$0 = L(v) = \inf_{w\in V} L(w)$$

so we have that $L'(v) = 0$. But $L'(w) = \widehat{A}(w) - \widehat{\lambda}_1 J(w), \forall w \in V$, where $\widehat{A} : V \to V^*$ and $J : V \to V^*$ are the operators defined by

$$((\widehat{A}(w), u)) = \int_0^b |w'(t)|^{p-2}w'(t)u'(t)\,dt\,, \forall\, w, u \in V\,,$$

and

$$J(w) = |w(\cdot)|^{p-2}w(\cdot)\,, \quad \forall\, w \in V\,.$$

Here by $((\cdot,\cdot))$ we denote the duality brackets for the pair (V^*, V). If by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}_{\mathrm{per}}(T)^*, W^{1,p}_{\mathrm{per}}(T))$ and $A : W^{1,p}_{\mathrm{per}}(T) \to W^{1,p}_{\mathrm{per}}(T)^*$ is the demicontinuous, nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\,dt\,, \forall\, x, y \in W^{1,p}_{\mathrm{per}}(T)\,,$$

then because $W^{1,p}_{\mathrm{per}}(T) = V \bigoplus \mathbb{R}$ (i.e. $V = W^{1,p}_{\mathrm{per}}(T)/\mathbb{R}$), we see that $A(x) = \widehat{A}(x)$ for all $x \in V$. Moreover we note that $J(w) \in L^q(T), \forall w \in V$, therefore we obtain $A(v) = \widehat{\lambda}_1 J(v) \in L^q(T)$. Then in particular for every $\eta \in C_0^\infty(T) = \{\eta \in C^\infty(T) :$ η has compact support in $(0, b)\}$ we have

$$\langle A(v), \eta\rangle = \widehat{\lambda}_1 (J(v), \eta)_{q,p}$$

(here by $(\cdot,\cdot)_{q,p}$ we denote the duality brackets for the pair $(L^q(T), L^p(T))$) which means that

$$\int_0^b |v'(t)|^{p-2}v'(t)\eta'(t)\,dt = \int_0^b \widehat{\lambda}_1 |v(t)|^{p-2}v(t)\eta(t)\,dt\,.$$

From Theorem 2.6.1, p. 89, of [15] we know that $(|v'|^{p-2}v')' \in W^{-1,q}(T) = W_0^{1,p}(T)^*$. So if by $\langle \cdot, \cdot \rangle_0$ we denote the duality brackets for the pair $(W_0^{1,p}(T), W^{-1,q}(T))$, we have

$$\langle -(|v'|^{p-2}v')', \eta \rangle_0 = \widehat{\lambda}_1 (J(v), \eta)_{q,p} \,.$$

Since $\eta \in C_0^\infty(T)$ was arbitrary and $C_0^\infty(T)$ is dense in $W^{1,p}_{\mathrm{per}}(T)$, we deduce that

$$\tag{4} \begin{aligned} -(|v'(t)|^{p-2}v'(t))' &= \widehat{\lambda}_1 |v(t)|^{p-2}v(t) \quad \text{a.e. on} \quad T\,, \\ v(0) &= v(b)\,. \end{aligned}$$

From this it follows that $|v'|^{p-2}v' \in W^{1,q}(T)$, hence $|v'|^{p-2}v' \in C(T)$.

Let now $j : \mathbb{R} \to \mathbb{R}$ be the function defined by $j(r) = |r|^{\,p-2}r$; it is simple to see that j is strictly monotone, continuous and so j^{-1} exists and it is easily seen to be continuous. Hence $j^{-1}(|v'(\cdot)|^{p-2}v'(\cdot)) = v'(\cdot) \in C(T)$ and so $v \in C^1(T)$.

Finally let $\eta \in C^\infty_{\mathrm{per}}(T)$ be such that $\eta(0) = \eta(b) = 1$. From Green's formula we have

$$\begin{aligned} \int_0^b (|v'(t)|^{p-2}v'(t))'\eta(t)\,dt = |v'(b)|^{p-2}v'(b)\eta(b) - |v'(0)|^{p-2}v'(0)\eta(0) \\ - \int_0^b |v'(t)|^{p-2}v'(t)\eta'(t)\,dt\,, \end{aligned}$$

and so from (4) and from the fact that $\eta(0) = \eta(b) = 1$ we obtain

$$\begin{aligned} -\widehat{\lambda}_1 \int_0^b |v(t)|^{p-2}v(t)\eta(t)\,dt = |v'(b)|^{p-2}v'(b) - |v'(0)|^{p-2}v'(0) \\ - \int_0^b |v'(t)|^{p-2}v'(t)\eta'(t)\,dt\,. \end{aligned}$$

Recalling the definitions of $A(v)$ and $J(v)$ and the fact that $A(v) = \widehat{\lambda}_1 J(v)$ we deduce that

$$-\widehat{\lambda}_1 (J(v), \eta)_{q,p} = |v'(b)|^{p-2}v'(b) - |v'(0)|^{p-2}v'(0) - \widehat{\lambda}_1 (J(v), \eta)_{q,p}\,,$$

and so $|v'(b)|^{p-2}v'(b) = |v'(0)|^{p-2}v'(0)$ from which using the action of j^{-1} we conclude that $v'(0) = v'(b)$. $\square$

Remark 2. In general $\widehat{\lambda}_1 > 0$ is not the first nonzero eigenvalue of negative ordinary scalar p-Laplacian with periodic boundary conditions. To identify that eigenvalue we have to minimize not on the linear subspace V but on the cone $K = \{x \in W^{1,p}_{\mathrm{per}}(T) : \int_0^b \|x(t)\|^{p-2}x(t)\,dt = 0\}$. Of course if $p = 2$, then $\widehat{\lambda}_1$ is indeed the first nonzero eigenvalue and in fact $\widehat{\lambda}_1 = \left(\frac{2\pi}{b}\right)^p$ (we thank the referee for bringing to our attention this issue).

4. Existence of two solutions

Now we are ready to deal with problem (2). We introduce the following hypotheses for the discontinuous nonlinearity $f(t,x)$:

$H(f)$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) f_1, f_2 are both N-measurable functions (i.e. for every measurable function $x : T \to \mathbb{R}$ the functions $t \to f_i(t, x(t))$, $i = 1, 2$, are measurable);

(ii) for every $M > 0$, there exists $\gamma_M \in L^q(T)_+$ such that for almost all $t \in T$ and all $x \in \mathbb{R}$ with $|x| \le M$ we have

$$|f(t,x)| \le \gamma_M(t)\,;$$

(iii) $\liminf\limits_{x \to +\infty} f_2(t,x) < 0$ uniformly for almost all $t \in T$ and $\limsup\limits_{x \to -\infty} f_1(t,x) > 0$ uniformly for almost all $t \in T$;

(iv) $\liminf\limits_{|x| \to 0} \dfrac{f(t,x)}{|x|^{p-2}x} > \widehat{\lambda}_1$ uniformly for almost all $t \in T$.

By virtue of hypothesis $H(f)$(iii) we can find $M, N > 1$ such that for almost all $t \in T$ we have

$$f_2(t, M) < 0\,, \tag{5}$$

and

$$f_1(t, -N) > 0\,. \tag{6}$$

Also from hypothesis $H(f)$(iv), we can find $\delta > 0$, $\delta < 1$ such that for almost all $t \in T$ we have

$$f_1(t,x) \ge \widehat{\lambda}_1 |x|^{p-2}x\,,\ \forall\, x \in (0,\delta) \tag{7}$$

while

$$f_2(t,x) \le \widehat{\lambda}_1 |x|^{p-2}x\,, \forall\, x \in (-\delta, 0)\,. \tag{8}$$

Let $v \in C^1(T)$ be as in Proposition 3 and set $u(t) = |v(t)|$, $t \in T$. Then we know that $u \in W^{1,p}_{\mathrm{per}}(T)$ (cf. [10], p.146 or [14], p. 866). So u is absolutely continuous and

$$u'(t) = \begin{cases} v'(t)\,, & \text{for a.a.} \quad t \in \{t \in T : v(t) > 0\}\,, \\ 0\,, & \text{for a.a.} \quad t \in \{t \in T : v(t) = 0\}\,, \\ -v'(t)\,, & \text{for a.a.} \quad t \in \{t \in T : v(t) < 0\}\,. \end{cases}$$

Evidently we can find $\xi > 0$ such that $0 \le \xi u(t) < \delta$ for all $t \in T$. Set $v_1 = \xi u$. Then $v_1 \in W^{1,p}_{\mathrm{per}}(T), v_1 \ne 0, v_1(t) \ge 0, \forall t \in T$ and by virtue of Proposition 3 and (7) we have

$$\begin{aligned} -(|v_1'(t)|^{p-2}v_1'(t))' &= -\,\xi^{p-1}(|u'(t)|^{p-2}u'(t))' \\ &= \xi^{p-1}\widehat{\lambda}_1 |u(t)|^{p-2}u(t) \\ &= \widehat{\lambda}_1 |v_1(t)|^{p-2}v_1(t) \le f_1(t, v_1(t))\,, \quad \text{a.e. on} \quad T\,, \end{aligned}$$

which means that v_1 is a lower solution of problem (2). Now, if we put $\varphi(t) = M, t \in T$, from (5) we have that $f_2(t, M) \le 0 = -(|\varphi'(t)|^{p-2}\varphi'(t))'$ for almost all $t \in T$. So φ is an upper solution of problem (2). Moreover note that $0 \le v_1(t) < \varphi(t), \forall t \in T$, and as in Proposition 3 we deduce that $v_1 \in C^1_{\text{per}}(T)$.

In the same way, using Proposition 3, (6) and (8), we can find $v_2, \psi \in C^1_{\text{per}}(T)$ which are respectively an upper and a lower solution of problem (2); moreover $\psi(t) < v_2(t) \le 0, \forall t \in T$ and $v_2 \ne 0$.

Now we are ready to prove the following

Theorem 4. *If hypotheses $H(f)$ hold, then problem* (2) *has a nontrivial solution $x \in C^1(T)$ such that* $0 \le v_1(t) \le x(t) \le \varphi(t), \forall t \in T$.

Proof. Our proof uses truncation and penalization techniques together with results from the theory of operators of monotone type.

We introduce the truncation map $\tau : W^{1,p}(T) \to W^{1,p}(T)$ defined by

$$\tau(x)(t) = \begin{cases} M\,, & M \le x(t)\,, \\ x(t)\,, & v_1(t) \le x(t) \le M\,, \\ v_1(t)\,, & x(t) \le v_1(t)\,. \end{cases}$$

It is easy to see that τ is continuous. Also we introduce the penalty function $\beta : T \times \mathbb{R} \to \mathbb{R}$ defined by

$$\beta(t, x) = \begin{cases} (x - M)^{p-1}\,, & M \le x\,, \\ 0\,, & v_1(t) \le x \le M\,, \\ -(v_1(t) - x)^{p-1}\,, & x \le v_1(t)\,. \end{cases}$$

Clearly β is a Carathéodory function and there exist $a_1 \in L^q(T)_+$ and $c_1 > 0$ such that

$$|\beta(t, x)| \le a_1(t) + c_1|x|^{p-1}\,, \quad \text{a.e. on} \quad T \quad \text{and for all} \quad x \in \mathbb{R}. \tag{9}$$

Moreover, it is easy to verify that it is possible to find $c_2, c_3, c_4 > 0$ such that

$$\int_0^b \beta(t, x(t))x(t)\,dt \ge c_2\|x\|_p^p - c_3\|x\|_1 - c_4\,, \quad \text{for all} \quad x \in L^p(T)\,. \tag{10}$$

We introduce the following auxiliary problem

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' \in \widehat{f}(t, \tau(x)(t)) - \beta(t, x(t))\,, \quad \text{a.e. on} \quad T \\ &x(0) = x(b)\,,\ x'(0) = x'(b) \end{aligned} \tag{11}$$

Let $A : W^{1,p}_{\text{per}}(T) \to W^{1,p}_{\text{per}}(T)^*$ be the operator defined, as in Proposition 3, by

$$\langle A(w), u\rangle = \int_0^b |w'(t)|^{p-2}w'(t)u'(t)\,dt\,,\ \forall\, w, u \in W^{1,p}_{\text{per}}(T)\,.$$

It is easy to see that, A is bounded, monotone, demicontinuous and so maximal monotone. Also let $B : L^p(T) \to L^q(T)$ be the Nemitsky operator corresponding to the penalty function β: for all $x \in L^p(T)$, we have

$$B(x)(t) = \beta(t, x(t)), \quad t \in T.$$

It is well-known that B is continuous (cf. [13], Theorem 7.26, p. 237) and, from (9) we obtain that B is bounded, while, directly from the definition of β it follows that B is monotone. Finally let $F : W^{1,p}(T) \to 2^{L^q(T)}$ defined by

$$F(x) = \{g \in L^q(T) : g(t) \in \widehat{f}(t, \tau(x)(t)), \quad \text{a.e. on } T\}, \text{ for all } \quad x \in W^{1,p}(T).$$

An easy application of the Yankov-von Neumann-Aumann selection theorem (cf. [13], p. 158) reveals that for every $x \in W^{1,p}(T)$, $F(x)$ is nonempty, weakly compact and convex in $L^q(T)$, while by virtue of hypothesis $H(f)$(ii) and the definition of truncation map we have that F is bounded.

Let $R : W^{1,p}_{\text{per}}(T) \to 2^{W^{1,p}_{\text{per}}(T)^*}$ be the multivalued operator defined by $R = A + B - F$.

Claim 1: *R is pseudomonotone.*

Since R is bounded and with nonempty convex and closed values it is sufficient to prove (cf. [13], Proposition 6.11, p. 366) that R is generalized pseudomonotone. Therefore let $(x_n)_n \in W^{1,p}_{\text{per}}(T)$ be a sequence such that $x_n \rightharpoonup x$ weakly in $W^{1,p}_{\text{per}}(T)$ and let $(u_n)_n \in W^{1,p}_{\text{per}}(T)^*$ be a sequence such that $u_n \rightharpoonup u$ weakly in $W^{1,p}_{\text{per}}(T)^*$, $u_n \in R(x_n), \forall n \in \mathbb{N}$ and suppose that $\limsup_{n\to\infty} \langle u_n, x_n - x\rangle \le 0$. Moreover let $w_n \in F(x_n)$, be such that $u_n = A(x_n) + B(x_n) - w_n$, $n \in \mathbb{N}$; therefore we have

$$\langle u_n, x_n - x\rangle = \langle A(x_n), x_n - x\rangle + (B(x_n), x_n - x)_{q,p} - (w_n, x_n - x)_{q,p}.$$

Recall that $W^{1,p}_{\text{per}}(T)$ is embedded compactly in $L^p(T)$, so, by passing to a subsequence if necessary, we have that $x_n \to x$ in $L^p(T)$ from which, recalling that B and F are bounded we deduce that $(B(x_n), x_n - x)_{q,p} \to 0$ and $(w_n, x_n - x)_{q,p} \to 0$. Therefore

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \le 0.$$

But A being maximal monotone is generalized pseudomonotone. So we have $A(x_n) \rightharpoonup A(x)$ weakly in $W^{1,p}_{\text{per}}(T)^*$ and $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$ as $n \to \infty$; which implies that $\|x_n'\|_p \to \|x'\|_p$ thus, since $L^p(T)$ is a uniformly convex space, we deduce that

$$x_n \to x \quad \text{in} \quad W^{1,p}_{\text{per}}(T) \quad \text{as} \quad n \to \infty.$$

Therefore we obtain that $B(x_n) \to B(x)$ in $L^q(T)$ and, for at least a subsequence, we have $w_n \rightharpoonup w$ weakly in $L^q(T)$ and by virtue of Proposition VII. 3.9 p. 694 of [13] we have that $w \in F(x)$. So $u_n \rightharpoonup u = A(x) + B(x) - w$ weakly in $W^{1,p}_{\text{per}}(T)^*$ which means that $u \in R(x)$ and $\langle u_n, x_n\rangle \to \langle u, x\rangle$ as $n \to \infty$. So R is generalized pseudomonotone and therefore it is pseudomonotone.

Claim 2: *R is coercive.*

Fixed $x \in W^{1,p}_{\text{per}}(T)$ and $u \in R(x)$, let $w \in F(x)$ such that $u = A(x)+B(x)-w$. So, taking into account (10) we have

$$\begin{aligned}\langle u,x\rangle &= \langle A(x),x\rangle + (B(x),x)_{q,p} - (w,x)_{q,p}\\ &\geq \langle A(x),x\rangle + (B(x),x)_{q,p} - \|w\|_q\|x\|_p\\ &\geq \|x'\|_p^p + c_2\|x\|_p^p - c_3\|x\|_1 - c_4 - \|w\|_q\|x\|_p\\ &\geq \|x'\|_p^p + c_2\|x\|_p^p - c_3\|x\|_1 - c_4 - \tfrac{\|w\|_q^q}{q\varepsilon^{q/p}} - \tfrac{\varepsilon\|x\|_p^p}{p}\,.\end{aligned}$$

Observe now that the set $F(W^{1,p}_{\text{per}}(T))$ is bounded in $L^q(T)$ and $W^{1,p}_{\text{per}}(T)$ is embedded in $L^1(T)$ so there exist $k_1, k_2 > 0$ such that

$$\langle u,x\rangle \geq \|x'\|_p^p + c_2\|x\|_p^p - k_1\|x\|_{1,p} - c_4 - \frac{k_2}{q\varepsilon^{q/p}} - \frac{\varepsilon\|x\|_p^p}{p}\,.$$

Let $\varepsilon > 0$ be such that $pc_2 - \varepsilon > 0$ then we can find $k_3, k_4 > 0$ with the property

$$\langle u,x\rangle \geq k_3\|x\|_{1,p}^p - k_1\|x\|_{1,p} - k_4\,,$$

from which we deduce the coercivity of R.

From Corollary 6.30 of [13], we have that R is surjective; so we can find $x \in W^{1,p}_{\text{per}}(T)$ such that $0 \in R(x)$. Arguing as in the proof of Proposition 3 we obtain that x is a solution of problem (11), $|x'|^{p-2}x' \in W^{1,q}(T)$ and $x \in C^1_{\text{per}}(T)$.

Claim 3: *For all $t \in T$ we have $0 \leq v_1(t) \leq x(t) \leq \varphi(t)$.*

We recall that $v_1 \in C^1_{\text{per}}(T)$ and

$$-(|v_1'(t)|^{p-2}v_1'(t))' \leq f_1(t, v_1(t)) \quad \text{a.e. on} \quad T\,. \tag{12}$$

Also x is a solution of problem (11) so it follows that there exists $w \in L^q(T)$ such that $w(t) \in \widehat{f}(t, \tau(x)(t))$, a.e. on T and

$$-(|x'(t)|^{p-2}x'(t))' = w(t) - \beta(t, x(t))\,, \quad \text{a.e. on} \quad T\,.$$

Subtract from (12) the previous equality and multiply with $(v_1 - x)_+(t)$; we obtain

$$\begin{aligned}&[(|x'(t)|^{p-2}x'(t))' - (|v_1'(t)|^{p-2}v_1'(t))'](v_1 - x)_+(t)\\ &\qquad \leq f_1[(t, v_1(t)) - w(t) + \beta(t, x(t))](v_1 - x)_+(t)\,, \quad \text{a.e. on} \quad T\,.\end{aligned}$$

Now integrating over T and using Green's formula (note that $(v_1 - x)_+ \in W^{1,p}_{\text{per}}(T)$, see [10], p. 145) we have

$$\begin{aligned}&\int_0^b (|v_1'(t)|^{p-2}v_1'(t) - |x'(t)|^{p-2}x'(t))(v_1 - x)_+'(t)\,dt\\ &\qquad \leq \int_0^b [f_1(t, v_1(t)) - w(t) + \beta(t, x(t))](v_1 - x)_+(t)\,dt\,.\end{aligned} \tag{13}$$

We know that (cf. [10])

$$(v_1 - x)_+'(t) = \begin{cases}(v_1 - x)'(t)\,, & \text{if} \quad x(t) < v_1(t)\\ 0\,, & \text{if} \quad x(t) \geq v_1(t)\,.\end{cases}$$

So we have

$$\begin{aligned}
&\int_0^b (|v_1'(t)|^{p-2}v_1'(t) - |x'(t)|^{p-2}x'(t))(v_1 - x)_+'(t)\,dt \\
\text{(14)}\qquad &\qquad = \int_A (|v_1'(t)|^{p-2}v_1'(t) - |x'(t)|^{p-2}x'(t))(v_1'(t) - x'(t))\,dt \geq 0\,,
\end{aligned}$$

where $A = \{t \in T : v_1(t) \geq x(t)\}$. Also we have

$$\begin{aligned}
&\int_0^b [f_1(t, v_1(t)) - w(t)](v_1 - x)_+(t)\,dt \\
\text{(15)}\qquad &\qquad = \int_A [f_1(t, v_1(t)) - w(t)](v_1(t) - x(t))\,dt \leq 0\,,
\end{aligned}$$

because on A, $\widehat{f}(t, \tau(x)(t)) = \widehat{f}(t, v_1(t))$ and so $w(t) \geq f_1(t, v_1(t))$.

Using (14) and (15) in (13) we obtain

$$0 \leq \int_0^b \beta(t, x(t))(v_1 - x)_+(t)\,dt = \int_A -(v_1(t) - x(t))^{p-1}(v_1(t) - x(t))\,dt\,,$$

which implies that

$$\int_0^b (v_1 - x)_+^p(t)\,dt = 0$$

and so $v_1(t) \leq x(t)$ for all $t \in T$.

Similarly we can show that $x(t) \leq \varphi(t) = M$ for all $t \in T$. Hence $0 \leq v_1(t) \leq x(t) \leq \varphi(t)$ for all $t \in T$, which implies that $\tau(x) = x$ and $\beta(t, x(t)) = 0$. Therefore x is a solution of problem (2). ☐

In the same way we obtain the following

Theorem 5. *If hypotheses $H(f)$ hold, then problem* (2) *has a nontrivial solution $x \in C^1(T)$ such that $\psi(t) \leq x(t) \leq v_2(t) \leq 0$, $\forall\, t \in T$.*

Putting together Theorems 4 and 5, we have the following multiplicity result

Theorem 6. *If hypotheses $H(f)$ hold, then problem* (2) *has at least two nontrivial solutions in $C^1(T)$: one positive and the second negative.*

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