ARCHIVUM MATHEMATICUM (BRNO)
Tomus 38 (2002), 209 – 217

HOMOMORPHISMS FROM THE UNITARY GROUP TO THE GENERAL LINEAR GROUP OVER COMPLEX NUMBER FIELD AND APPLICATIONS

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ABSTRACT. Let M_n be the multiplicative semigroup of all $n \times n$ complex matrices, and let U_n and GL_n be the n–degree unitary group and general linear group over complex number field, respectively. We characterize group homomorphisms from U_n to GL_m when $n > m \geq 1$ or $n = m \geq 3$, and thereby determine multiplicative homomorphisms from U_n to M_m when $n > m \geq 1$ or $n = m \geq 3$. This generalize Hochwald's result in [*Lin. Alg. Appl. 212/213:339-351(1994)*]: if $f : U_n \to M_n$ is a spectrum–preserving multiplicative homomorphism, then there exists a matrix R in GL_n such that $f(A) = R^{-1}AR$ for any $A \in U_n$.

1. Introduction

Let $\mathbb{C}$ be the complex number field and I_n the $n \times n$ identity matrix over $\mathbb{C}$. We denote the n-degree unitary group ($\{A|A^*A = I_n\}$), the n-degree general linear group and the multiplicative semigroup of all $n \times n$ matrices over $\mathbb{C}$ by U_n, GL_n and M_n, respectively.

In the last few decades, some authors have determined multiplicative homomorphisms or isomorphisms between matrix (semi)groups (see [2], [3], [4], [6], [7], [8], [9], [10], [11] and [12]). Hochwald in [5] has studied a similar problem: characterizing the spectrum–preserving multiplicative homomorphisms from U_n to M_n. In this paper we characterize group homomorphisms from U_n to GL_m when $n > m \geq 1$ or $n = m \geq 3$, As applications, we also determine multiplicative homomorphisms from U_n to M_m when $n > m \geq 1$ or $n = m \geq 3$, and thereby generalize the mentioned result in [5].

We denote by $\mathrm{Hom}\,(U_n, \Gamma_m)$ the set of the multiplicative homomorphisms from U_n to Γ_m, where Γ_m is either GL_m or M_m. Let $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ be the set $\{c \in \mathbb{C} \big| |c| \leq 1\}$, $\{c \in \mathbb{C} \big| |c| = 1\}$ and $\{(a,b) | a, b \in \mathbb{C}_0,\ |a|^2 + |b|^2 = 1\}$, respectively. Let

2000 *Mathematics Subject Classification*: 20G15.

Key words and phrases: homomorphism, unitary group, general linear group.

Partially supported by the Natural Foundation of Heilongjiang Province under grant No. A01-07 and the N. S. F. of Heilongjiang Education Committee under grant No. 15011014.

Received December 8, 2000.

E_{pq} denote the matrix with 1 at the (p,q) position and 0 elsewhere. For positive integers p and q, and $c \in \mathbb{C}_1$, we denote by $D_p(c)$ and Z_{pq} the matrix $I_n-(1-c)E_{pp}$ and $I_n - E_{pp} - E_{qq} + E_{pq} + E_{qp}$, respectively. In particular, if $p < q$, we denote by $V_{pq}(a,b)$ the matrix $I_n + (\overline{a}-1)E_{pp} + (a-1)E_{qq} - bE_{qp} + \overline{b}E_{pq}$ for $(a,b) \in \mathbb{C}_2$ and write $V_{pq}(x)$ for $V_{pq}(x, \sqrt{1-x^2})$.

2. Preliminaries

In this section, we assume that $n \geq m$, $n \geq 2$ and $\phi \in \mathrm{Hom}\,(U_n, GL_m)$. Since $\phi(AB) = \phi(A)\phi(B)$ for any A and B in U_n, ϕ have the next propositions.

Proposition 1. $\phi(I_n) = I_m$.

Proposition 2. $\phi(A)^{-1} = \phi(A^*)$ *for any* $A \in U_n$.

Proposition 3. *For* A *and* B *in* U_n, *if* $A = PBP^*$ *for some* $P \in U_n$, *then* $\phi(A) = \phi(P)\phi(B)\phi(P)^{-1}$.

Proposition 4. *For* A *and* B *in* U_n, *if* A *and* B *are commutative, then* $\phi(A)$ *and* $\phi(B)$ *are also.*

Proposition 5. *For any mutually distinct positive integers* p, q *and* k *with* $p < q$, $x, y \in [-1,\ 1]$, $c \in \mathbb{C}_1$ *and* $(a,b) \in \mathbb{C}_2$, *the following equations hold.*

$$\phi(D_k(c))\phi(Z_{pq}) = \phi(Z_{pq})\phi(D_k(c))\,; \tag{1}$$

$$\phi(D_p(c))\phi(Z_{pq}) = \phi(Z_{pq})\phi(D_q(c))\,; \tag{2}$$

$$\phi(Z_{pq})^2 = I_m\,; \tag{3}$$

$$\phi(Z_{pq})\phi(Z_{pk})\phi(Z_{qk}) = \phi(Z_{pk})\,; \tag{4}$$

$$\phi(D_k(c))\phi(V_{pq}(x)) = \phi(V_{pq}(x))\phi(D_k(c))\,; \tag{5}$$

$$[\phi(D_p(-1))\phi(V_{pq}(x))]^2 = I_m\,; \tag{6}$$

$$\phi(V_{pq}(x)) = \phi\Big(V_{pq}\Big(\sqrt{\tfrac{x+1}{2}}\Big)\Big)^2\,; \tag{7}$$

$$[\phi(Z_{pq})\phi(V_{pq}(x))]^2 = I_m\,; \tag{8}$$

$$\begin{aligned}\phi\left(V_{pq}\Big(xy - \sqrt{(1-x^2)(1-y^2)}\Big)\right) &= \phi(V_{pq}(x))\phi(V_{pq}(y)) \\ &= \phi(V_{pq}(y))\phi(V_{pq}(x))\,;\end{aligned} \tag{9}$$

$$\phi(D_p(-1))\phi(V_{pq}(x)) = \phi(V_{pq}(-x))\phi(D_q(-1))\,; \tag{10}$$

$$\phi(V_{pq}(a,b)) = \phi\big(D_p\big(\tfrac{\overline{ab}}{|ab|}\big)\big)\phi(V_{pq}(|a|))\phi\big(D_p\big(\tfrac{b}{|b|}\big)\big)\phi\big(D_q\big(\tfrac{a}{|a|}\big)\big)\,; \tag{11}$$

$$\phi(D_k(-1))^2 = I_m\,. \tag{12}$$

Lemma 1. *Let $\{R_1, R_2, \cdots, R_t\}$ be a set of t mutually commutative involutory matrices in GL_n. Then there exists $Q \in GL_n$ such that $Q^{-1}R_iQ = \Lambda_i$ for any $1 \le i \le t$, where $\Lambda_1 = -I_r \oplus I_{n-r}$ for some $0 \le r \le n$ and $\Lambda_2, \cdots, \Lambda_t$ are diagonal involutory matrices.*

Proof. It is easy to see that $\left\{\frac{1}{2}(I_n+R_1), \frac{1}{2}(I_n+R_2), \cdots, \frac{1}{2}(I_n+R_t)\right\}$ is a set of t mutually commutative idempotent matrices. By a similar argument to [1, Lemma 3.1], the lemma can be obtained. □

Lemma 2. *Suppose $A = (a_{st}) \in U_n$. Then $A = \prod_{k=p}^{n} V_{1k}(a_k, b_k)(1 \oplus A_1)$ for some $p \ge 2$, $A_1 \in U_{n-1}$ and (a_p, b_p), (a_{p+1}, b_{p+1}), $\cdots$, $(a_n, b_n) \in \mathbb{C}_2$.*

Proof. Case 1. Suppose $a_{21} = \cdots = a_{n1} = 0$. Then $A = a_{11} \oplus B$ from $A \in U_n$, where $a_{11} \in \mathbb{C}_1$ and $B \in U_{n-1}$. Let $p = n$, $a_n = \overline{a_{11}}$ and $b_n = 0$. Then $A = V_{1n}(a_n, b_n)(1 \oplus A_1)$ for some $A_1 \in U_{n-1}$.

Case 2. Suppose a_{p1} is the first nonzero element of $a_{21}, \cdots, a_{n1}$. Let $b_k = -\frac{a_{k1}}{r_k}$ and

$$a_k = \begin{cases} \frac{\overline{a_{11}}}{r_p} & \text{if} \quad k = p \\ \frac{r_{k-1}}{r_k} & \text{if} \quad k > p \end{cases}$$

for any $k \ge p$, where $r_k = \sqrt{\sum_{j=1}^{k} |a_{j1}|^2}$. Then

$$V_{1n}(\overline{a_n}, -b_n)V_{1\ n-1}(\overline{a_{n-1}}, -b_{n-1}) \cdots V_{1p}(\overline{a_p}, -b_p)A = \begin{pmatrix} 1 & \Delta \\ 0 & A_1 \end{pmatrix}.$$

Noting $V_{1k}(\overline{a_k}, -b_k) \in U_n$ for any $k \ge p$, we have $\Delta = 0$ and $A_1 \in U_{n-1}$. Thus the lemma follows. □

From which, we can obtain the next corollary by induction.

Corollary 1. *Suppose $A \in U_n$. Then $A = \Delta_1\Delta_2 \cdots \Delta_t(I_{n-1} \oplus \det A)$ for some $\Delta_1, \Delta_2, \cdots, \Delta_t \in \{V_{pq}(a, b) | 1 \le p < q \le n,\ (a, b) \in \mathbb{C}_2\}$.*

By a similar argument to Lemma 2, we have

Lemma 3. *Suppose $B = (b_{st}) \in U_n$. Then*

$$B = (1 \oplus B_1)V_{1n}(c_n, d_n)V_{1\ n-1}(c_{n-1}, d_{n-1}) \cdots V_{1p}(c_p, d_p)$$

for some (c_p, d_p), (c_{p+1}, d_{p+1}), $\cdots$, $(c_n, d_n) \in \mathbb{C}_2$ and $B_1 \in U_{n-1}$.

Lemma 4. (a) *If $\phi(V_{st}(x)) = I_m$ for all $x \in [-1, 1]$ and some pairs positive integers s and t with $s < t$, then*

$$\phi(A) = \sigma(\det A), \qquad \forall A \in U_n, \tag{13}$$

where σ is a multiplicative group homomorphism from $\mathbb{C}_1$ to GL_m.

(b) *If $\phi(D_k(-1)) = \pm I_m$ for some positive integer k, then ϕ is the form* (13).

(c) *If $n \ge 3$ and $\phi(D_s(-1)) = \phi(D_t(-1))$ for some pairs positive integers s and t with $s < t$, then ϕ is the form* (13).

Proof. (a) For any positive integers p and q with $p < q$, it follows from $V_{pq}(x) = Z_{ps}Z_{qt}V_{st}(x)Z_{qt}Z_{ps}$ and Proposition 3 that $\phi(V_{pq}(x)) = I_m$. By applying ϕ to the equation $Z_{pq} = D_q(-1)V_{pq}(0)$, we have $\phi(Z_{pq}) = \phi(D_q(-1))$. Futher

$$\begin{aligned}\phi(D_p(c)) = & \ \phi(Z_{pq}D_q(c)Z_{pq}) = \phi(Z_{pq})\phi(D_q(c))\phi(Z_{pq}) \\ = & \ \phi(D_q(-1))\phi(D_q(c))\phi(D_q(-1)) \\ = & \ \phi(D_q(-1)D_q(c)D_q(-1)) \\ = & \ \phi(D_q(c)), \qquad \forall c \in \mathbb{C}_1\end{aligned} \tag{14}$$

and

$$\begin{aligned}\phi(V_{pq}(a,b)) = & \ \phi(D_p(\tfrac{\overline{ab}}{|ab|}))\phi(V_{pq}(|a|))\phi(D_p(\tfrac{b}{|b|}))\phi(D_q(\tfrac{a}{|a|})) \\ = & \ \phi(D_q(\tfrac{\overline{ab}}{|ab|}))\phi(D_q(\tfrac{b}{|b|}))\phi(D_q(\tfrac{a}{|a|})) \\ = & \ \phi(D_q(\tfrac{\overline{ab}}{|ab|})D_q(\tfrac{b}{|b|})D_q(\tfrac{a}{|a|})) \\ = & \ \phi(I_n) = I_m, \qquad \forall (a,b) \in \mathbb{C}_2\end{aligned} \tag{15}$$

from (11), (14) and Proposition 1. Let $\sigma(c) = \phi(D_n(c))$ for any $c \in \mathbb{C}_1$. Then ϕ is the form (13) from Corollary 1 and (15).

(b) We only prove the result for $k < n$ (because the proof is similar when $k = n$). Applying (7) and (6), we obtain

$$\phi(V_{kn}(x)) = \phi\Big(V_{kn}\Big(\sqrt{\tfrac{x+1}{2}}\Big)\Big)^2 = \Big[\phi(D_k(-1))\phi\Big(V_{kn}\Big(\sqrt{\tfrac{x+1}{2}}\Big)\Big)\Big]^2 = I_m$$

for any $x \in [-1, 1]$. Thus ϕ is the form (13) from (a).

(c) We only prove the result for $t < n$(because the proof is similar when $t = n$). Applying (7), (12), (5), $\phi(D_s(-1)) = \phi(D_t(-1))$ and (6), we have

$$\begin{aligned}\phi(V_{tn}(x)) = & \ \phi\Big(V_{tn}\Big(\sqrt{\tfrac{x+1}{2}}\Big)\Big)^2 = \phi(D_s(-1))^2\phi\Big(V_{tn}\Big(\sqrt{\tfrac{x+1}{2}}\Big)\Big)^2 \\ = & \ \Big[\phi(D_s(-1))\phi\Big(V_{tn}\Big(\sqrt{\tfrac{x+1}{2}}\Big)\Big)\Big]^2 = \Big[\phi(D_t(-1))\phi\Big(V_{tn}\Big(\sqrt{\tfrac{x+1}{2}}\Big)\Big)\Big]^2 \\ = & \ I_m, \quad \forall x \in [-1, 1].\end{aligned}$$

Thus ϕ is the form (13) from (a). □

3. Homomorphisms from U_n to GL_m

Theorem 1. *Suppose $n > m \geq 1$. Then $\phi \in \mathrm{Hom}(U_n, GL_m)$ if and only if ϕ is the form* (13).

Proof. The "if" part is obvious, we only need to prove the "only if" part.

We proceed by induction on m. If $m = 1$, the result is obvious by applying (b) of Lemma 4. Suppose the theorem is true when $m < k(k \geq 2)$, we will prove that it is true when $m = k$. Without loss of generality, let $\phi(D_1(-1)) = -I_r \oplus I_{k-r}$ for some $0 \leq r \leq k$ from Lemma 1 and (12).

Case 1. Suppose $r = 0$ or $r = k$. The theorem can be proved by applying (b) of Lemma 4.

Case 2. Suppose $1 \leq r \leq k-1$. For any $B \in U_{n-1}$, since $D_1(-1)$ and $1 \oplus B$ are commutative, it follows that $\phi(D_1(-1))$ and $\phi(1 \oplus B)$ are also from Proposition 4. Thus

$$\phi(1 \oplus B) = f_1(B) \oplus f_2(B), \qquad \forall B \in U_{n-1},$$

where $f_1(B) \in GL_r$ and $f_2(B) \in GL_{k-r}$. It is easy to see that $f_1 \in \mathrm{Hom}(U_{n-1}, GL_r)$ and $f_2 \in \mathrm{Hom}(U_{n-1}, GL_{k-r})$. By the inductive hypothesis,

$$f_1(B) = \sigma_1(\det B), \qquad f_2(B) = \sigma_2(\det B), \qquad \forall B \in U_{n-1},$$

where $\sigma_1 : \mathbb{C}_1 \to GL_r$ and $\sigma_2 : \mathbb{C}_1 \to GL_{k-r}$ are multiplicative group homomorphisms. Thus, $\phi(D_2(-1)) = \phi(D_3(-1))$ by choosing $1 \oplus B = D_2(-1)$ and $D_3(-1)$, respectively. The theorem now follows by (c) of Lemma 4. □

Definition 1. Suppose S_1 and S_2 are two sets containing $1, 0$ and -1. We say that a map $g : S_1 \to S_2$ is a *almost homomorphism* if g satisfies $g(a+b) = g(a) + g(b)$ for any $a, b, a+b \in S_1$, $g(ab) = g(a)g(b)$ for any $a, b \in S_1$ and $g(\xi) = \xi$ for $\xi \in \{1, 0, -1\}$.

Lemma 5. *Suppose $n \geq 3$, $\phi \in \mathrm{Hom}(U_n, GL_n)$ and $\phi(D_k(-1)) = \eta D_k(-1)$ for any $1 \leq k \leq n$, where $\eta = \pm 1$. Then there exists $P \in GL_n$ such that*

(I) *$\phi(Z_{pq}) = \epsilon P Z_{pq} P^{-1}$ and $\phi(D_k(-1)) = \eta P D_k(-1) P^{-1}$ for any p, q and k with $p < q$, where $\epsilon = \pm 1$.*

(II) *$\phi(V_{pq}(x)) = P V_{pq}\left(\psi(x), \psi\left(\sqrt{1-x^2}\right)\right) P^{-1}$ for any $p < q$ and $x \in [-1, 1]$, where ψ is a map from $[-1, 1]$ to $\mathbb{C}$ with $\psi(\xi) = \xi$ for $\xi \in \{1, 0, -1\}$.*

(III) *$\phi(D_k(c)) = \lambda(c) P D_k(\delta(c)) P^{-1}$ for any $1 \leq k \leq n$ and $c \in \mathbb{C}_1$, where λ and δ are multiplicative homomorphisms from $\mathbb{C}_1$ to $\mathbb{C}$.*

(IV) *$\phi(V_{pq}(a, b)) = P V_{pq}(\tau(a), \tau(b)) P^{-1}$ for any $p < q$ and $(a, b) \in \mathbb{C}_2$, where τ is a almost homomorphism from $\mathbb{C}_0$ to $\mathbb{C}$.*

Proof. It follows that $\phi(Z_{pq}) = \epsilon_{pq} E_{pq} + \epsilon_{qp} E_{qp} + \sum_{t \neq p,q} \epsilon_t^{(p,q)} E_{tt}$ for any $p < q$ by choosing $c = -1$ in (1) and (2), and hence $\phi(Z_{pq}) = \epsilon_{pq} E_{pq} + \epsilon_{pq}^{-1} E_{qp} + \sum_{t \neq p,q} \epsilon_t^{(p,q)} E_{tt}$ from (3), where $\epsilon_t^{(p,q)} = \pm 1$. Again applying (4), we have $\phi(Z_{pq}) = \epsilon(I_n - E_{pp} - E_{qq}) + \epsilon_{pq} E_{pq} + \epsilon_{pq}^{-1} E_{qp}$ and $\epsilon_{pk} = \epsilon \epsilon_{pq} \epsilon_{qk}$ for any mutually distinct p, q and k, where $\epsilon = \pm 1$. Let $P = \mathrm{diag}(\epsilon, \epsilon_{12}^{-1}, \cdots, \epsilon_{1n}^{-1})$. Then (I) holds.

(II) It follows from (5) and (I) that $P^{-1}\phi\big(V_{12}(x)\big)P = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \oplus a_3(x) \oplus \cdots \oplus a_n(x)$ for any $x \in [-1, 1]$, where P is as in (I). Again applying (6) and (7), we have

$$P^{-1}\phi(V_{12}(x))P = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \oplus I_{n-2}, \quad \forall x \in [-1, 1],$$

where

$$\left\{ \begin{array}{l} a(x)^2 - b(x)c(x) = d(x)^2 - b(x)c(x) = 1 \\ [a(x) - d(x)]\, b(x) = [a(x) - d(x)]\, c(x) = 0 \end{array} \right. . \tag{16}$$

Case 1. Suppose $a(x_0) \neq d(x_0)$ for some $x_0 \in [-1, 1]$. Then $b(x_0) = c(x_0) = 0$ and $a(x_0) = -d(x_0) = \pm 1$ from (16), and hence $[\phi(Z_{12})\phi(V_{12}(x_0))]^2 = D_1(-1)D_2(-1)$ from (I), which contradicts to (8).

Case 2. Suppose $a(x_0) = d(x_0)$ and $b(x_0) + c(x_0) \neq 0$ for some $x_0 \in [-1,\ 1]$. Then $a(x_0) = 0$ and $b(x_0) = c(x_0) = \pm 1$ from (8) and (I), which contradicts to (16).

Case 3. Suppose $a(x) = d(x)$ and $b(x) = -c(x)$ for any $x \in [-1,1]$. That is

$$P^{-1}\phi(V_{12}(x))P = \begin{pmatrix} a(x) & b(x) \\ -b(x) & a(x) \end{pmatrix} \oplus I_{n-2}\,, \qquad \forall x \in [-1,1]\,. \tag{17}$$

Again applying (9), (10) and (17), we obtain

$$\begin{cases} a(x)^2 + b(x)^2 = 1 \\ a(-x) = -a(x) \\ b(-x) = b(x) \\ a\Big(xy - \sqrt{(1-x^2)(1-y^2)}\Big) = a(x)a(y) - b(x)b(y) \end{cases} \qquad \forall x, y \in [-1,1]\,. \tag{18}$$

It follows from $Z_{12} = D_2(-1)V_{12}(0)$ that $\phi(Z_{12}) = \phi(D_2(-1))\phi(V_{12}(0))$, and hence

$$a(0) = 0\,, \qquad b(0) = 1 \tag{19}$$

by applying (I) and (17).

Let $\psi(x) = a(x)$ for any $x \in [-1,1]$. Then $b(x) = -[a(x)a(0) - b(x)b(0)] = -a(-\sqrt{1-x^2}) = a(\sqrt{1-x^2}) = \psi(\sqrt{1-x^2})$ for any $x \in [-1,1]$ and $\psi(\xi) = \xi$ for $\xi \in \{1,0,-1\}$ from (18) and (19). Hence (II) holds.

(III) For any $c \in \mathbb{C}_1$ and $1 \le j \le n$, since $D_n(c)$ and $D_j(-1)$ are commutative, it follows from (I) and Proposition 4 that $P^{-1}\phi(D_n(c))P = \sum_{k=1}^{n} d_k(c)E_{kk}$, and hence

$$P^{-1}\phi\big(D_n(c)\big)P = d_1(c)\Big(\sum_{k=1}^{n-1} E_{kk}\Big) + d_n(c)E_{nn} = d_1(c)D_n\big(d_1(c)^{-1}d_n(c)\big)$$

from (I) and (1). Let $\lambda(c) = d_1(c)$ and $\delta(c) = d_1(c)^{-1}d_n(c)$ for any $c \in \mathbb{C}_1$. Then λ and δ are multiplicative homomorphisms from $\mathbb{C}_1$ to $\mathbb{C}$ by applying $D_n(c_1)D_n(c_2) = D_n(c_1c_2)$ for any c_1 and c_2 in $\mathbb{C}_1$ and Propositions 1 and 2. Again applying (I), (2) and (3), we have $P^{-1}\phi(D_k(c))P = \lambda(c)D_k(\delta(c))$ for any $1 \le k \le n$ and $c \in \mathbb{C}_1$.

(IV) Let $\tau(a) = \begin{cases} \psi(|a|)\delta(\frac{a}{|a|}) & a \neq 0 \\ 0 & a = 0 \end{cases}$ for any $a \in \mathbb{C}_0$, where ψ and δ are as in (II) and (III) respectively. Then $\tau(\xi) = \xi$ for $\xi \in \{1,0,-1\}$ and

$$\phi(V_{pq}(a,b)) = PV_{pq}(\tau(a),\tau(b))P^{-1}\,, \qquad \forall p < q\,,\ \ (a,b) \in \mathbb{C}_2 \tag{20}$$

by applying (II), (III) and (11).

Let

$$P^{-1}\phi(A)P = \lambda(\det A)\begin{pmatrix} f(A) & \star \\ \star & \star \end{pmatrix}, \qquad \forall A = (a_{ij}) \in U_n\,, \tag{21}$$

where f is a map from U_n to $\mathbb{C}$. Then $f(A)$ only depends on the 1–th column of A from Lemma 2, (III) and (20). On the other hand, $f(A)$ only depends on the 1–th row of A from Lemma 3, (III) and (20). Hence $f(A)$ only depends on a_{11}, i.e., we may write $f(A) = g(a_{11})$, where g is a map from $\mathbb{C}_0$ to $\mathbb{C}$. Again applying (20) and (21), we have

$$P^{-1}\phi(A)P = \lambda(\det A)\begin{pmatrix} \tau(a_{11}) & \star \\ \star & \star \end{pmatrix}, \qquad \forall A = (a_{ij}) \in U_n\,. \tag{22}$$

For any $a, b \in \mathbb{C}_0$, it follows from $V_{12}\big(\overline{a}, \sqrt{1-|a|^2}\big)V_{13}\big(\overline{b}, \sqrt{1-|b|^2}\big) = \begin{pmatrix} ab & \star \\ \star & \star \end{pmatrix}$ that $\phi\big(V_{12}\big(\overline{a}, \sqrt{1-|a|^2}\big)\big)\phi\big(V_{13}\big(\overline{b}, \sqrt{1-|b|^2}\big)\big) = \phi \begin{pmatrix} ab & \star \\ \star & \star \end{pmatrix}$, and hence

$$\tau(ab) = \tau(a)\tau(b)\,, \qquad \forall a, b \in \mathbb{C}_0 \tag{23}$$

from $\lambda(1) = 1$, (20) and (22).

For any $a, b, a+b \in \mathbb{C}_0$, without loss of generality, we can assume that $|a| \geq |b|$. Now we will prove that

$$\tau(a+b) = \tau(a) + \tau(b)\,, \qquad \forall a, b, a+b \in \mathbb{C}_0\,. \tag{24}$$

Case 1. Suppose $|a|^2 + |b|^2 \leq 1$. Let $x_{ab} = \frac{\overline{a}}{\sqrt{1-|b|^2}}$ and $y_{ab} = \frac{\sqrt{1-|b|^2-|a|^2}}{\sqrt{1-|b|^2}}$. Then

$$\begin{pmatrix} \frac{\sqrt{2}}{2}(a+b) & \star \\ \star & \star \end{pmatrix} = V_{12}\big(\sqrt{1-|b|^2}, \overline{b}\big)V_{13}(x_{ab}, y_{ab})V_{12}\big(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\big)\,,$$

and hence

$$\phi\begin{pmatrix} \frac{\sqrt{2}}{2}(a+b) & \star \\ \star & \star \end{pmatrix} = \phi\big(V_{12}\big(\sqrt{1-|b|^2}, \overline{b}\big)\big)\phi\big(V_{13}(x_{ab}, y_{ab})\phi\big(V_{12}\big(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\big)\big)\,.$$

Again applying $\lambda(1) = 1$, (20), (22) and (23), we can obtain that (24) holds.

Case 2. Suppose $|a|^2 + |b|^2 > 1$. Then c, d, $c+d \in \mathbb{C}_0$ and $|c|^2 + |d|^2 \leq 1$ by letting $c = \dfrac{a}{|a|^2 + |b|^2}$ and $d = \dfrac{b}{|a|^2 + |b|^2}$, and hence $\tau(c+d) = \tau(c) + \tau(d)$ from Case 1. Again applying (23), we have that (24) holds.

Summarizing, (IV) follows from (20), (23) and (24).

The lemma follows. ☐

Theorem 2. *Suppose $n \geq 3$. Then $\phi \in \mathrm{Hom}\,(U_n, GL_n)$ if and only if ϕ has one of the following forms.*

i) *$\phi(A) = \sigma(\det A)$ for any $A \in U_n$, and some multiplicative group homomorphism σ from $\mathbb{C}_1$ to GL_n.*

ii) *$\phi(A) = \lambda(\det A)PA^{\tau}P^{-1}$ for any $A = (a_{pq}) \in U_n$, and some $P \in U_n$, almost homomorphism τ from $\mathbb{C}_0$ to $\mathbb{C}$ and multiplicative homomorphism λ from $\mathbb{G}$ to $\mathbb{C}$, where $A^{\tau} = (\tau(a_{pq}))$.*

Proof. The "if" part is obvious, we only need to prove the "only if" part.

It is easy to see that $\{\phi(D_1(-1)),\ \phi(D_2(-1)),\ \cdots,\ \phi(D_n(-1))\}$ satisfy the assumption of Lemma 1, and hence $\phi(D_k(-1)) = P_1\Lambda_k P_1^{-1}$ for any k and some $P_1 \in U_n$, where $\Lambda_1 = -I_r \oplus I_{n-r}$ for some $0 \leq r \leq n$ and $\Lambda_2, \cdots, \Lambda_n$ are diagonal involutory matrices.

Case 1. Suppose $r = 0$ or $r = n$. Then ϕ is the form i) by (b) of Lemma 4.

Case 2. Suppose $2 \leq r \leq n-2$. Then ϕ is the form i) by a similar argument to Theorem 1.

Case 3. Suppose $r = 1$. Then $\Lambda_k = D_{g(k)}(-1)$ by applying Proposition 3, where g is a map from the set $\{1, 2, \cdots, n\}$ to itself.

a) If there exist distinct positive integers p and q such that $\Lambda_p = \Lambda_q$, then ϕ is the form i) by (c) of Lemma 4.

b) If $\Lambda_s \neq \Lambda_t$ for any distinct positive integers s and t, then there exists $P_2 \in U_n$ such that $\Lambda_k = P_2 D_k(-1) P_2^{-1}$ for any k. Let $P = P_1 P_2$. Then $\phi(D_k(-1)) = PD_k(-1)P^{-1}$ for any i, and hence ϕ is the form ii) from (III) and (IV) of Lemma 5 and Corollary 1.

Case 4. Suppose $r = n - 1$. By a similar argument to the Case 3, ϕ is the form i) or ii). □

4. Aplications

Theorem 3. *Suppose $n > m \geq 1$ or $n = m \geq 3$. Then $\phi \in \mathrm{Hom}\,(U_n, M_m)$ if and only if ϕ has one of the following forms.*

i) *$\phi(A) = Q(\rho(\det A) \oplus O)Q^{-1}$ for any $A \in U_n$, where $Q \in GL_m$ and ρ is a multiplicative homomorphism from $\mathbb{C}_1$ to GL_s for some $0 \leq s \leq m$.*

ii) *$\phi(A) = \lambda(\det A) P A^\tau P^{-1}$ for any $A \in U_n$, where P, A^τ, τ and λ are as in Theorem 2.*

Proof. It follows from $I_n^2 = I_n$ that $\phi(I_n)^2 = \phi(I_n)$, and hence $\phi(I_n) = Q(I_s \oplus O)Q^{-1}$ for some $0 \leq s \leq m$ and $Q \in GL_m$. Again applying ϕ to the equation $A = AI_n = I_n A$, we have $Q^{-1}\phi(A)Q = f(A) \oplus O$ for any $A \in U_n$, where $f(A) \in M_s$. Obviously, f is a multiplicative homomorhism from U_n to M_s. Thus, $\phi(A) f(A^*) = I_s$ for any $A \in U_n$ from $AA^* = I_n$, i.e., $f \in \mathrm{Hom}\,(U_n, GL_s)$. The theorem now follows by Theorems 1 and 2. □

Theorem 4. (see [5, Theorem 3]) *Suppose $n \geq 3$. If $\phi : U_n \to M_n$ is a spectrum–preserving multiplicative map, then there exists a nonsingular matrix R in M_n such that $\phi(U) = R^{-1} U R$ for any $U \in U_n$.*

Proof. It is easy to see that i) of Theorem 3 can not happen by choosing $A = D_1(2)D_2(\frac{1}{2})$. For any $x \in \mathbb{C}_1$, choosing $A = I_n + (x-1)E_{11}$ in ii) Theorem 3, we conclude that $\lambda(x)$ is a multiple eigenvalue of $\phi(A)$, and hence $\lambda(x)$ is a multiple eigenvalue of A. This implies $\lambda(x) = 1$ for any $x \in \mathbb{C}_1$, i.e.,

$$\phi(A) = P A^\tau P^{-1}, \qquad \forall\ A \in U_n, \tag{25}$$

where P, A^τ and τ are as in Theorem 3. For any $b \in \mathbb{C}_0$, let $A = bI_n$ in (25), then $\tau(b)$ is a eigenvalue of $\phi(A)$, and hence $\tau(b)$ is a eigenvalue of A. This implies $\tau(b) = b$ for any $b \in \mathbb{C}_0$. Hence the theorem follows. □

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