Archivum Mathematicum

Jan Kubalčík

On a two point linear boundary value problem for system of ODEs with deviating arguments

Archivum Mathematicum, Vol. 38 (2002), No. 2, 101--118

Persistent URL: `http://dml.cz/dmlcz/107825`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 38 (2002), 101 – 118

ON A TWO POINT LINEAR BOUNDARY VALUE PROBLEM FOR SYSTEM OF ODES WITH DEVIATING ARGUMENTS

JAN KUBALČíK

Abstract. Two point boundary value problem for the linear system of ordinary differential equations with deviating arguments

$$\begin{aligned}\mathbf{x}'(t) &= \mathbf{A}(t)\mathbf{x}(\tau_{11}(t)) + \mathbf{B}(t)\mathbf{u}(\tau_{12}(t)) + \mathbf{q}_1(t)\,,\\ \mathbf{u}'(t) &= \mathbf{C}(t)\mathbf{x}(\tau_{21}(t)) + \mathbf{D}(t)\mathbf{u}(\tau_{22}(t)) + \mathbf{q}_2(t)\,,\\ \alpha_{11}\mathbf{x}(0) + \alpha_{12}\mathbf{u}(0) = \mathbf{c}_0, &\quad \alpha_{21}\mathbf{x}(T) + \alpha_{22}\mathbf{u}(T) = \mathbf{c}_T\end{aligned}$$

is considered. For this problem the sufficient condition for existence and uniqueness of solution is obtained. The same approach as in [2], [3] is applied.

Notation

$\Re = (-\infty, +\infty)$ is the set of real numbers.

$\Re_+ = \langle 0, +\infty)$ is the set of real nonnegative numbers.

$I = \langle 0, T\rangle$ is the finite segment of $\Re$, $T > 0$.

$\Re^{m\times n}$ is the space of $m \times n$ matrices $\mathbf{X} = (x_{ij})_{i,j=1}^{m,n}$ with elements $x_{ij} \in \Re \quad (i = 1, \dots, m),\ (j = 1, \dots, n)$ and the norm

$$\|\mathbf{X}\| = \sum_{i=1}^{m}\sum_{j=1}^{n} |x_{ij}| \quad \text{and} \quad |\mathbf{X}| = (|x_{ij}|)_{i,j=1}^{m,n}\,.$$

$\Re_+^{m\times n}$ is the space of $m \times n$ matrices $\mathbf{X} = (x_{ij})_{i,j=1}^{m,n}$ with elements $x_{ij} \in \Re_+$, $(i = 1, \dots, m),\ (j = 1, \dots, n)$.

If $\mathbf{X},\ \mathbf{Y}$ are $m \times n$ matrices, then

$$\mathbf{X} \geq \mathbf{Y} \quad \text{if and only if} \quad \mathbf{X} - \mathbf{Y} \in \Re_+^{m\times n}\,.$$

$\mathbf{E}_n$ is the unit $n \times n$ matrix, $\mathbf{\Theta}_n$ is the zero $n \times n$ matrix.

2000 *Mathematics Subject Classification*: 34B10, 34B05, 34K10.

Key words and phrases: existence and uniqueness of solution, two point linear boundary value problem, linear system of ordinary differential equations, deviating argument, delay.

Received September 25, 2000.

If $\mathbf{X}$, $\mathbf{Y}$ are $n \times n$ matrices, then $[\mathbf{X}, \mathbf{Y}]$ is the $n \times 2n$ matrix

$$\begin{pmatrix} x_{11} & . & . & . & x_{1n} & y_{11} & . & . & . & y_{1n} \\ . & . & & & . & . & . & & & . \\ . & & . & & . & . & & . & & . \\ . & & & . & . & . & & & . & . \\ x_{n1} & . & . & . & x_{nn} & y_{n1} & . & . & . & y_{nn} \end{pmatrix}.$$

If $\mathbf{X} = (x_{ij})_{i,j=1}^{n,n} \in \Re^{n\times n}$ then $\mathbf{X}^{-1}$ is the inverse matrix to $\mathbf{X} \in \Re^{n\times n}$, $\det(\mathbf{X})$ is the determinant of the matrix $\mathbf{X} \in \Re^{n\times n}$ and $r(\mathbf{X})$ is the spectral radius of the matrix $\mathbf{X} \in \Re^{n\times n}$.

$C(I;\Re^{m\times n})$ is the space of continous matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ with the norm

$$\|\mathbf{X}\|_C = \sup\{\|\mathbf{X}(t)\| : t \in I\}\ .$$

$\tilde{C}(I;\Re^{m\times n})$ is the space of absolutely continous matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ with the norm

$$\|\mathbf{X}\|_{\tilde{C}} = \|\mathbf{X}(0)\| + \int_I \|\mathbf{X}'(t)\|\ dt\,.$$

$\tilde{C}_{\mathrm{loc}}(I;\Re^{m\times n})$ is the set of absolutely continous matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ on arbitrary subinterval of interval I.

$L^\alpha(I;\Re^{m\times n})$, where $1 \le \alpha < +\infty$ is the space of matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ integrable in the $\alpha-$th power with the norm

$$\|\mathbf{X}\|_{L^\alpha} = \left(\int_I \|\mathbf{X}(t)\|^\alpha\ dt\right)^{\frac{1}{\alpha}}\ ,$$
$$L(I;\Re^{m\times n}) = L^1(I;\Re^{m\times n})\,.$$

$L^{+\infty}(I;\Re^{m\times n})$ is the space of mesurable and bounded matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ with the norm

$$\|\mathbf{X}\|_{L^{+\infty}} = \ \text{ess}\ \sup\{\|\mathbf{X}(t)\| :\ t \in I\}\,.$$

$\Re^n = \Re^{n\times 1}$ is the space of n dimensional column vectors $\mathbf{x} = (x_i)_{i=1}^n$ with elements $x_i \in \Re \quad (i = 1, \dots, n)$.

$\boldsymbol{\theta}_n$ is the zero n vector.

$C(I;\Re^n) = C(I;\Re^{n\times 1})$, $\tilde{C}(I;\Re^n) = \tilde{C}(I;\Re^{n\times 1})$, $\tilde{C}_{\mathrm{loc}}(I;\Re^n) = \tilde{C}_{\mathrm{loc}}(I;\Re^{n\times 1})$ and $L^\alpha(I;\Re^n) = L^\alpha(I;\Re^{n\times 1})$.

Formulation of problem

We consider the system of ordinary differential equations with deviating arguments defined on interval $I = \langle 0,\ T\rangle$

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{A}(t)\mathbf{x}(\tau_{11}(t)) + \mathbf{B}(t)\mathbf{u}(\tau_{12}(t)) + \mathbf{q}_1(t)\,, \\ \mathbf{u}'(t) &= \mathbf{C}(t)\mathbf{x}(\tau_{21}(t)) + \mathbf{D}(t)\mathbf{u}(\tau_{22}(t)) + \mathbf{q}_2(t)\,, \end{aligned} \tag{1}$$

where $\mathbf{q}_1,\ \mathbf{q}_2 \in L(I;\Re^n)$, $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D} \in L(I;\Re^{n\times n})$ and τ_{ij} is mesurable for $i,j \in \{1,2\}$. In addition, let

$$\mathbf{x}(s)=\varphi(s),\quad \mathbf{u}(s)=\psi(s)\,,\quad \text{for } s\in\Re\setminus I\,, \tag{2}$$

where $\varphi,\ \psi : \Re\setminus I \to \Re^n$ are the continous and bounded functions. By the pair $(\mathbf{x},\ \mathbf{u})$ an absolutely continous n-dimensional vector functions which satisfies (1) almost everywhere on I and satisfies (2), we understand a solution of (1), (2). Denote

$$\mathbf{c}_0=(c_{0i})_{i=1}^n\in\Re^n\,,\quad \mathbf{c}_T=(c_{Ti})_{i=1}^n\in\Re^n$$

and consider the boundary value conditions for the solution of system (1), (2)

$$\alpha_{11}\mathbf{x}(0)+\alpha_{12}\mathbf{u}(0)=\mathbf{c}_0\,,\quad \alpha_{21}\mathbf{x}(T)+\alpha_{22}\mathbf{u}(T)=\mathbf{c}_T\,, \tag{3}$$

where $\alpha_{ij}\in\Re$ for $i,j\in\{1,2\}$. Define $\tau_{ij}^0 : I\to I,\ \chi_I : C(I;\Re)\to\{0,1\}$,

$$\tau_{ij}^0(t)=\begin{cases} 0 & \tau_{ij}(t)<0\\ \tau_{ij}(t) & \tau_{ij}(t)\in I\\ T & \tau_{ij}(t)>T\end{cases}\;;\quad t\in I,\ i,j\in\{1,2\} \tag{4}$$

$$\chi_I(\tau(t))=\begin{cases}1 & \tau(t)\in I\\ 0 & \tau(t)\notin I\end{cases}\;;\quad t\in I\,. \tag{5}$$

By (2), (4) and (5) we can the system (1) rewrite as follows

$$\begin{aligned}\mathbf{x}'(t)&=\chi_I(\tau_{11}(t))\mathbf{A}(t)\mathbf{x}(\tau_{11}^0(t))+\chi_I(\tau_{12}(t))\mathbf{B}(t)\mathbf{u}(\tau_{12}^0(t))\\ &\quad+\left[1-\chi_I(\tau_{11}(t))\right]\mathbf{A}(t)\varphi(\tau_{11}(t))\\ &\quad+\left[1-\chi_I(\tau_{12}(t))\right]\mathbf{B}(t)\psi(\tau_{12}(t))+\mathbf{q}_1(t)\,,\\ \mathbf{u}'(t)&=\chi_I(\tau_{21}(t))\mathbf{C}(t)\mathbf{x}(\tau_{21}^0(t))+\chi_I(\tau_{22}(t))\mathbf{D}(t)\mathbf{u}(\tau_{22}^0(t))\\ &\quad+\left[1-\chi_I(\tau_{21}(t))\right]\mathbf{C}(t)\varphi(\tau_{21}(t))\\ &\quad+\left[1-\chi_I(\tau_{22}(t))\right]\mathbf{D}(t)\psi(\tau_{22}(t))+\mathbf{q}_2(t)\,.\end{aligned} \tag{6}$$

Remark 1. The optimization theory is the motivation for the study of the boundary condition (3). The existence and uniqueness of the solution of a problem similar to (1), (2), (3) is studied for example by I. Öztürk in [6], L. Jodar in [7], [8] or I. Kiguradze and B. Půža in [9]. The junction of optimization theory and theory of boundary value problems is considered in [10], [11] and [12].

Existence and uniqueness of the solution

Along with system (6) we have to consider the corresponding homogeneous system

$$\begin{aligned}\mathbf{x}'(t)&=\chi_I(\tau_{11}(t))\mathbf{A}(t)\mathbf{x}(\tau_{11}^0(t))+\chi_I(\tau_{12}(t))\mathbf{B}(t)\mathbf{u}(\tau_{12}^0(t))\,,\\ \mathbf{u}'(t)&=\chi_I(\tau_{21}(t))\mathbf{C}(t)\mathbf{x}(\tau_{21}^0(t))+\chi_I(\tau_{22}(t))\mathbf{D}(t)\mathbf{u}(\tau_{22}^0(t))\end{aligned} \tag{6_0}$$

and in accordance with conditions (3) we have to consider the homogeneous boundary conditions

$$(3_0)\qquad \alpha_{11}\mathbf{x}(0)+\alpha_{12}\mathbf{u}(0)=\theta_n\,,\quad \alpha_{21}\mathbf{x}(T)+\alpha_{22}\mathbf{u}(T)=\theta_n\,.$$

Theorem 1. *The problem* (6), (3) *is uniquely solvable if and only if the corresponding homogeneous problem* (6_0), (3_0) *has only the trivial solution.*

Proof See [1] p. 345. □

The pair $(\mathbf{x},\ \mathbf{u})$ an absolutely continous n-dimensional vector functions is the solution of the problem (6_0), (3_0) if and only if $(\mathbf{x},\ \mathbf{u})$ is the solution of following integral equation on I

$$(7)\qquad \begin{aligned}\mathbf{x}(t)&=\mathbf{x}(0)+\int_0^t\left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right]\,ds\,,\\ \mathbf{u}(t)&=\mathbf{u}(0)+\int_0^t\left[\chi_I(\tau_{21}(s))\mathbf{C}(s)\mathbf{x}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))\mathbf{D}(s)\mathbf{u}(\tau_{22}^0(s))\right]\,ds\,.\end{aligned}$$

Clearly

$$(8)\qquad \begin{aligned}\mathbf{x}(\tau_{i1}^0(t))&=\mathbf{x}(0)+\int_0^{\tau_{i1}^0(t)}\mathbf{x}'(s)\,ds\,,\\ \mathbf{u}(\tau_{i2}^0(t))&=\mathbf{u}(0)+\int_0^{\tau_{i2}^0(t)}\mathbf{u}'(s)\,ds\,,\quad (i=1,2)\,.\end{aligned}$$

Denote

$$(9)\qquad A^0(t)=\mathbf{E}_n,\quad B^0(t)=\mathbf{\Theta}_n\,,\quad C^0(t)=\mathbf{\Theta}_n,\quad D^0(t)=\mathbf{E}_n\,,$$

$$\begin{aligned}A^i(t)&=\int_0^t\left[\chi_I(\tau_{11}(s))\mathbf{A}(s)A^{i-1}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))\mathbf{B}(s)C^{i-1}(\tau_{12}^0(s))\right]\,ds\,,\\ B^i(t)&=\int_0^t\left[\chi_I(\tau_{11}(s))\mathbf{A}(s)B^{i-1}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))\mathbf{B}(s)D^{i-1}(\tau_{12}^0(s))\right]\,ds\,,\\ C^i(t)&=\int_0^t\left[\chi_I(\tau_{21}(s))\mathbf{C}(s)A^{i-1}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))\mathbf{D}(s)C^{i-1}(\tau_{22}^0(s))\right]\,ds\,,\\ D^i(t)&=\int_0^t\left[\chi_I(\tau_{21}(s))\mathbf{C}(s)B^{i-1}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))\mathbf{D}(s)D^{i-1}(\tau_{22}^0(s))\right]\,ds\,,\\ &\qquad (i=1,2,\dots)\end{aligned}$$

and

$$
\begin{aligned}
&(AB)^1(\mathbf{x},\mathbf{u})^T(t)\\
&=\int_0^t \left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right]\,ds\,,\\
&(CD)^1(\mathbf{x},\mathbf{u})^T(t)\\
&=\int_0^t \left[\chi_I(\tau_{21}(s))\mathbf{C}(s)\mathbf{x}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))\mathbf{D}(s)\mathbf{u}(\tau_{22}^0(s))\right]\,ds\,,
\end{aligned}
\tag{10}
$$

$$
(i=1,2,\dots)\,,
$$

$$
\begin{aligned}
(AB)^i(\mathbf{x},\mathbf{u})^T(t)=\int_0^t &\left[\chi_I(\tau_{11}(s))\mathbf{A}(s)(AB)^{i-1}(\mathbf{x},\mathbf{u})^T(\tau_{11}^0(s))\right.\\
&\left.+\chi_I(\tau_{12}(s))\mathbf{B}(s)(CD)^{i-1}(\mathbf{x},\mathbf{u})^T(\tau_{12}^0(s))\right]\,ds\,,\\
(CD)^i(\mathbf{x},\mathbf{u})^T(t)=\int_0^t &\left[\chi_I(\tau_{21}(s))\mathbf{C}(s)(AB)^{i-1}(\mathbf{x},\mathbf{u})^T(\tau_{21}^0(s))\right.\\
&\left.+\chi_I(\tau_{22}(s))\mathbf{D}(s)(CD)^{i-1}(\mathbf{x},\mathbf{u})^T(\tau_{22}^0(s))\right]\,ds\,,
\end{aligned}
$$

Note 1. For an arbitrary matrix function $\mathbf{Z}\in C(I;\Re^{2n\times 2n})$ with column $\mathbf{z}_1,\dots,\mathbf{z}_{2n}$ we understand by $(AB)^i(\mathbf{Z})$, $(CD)^i(\mathbf{Z})$ the matrix with column $(AB)^i(\mathbf{z}_1),\dots,(AB)^i(\mathbf{z}_{2n})$, $(CD)^i(\mathbf{z}_1),\dots,(CD)^i(\mathbf{z}_{2n})$, respectively. $(AB)^k(\mathbf{E})(t) = \left[A^k(t),B^k(t)\right]$, $(CD)^k(\mathbf{E})(t)=\left[C^k(t),D^k(t)\right]$, $(k=1,2,\dots)$.

According to (9) we denote

$$
|A|^0(t)=\mathbf{E}_n,\quad |B|^0(t)=\mathbf{\Theta}_n\,,\quad |C|^0(t)=\mathbf{\Theta}_n,\quad |D|^0(t)=\mathbf{E}_n\,,
\tag{9a}
$$

$$
\begin{aligned}
|A|^i(t)&=\int_0^t \left[\chi_I(\tau_{11}(s))|\mathbf{A}(s)||A|^{i-1}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))|\mathbf{B}(s)||C|^{i-1}(\tau_{12}^0(s))\right]\,ds\,,\\
|B|^i(t)&=\int_0^t \left[\chi_I(\tau_{11}(s))|\mathbf{A}(s)||B|^{i-1}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))|\mathbf{B}(s)||D|^{i-1}(\tau_{12}^0(s))\right]\,ds\,,\\
|C|^i(t)&=\int_0^t \left[\chi_I(\tau_{21}(s))|\mathbf{C}(s)||A|^{i-1}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))|\mathbf{D}(s)||C|^{i-1}(\tau_{22}^0(s))\right]\,ds\,,\\
|D|^i(t)&=\int_0^t \left[\chi_I(\tau_{21}(s))|\mathbf{C}(s)||B|^{i-1}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))|\mathbf{D}(s)||D|^{i-1}(\tau_{22}^0(s))\right]\,ds\,,\\
&\qquad (i=1,2,\dots)\,.
\end{aligned}
$$

So, by (10) we can rewrite (8) as follows

$$
\begin{aligned}
&\mathbf{x}(\tau_{i1}^0(t))=\mathbf{x}(0)+(AB)^1(\mathbf{x},\mathbf{u})^T(\tau_{i1}^0(t))\,,\\
&\mathbf{u}(\tau_{i2}^0(t))=\mathbf{u}(0)+(CD)^1(\mathbf{x},\mathbf{u})^T(\tau_{i2}^0(t))\,,\quad (i=1,2)\,.
\end{aligned}
$$

If we use last equalities in (7) then we obtain

$$
\begin{aligned}
\mathbf{x}(t) &= \mathbf{x}(0) + \int_0^t \big[\chi_I(\tau_{11}(s))\mathbf{A}(s)\big[\mathbf{x}(0) + (AB)^1(\mathbf{x},\mathbf{u})^T(\tau_{11}^0(s))\big] \\
&\quad + \chi_I(\tau_{12}(s))\mathbf{B}(s)\big[\mathbf{u}(0) + (CD)^1(\mathbf{x},\mathbf{u})^T(\tau_{12}^0(s))\big]\big]\, ds \\
&= \mathbf{x}(0) + \left[\int_0^t \chi_I(\tau_{11}(s))\mathbf{A}(s)\, ds\right]\mathbf{x}(0) \\
&\quad + \left[\int_0^t \chi_I(\tau_{12}(s))\mathbf{B}(s)\, ds\right]\mathbf{u}(0) + (AB)^2(\mathbf{x},\mathbf{u})^T(t) \\
&= \big[A^0(t) + A^1(t)\big]\,\mathbf{x}(0) + B^1(t)\mathbf{u}(0) + (AB)^2(\mathbf{x},\mathbf{u})^T(t)\,, \\
\mathbf{u}(t) &= \mathbf{u}(0) + \int_0^t \big[\chi_I(\tau_{21}(s))\mathbf{C}(s)\big[\mathbf{x}(0) + (AB)^1(\mathbf{x},\mathbf{u})^T(\tau_{21}^0(s))\big] \\
&\quad + \chi_I(\tau_{22}(s))\mathbf{D}(s)\big[\mathbf{u}(0) + (CD)^1(\mathbf{x},\mathbf{u})^T(\tau_{22}^0(s))\big]\big]\, ds \\
&= \left[\int_0^t \chi_I(\tau_{21}(s))\mathbf{C}(s)\, ds\right]\mathbf{x}(0) \\
&\quad + \left[\mathbf{E}_n + \int_0^t \chi_I(\tau_{22}(s))\mathbf{D}(s)\, ds\right]\mathbf{u}(0) + (CD)^2(\mathbf{x},\mathbf{u})^T(t) \\
&= C^1(t)\mathbf{x}(0) + \big[D^0(t) + D^1(t)\big]\,\mathbf{u}(0) + (CD)^2(\mathbf{x},\mathbf{u})^T(t)
\end{aligned}
$$

and by the same way

$$
\begin{aligned}
\mathbf{x}(t) &= \big[A^0(t) + A^1(t) + A^2(t)\big]\,\mathbf{x}(0) + \big[B^1(t) + B^2(t)\big]\,\mathbf{u}(0) + (AB)^3(\mathbf{x},\mathbf{u})^T(t)\,, \\
\mathbf{u}(t) &= \big[C^1(t) + C^2(t)\big]\,\mathbf{x}(0) + \big[D^0(t) + D^1(t) + D^2(t)\big]\,\mathbf{u}(0) + (CD)^3(\mathbf{x},\mathbf{u})^T(t)\,.
\end{aligned}
$$

If we continue this process for $k = 1, 2, \dots$ then we obtain

$$
\begin{aligned}
\mathbf{x}(t) &= \left[\sum_{i=0}^{k-1} A^i(t)\right]\mathbf{x}(0) + \left[\sum_{i=0}^{k-1} B^i(t)\right]\mathbf{u}(0) + (AB)^k(\mathbf{x},\mathbf{u})^T(t)\,, \\
\mathbf{u}(t) &= \left[\sum_{i=0}^{k-1} C^i(t)\right]\mathbf{x}(0) + \left[\sum_{i=0}^{k-1} D^i(t)\right]\mathbf{u}(0) + (CD)^k(\mathbf{x},\mathbf{u})^T(t)\,.
\end{aligned}
\tag{11}
$$

From (3_0) and (11)

$$
\begin{aligned}
&\theta_{2n} \\
&= \begin{pmatrix} \alpha_{11}\mathbf{E}_n & \alpha_{12}\mathbf{E}_n \\ \alpha_{21}\sum_{i=0}^{k-1} A^i(T) + \alpha_{22}\sum_{i=0}^{k-1} C^i(T) & \alpha_{21}\sum_{i=0}^{k-1} B^i(T) + \alpha_{22}\sum_{i=0}^{k-1} D^i(T) \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \mathbf{x}(0) \\ \mathbf{u}(0) \end{pmatrix} + \begin{pmatrix} \theta_n \\ \alpha_{21}(AB)^k(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^k(\mathbf{x},\mathbf{u})^T(T) \end{pmatrix}
\end{aligned}
\tag{12}
$$

and if we assume the invertibility of the matrix

$$\begin{aligned}\mathbf{S}_k = \alpha_{12}\left(\alpha_{21}\sum_{i=0}^{k-1} A^i(T) + \alpha_{22}\sum_{i=0}^{k-1} C^i(T)\right) \\ - \alpha_{11}\left(\alpha_{21}\sum_{i=0}^{k-1} B^i(T) + \alpha_{22}\sum_{i=0}^{k-1} D^i(T)\right)\end{aligned} \tag{13}$$

then

$$\mathbf{x}(0) = -\alpha_{12}\mathbf{R}_k\,, \quad \mathbf{u}(0) = \alpha_{11}\mathbf{R}_k \tag{14}$$

is valid, where

$$\mathbf{R}_k = \mathbf{S}_k^{-1}\left[\alpha_{21}(AB)^k(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^k(\mathbf{x},\mathbf{u})^T(T)\right]\,. \tag{15}$$

Using (14) in (11) we obtain

$$\begin{aligned}\mathbf{x}(t) &= -\alpha_{12}\left[\sum_{i=0}^{m-1} A^i(t)\right]\mathbf{R}_k + \alpha_{11}\left[\sum_{i=0}^{m-1} B^i(t)\right]\mathbf{R}_k + (AB)^m(\mathbf{x},\mathbf{u})^T(t)\,, \\ \mathbf{u}(t) &= -\alpha_{12}\left[\sum_{i=0}^{m-1} C^i(t)\right]\mathbf{R}_k + \alpha_{11}\left[\sum_{i=0}^{m-1} D^i(t)\right]\mathbf{R}_k + (CD)^m(\mathbf{x},\mathbf{u})^T(t)\,.\end{aligned} \tag{16}$$

Henceforth

$$\begin{aligned}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} = &\left[\sum_{i=0}^{m-1}\begin{pmatrix}-\alpha_{12}A^i(t) & \alpha_{11}B^i(t)\\ -\alpha_{12}C^i(t) & \alpha_{11}D^i(t)\end{pmatrix}\right] \\ &\left[\begin{pmatrix}\mathbf{\Theta}_n & \mathbf{S}_k^{-1}\\ \mathbf{\Theta}_n & \mathbf{S}_k^{-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{\theta}_n\\ \alpha_{21}(AB)^k(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^k(\mathbf{x},\mathbf{u})^T(T)\end{pmatrix}\right] \\ &+ \begin{pmatrix}(AB)^m(\mathbf{x},\mathbf{u})^T(t)\\ (CD)^m(\mathbf{x},\mathbf{u})^T(t)\end{pmatrix}.\end{aligned} \tag{17}$$

From (17) by the notation (9a), Note 1 and properties of the norm we obtain

$$\left|\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}\right|_C \le \mathbf{M}_{k,m}\left|\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}\right|_C\,, \tag{18}$$

where $|\mathbf{y}|_C = (\|y_i\|_C)_{i=1}^{2n}$ and

$$\begin{aligned}\mathbf{M}_{k,m} = &\left[\sum_{i=0}^{m-1}\begin{pmatrix}\alpha_{12}|A|^i(T) & \alpha_{11}|B|^i(T)\\ \alpha_{12}|C|^i(T) & \alpha_{11}|D|^i(T)\end{pmatrix}\right] \\ &\left|\begin{pmatrix}\mathbf{\Theta}_n & \mathbf{S}_k^{-1}\\ \mathbf{\Theta}_n & \mathbf{S}_k^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{\Theta}_n & \mathbf{\Theta}_n\\ \alpha_{21}A^k(T) + \alpha_{22}C^k(T) & \alpha_{21}B^k(T) + \alpha_{22}D^k(T)\end{pmatrix}\right| \\ &+ \begin{pmatrix}|A|^m(T) & |B|^m(T)\\ |C|^m(T) & |D|^m(T)\end{pmatrix}\end{aligned} \tag{19}$$

and from inequality (18) evidently

$$(\mathbf{E}_{2n} - \mathbf{M}_{k,m}) \left| \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} \right|_C \le \theta_{2n} \,.$$

Now, if $m \in \{1, 2, \dots\}$ exists, such that

$$r(\mathbf{M}_{k,m}) < 1 \,, \tag{20}$$

then the inverse $(\mathbf{E}_{2n} - \mathbf{M}_{k,m})^{-1}$ of the matrix $(\mathbf{E}_{2n} - \mathbf{M}_{k,m})$ exists and

$$\left| \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} \right|_C \le (\mathbf{E}_{2n} - \mathbf{M}_{k,m})^{-1} \theta_{2n} = \theta_{2n} \,,$$

holds, so

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} \equiv \theta_{2n}$$

and the problem $(6_0), (3_0)$ has only trivial solution. According to Theorem 1, we have obtained, that the condition of invertibility of the matrix (13) for any $k = 1, 2, \dots$ and condition (20) for any $m = 1, 2, \dots$ are sufficient conditions for existence and uniqueness of the solution of the problem (1), (2), (3). By this result we can formulate the following theorem: Sufficient condition for existence and uniqueness of the solution in effective form.

Theorem 2. *Let positive integers k, m exist, such that the matrix* (13) *is invertible and the matrix* (19) *the condition* (20) *holds. Then the problem* (1), (2), (3) *is uniquely solvable.*

Consider now the differential system on I

$$\begin{aligned} \mathbf{x}'(t) &= \epsilon \mathbf{A}(t)\mathbf{x}(\tau_{11}(t)) + \epsilon \mathbf{B}(t)\mathbf{u}(\tau_{12}(t)) + \mathbf{q}_1(t) \,, \\ \mathbf{u}'(t) &= \epsilon \mathbf{C}(t)\mathbf{x}(\tau_{21}(t)) + \epsilon \mathbf{D}(t)\mathbf{u}(\tau_{22}(t)) + \mathbf{q}_2(t) \,, \end{aligned} \tag{21}$$

which is dependent on small positive parametr ϵ. Again $\mathbf{q}_1,\ \mathbf{q}_2 \in L(I; \Re^n)$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D} \in L(I; \Re^{n \times n})$ and τ_{ij} is mesurable for $i, j \in \{1, 2\}$ and consider the boundary value conditions for the solution of system (21), (2)

$$\alpha_1 \mathbf{x}(0) + \alpha_2 \mathbf{u}(0) = \mathbf{c}_0 \,, \quad \alpha_1 \mathbf{x}(T) + \alpha_2 \mathbf{u}(T) = \mathbf{c}_T \,, \tag{3'}$$

where $\alpha_i \in \Re$ for $i \in \{1, 2\}$. We proved, that Theorem 2 implies for problem (21), (2), (3') following corollary.

Corollary 1. *Let there exist a $j \in \{1, 2, \dots\}$ such that*

$$\begin{gathered} A^i(T) = \mathbf{\Theta}_n \,, \quad D^i(T) = \mathbf{\Theta}_n \qquad i = 1, \dots, j-1 \\ B^i(T) = \mathbf{\Theta}_n, \quad C^i(T) = \mathbf{\Theta}_n \qquad i = 0, \dots, j-1 \\ \det \left[\alpha_2 \left(\alpha_1 A^j(T) + \alpha_2 C^j(T) \right) - \alpha_1 \left(\alpha_1 B^j(T) + \alpha_2 D^j(T) \right) \right] \neq 0 \,. \end{gathered} \tag{22}$$

Then $\epsilon_0 > 0$ exists, such that the problem (21), (2), (3') *is uniquely solvable for arbitrary $\epsilon \in (0, \epsilon_0)$.*

Proof. We apply Theorem 2. Let $k = j+1,\ m = 1$. In addition we denote

$$\mathbf{A}(t;\epsilon) = \epsilon\mathbf{A}(t),\ \mathbf{B}(t;\epsilon) = \epsilon\mathbf{B}(t),\ \mathbf{C}(t;\epsilon) = \epsilon\mathbf{C}(t),\ \mathbf{D}(t;\epsilon) = \epsilon\mathbf{D}(t),$$

$$\begin{aligned}\mathbf{S}_k(\epsilon) = \alpha_2 &\left(\alpha_1 \sum_{i=0}^{k-1} A^i(T;\epsilon) + \alpha_2 \sum_{i=0}^{k-1} C^i(T;\epsilon)\right) \\ &- \alpha_1 \left(\alpha_1 \sum_{i=0}^{k-1} B^i(T;\epsilon) + \alpha_2 \sum_{i=0}^{k-1} D^i(T;\epsilon)\right),\end{aligned}$$

$$\begin{aligned}\mathbf{M}_{k,1}(\epsilon) = &\begin{pmatrix} \alpha_2\mathbf{E}_n & \mathbf{\Theta}_n \\ \mathbf{\Theta}_n & \alpha_1\mathbf{E}_n \end{pmatrix} \\ &\left|\begin{pmatrix} \mathbf{\Theta}_n & \mathbf{S}_k^{-1}(\epsilon) \\ \mathbf{\Theta}_n & \mathbf{S}_k^{-1}(\epsilon) \end{pmatrix} \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{\Theta}_n \\ \alpha_1 A^k(T;\epsilon) + \alpha_2 C^k(T;\epsilon) & \alpha_1 B^k(T;\epsilon) + \alpha_2 D^k(T;\epsilon) \end{pmatrix}\right| \\ &+ \begin{pmatrix} |A|^1(T;\epsilon) & |B|^1(T;\epsilon) \\ |C|^1(T;\epsilon) & |D|^1(T;\epsilon) \end{pmatrix},\end{aligned}$$

where matrix functions $A^i(.;\epsilon),\ B^i(.;\epsilon),\ C^i(.;\epsilon),\ D^i(.;\epsilon)$ are defined analogously as in (9). Evidently

$$\begin{aligned}&A^i(t;\epsilon) = \epsilon^i A^i(t), \quad B^i(t;\epsilon) = \epsilon^i B^i(t), \\ &C^i(t;\epsilon) = \epsilon^i C^i(t), \quad D^i(t;\epsilon) = \epsilon^i D^i(t), \qquad (i = 0,1,2,\dots)\end{aligned}$$

is valid. Hence

$$\begin{aligned}\mathbf{S}_k(\epsilon) &= \alpha_2 \left(\alpha_1 \sum_{i=0}^{k-1} \epsilon^i A^i(T) + \alpha_2 \sum_{i=0}^{k-1} \epsilon^i C^i(T)\right) \\ &\quad - \alpha_1 \left(\alpha_1 \sum_{i=0}^{k-1} \epsilon^i B^i(T) + \alpha_2 \sum_{i=0}^{k-1} \epsilon^i D^i(T)\right) \\ &= \epsilon^{k-1}\left[\alpha_2\left(\alpha_1 A^{k-1}(T) + \alpha_2 C^{k-1}(T)\right) - \alpha_1\left(\alpha_1 B^{k-1}(T) + \alpha_2 D^{k-1}(T)\right)\right] \\ &\quad + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\mathbf{E}_n\end{aligned}$$

and in accordance with $k = j+1$, (22) implies

$$\det\left(\mathbf{S}_k(\epsilon)\right) \neq 0.$$

So

$$\mathbf{M}_{k,1}(\epsilon) = \begin{pmatrix} \alpha_2\mathbf{E}_n & \mathbf{\Theta}_n \\ \mathbf{\Theta}_n & \alpha_1\mathbf{E}_n \end{pmatrix} \left| \frac{1}{\epsilon^{k-1}} \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{S}_k^{-1} \\ \mathbf{\Theta}_n & \mathbf{S}_k^{-1} \end{pmatrix} \epsilon^k \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{\Theta}_n \\ \alpha_1 A^k(T) + \alpha_2 C^k(T) & \alpha_1 B^k(T) + \alpha_2 D^k(T) \end{pmatrix} \right| + \epsilon \begin{pmatrix} |A|^1(T) & |B|^1(T) \\ |C|^1(T) & |D|^1(T) \end{pmatrix} = \epsilon \mathbf{M}_{k,1} \,.$$

If we get

$$\epsilon_0 = \frac{1}{r\left(\mathbf{M}_{k,1}\right)}\,,$$

then $r\left(\mathbf{M}_{k,1}(\epsilon)\right) < 1$ is valid for arbitrary $\epsilon \in (0, \epsilon_0)$. So, the assumptions of Theorem 2 are valid and in this way Corollary 1 is proved. □

Let now $\alpha_{11} = \alpha_{22} = 1$ and $\alpha_{12} = \alpha_{21} = 0$. Then the problem (1), (2), (3) call Cauchy-Nicoletti. By Theorem 2 we can formulated the following corollary for this problem.

Corollary 2. *Let $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = \alpha_{21} = 0$ and positive integers $k,\ m$ exist, such that the matrix*

$$-\left[\sum_{i=0}^{k-1} D^i(T)\right]$$

is invertible and the matrix

$$\mathbf{M}_{k,m} = \left[\sum_{i=0}^{m-1} \begin{pmatrix} \mathbf{\Theta}_n & |B|^i(T) \\ \mathbf{\Theta}_n & |D|^i(T) \end{pmatrix}\right] \left[\left| \begin{pmatrix} \mathbf{\Theta}_n & \left[\sum_{i=0}^{k-1} D^i(T)\right]^{-1} \\ \mathbf{\Theta}_n & \left[\sum_{i=0}^{k-1} D^i(T)\right]^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{\Theta}_n \\ C^k(T) & D^k(T) \end{pmatrix} \right|\right] + \begin{pmatrix} |A|^m(T) & |B|^m(T) \\ |C|^m(T) & |D|^m(T) \end{pmatrix}$$

the condition (20) *holds. Then the Cauchy-Nicoletti problem* (1), (2), (3) *is uniquely solvable.*

Remark 2. I. Öztürk study the similar problem (but with different boundary condition) in [6], where $\tau_{ij}(t) = t_{ij},\ t_{ij} \in I,\ i,j \in \{1,2\}$ ("the problem of interpolation"). The question of existence and uniqueness of problem's solution is solvable by Theorem 2 too.

Let now

$$\left(\tau_{ij}^0(t)\right)' \quad \text{exists almost everywhere on} \quad I, \qquad i,j \in \{1,2\} \tag{23}$$

and

$$0 < \inf_I \left\{ \left| \left(\tau_{ij}^0(t)\right)' \right| \right\} , \qquad i,j \in \{1,2\} . \tag{24}$$

Denote

$$\tau_{ij}^* = \inf_I \left\{ \left| \left(\tau_{ij}^0(t)\right)' \right| \right\} , \qquad i,j \in \{1,2\} . \tag{25}$$

Lemma 1 (Hölder inequality - see [4], p. 462)**.** *Let* $p,\ q \in \langle 1, +\infty\rangle$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f \in L^p(I; \Re^{n_1 \times n_2})$, $g \in L^q(I; \Re^{n_2 \times n_3})$*. Then*

$$\int_I \|f(s)g(s)\|\, ds = \left(\int_I \|f(s)\|^p\, ds\right)^{\frac{1}{p}} \left(\int_I \|g(s)\|^q\, ds\right)^{\frac{1}{q}}$$

is valid.

Lemma 2 (Wirtinger inequality - see [5], p. 99)**.** *Let* $u : \langle a, b\rangle \to \Re$ *is absolutely continous function, such that its derivation in second power is integrable. If exists* $t_0 \in \langle a, b\rangle$ *such that*

$$u(t_0) = 0 ,$$

then

$$\int_a^b u^2(t)\, dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b \left(u'(t)\right)^2\, dt .$$

holds.

So, from (7)

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s)) + \chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right]\, ds$$

and

$$\begin{aligned} \|\mathbf{x}\|_{L^\beta} \le\ & \|\mathbf{x}(0)\|_{L^\beta} \\ & + \left\| \int_0^t \left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s)) + \chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right]\, ds \right\|_{L^\beta} . \end{aligned} \tag{26}$$

Now

$$\begin{aligned} \left\|(AB)^1(\mathbf{x},\mathbf{u})^T(t)\right\|_{L^\beta} \le & \left(\int_0^T \left(\int_0^t \left\|\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\right\|\, ds\right)^\beta dt\right)^{\frac{1}{\beta}} \\ & + \left(\int_0^T \left(\int_0^t \left\|\chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right\|\, ds\right)^\beta dt\right)^{\frac{1}{\beta}} . \end{aligned}$$

By Lemma 1, where $f=\chi_I(\tau_{11})\mathbf{A},\ \chi_I(\tau_{12})\mathbf{B}$ and $g=\mathbf{x}(\tau_{11}^0),\ \mathbf{u}(\tau_{12}^0)$, respectively and $\frac{1}{\alpha}+\frac{2}{\beta}=1$ we obtain

$$\left\|(AB)^1(\mathbf{x},\mathbf{u})^T(t)\right\|_{L^\beta} \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}\left[\int_0^T\left(\int_0^t \left\|\mathbf{x}(\tau_{11}^0(s))\right\|^{\frac{\beta}{2}}\,ds\right)^2 dt\right]^{\frac{1}{\beta}} + \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}\left[\int_0^T\left(\int_0^t \left\|\mathbf{u}(\tau_{12}^0(s))\right\|^{\frac{\beta}{2}}\,ds\right)^2 dt\right]^{\frac{1}{\beta}}.$$

Let $\xi_{1j}=\tau_{1j}^0(s),\ j\in\{1,2\}$. Then, by (25)

$$d\xi_{1j}=\left(\tau_{1j}^0(s)\right)'\,ds\,,\quad ds=\frac{d\xi_{1j}}{\left(\tau_{1j}^0(s)\right)'}\le\frac{d\xi_{1j}}{\tau_{1j}^*}\qquad j\in\{1,2\}\,.$$

Hence

$$\left\|(AB)^1(\mathbf{x},\mathbf{u})^T\ (t)\right\|_{L^\beta} \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}\left(\frac{1}{\tau_{11}^*}\right)^{\frac{2}{\beta}}\left[\int_0^T\left(\int_{\tau_{11}^0(0)}^{\tau_{11}^0(t)} \|\mathbf{x}(\xi_{11})\|^{\frac{\beta}{2}}\,d\xi_{11}\right)^2 dt\right]^{\frac{1}{\beta}} + \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}\left(\frac{1}{\tau_{12}^*}\right)^{\frac{2}{\beta}}\left[\int_0^T\left(\int_{\tau_{12}^0(0)}^{\tau_{12}^0(t)} \|\mathbf{u}(\xi_{12})\|^{\frac{\beta}{2}}\,d\xi_{12}\right)^2 dt\right]^{\frac{1}{\beta}}.$$

From Lemma 2, where $u=\int_{\tau_{11}^0(0)}^{\tau_{11}^0(t)}\|\mathbf{x}(\xi_{11})\|^{\frac{\beta}{2}}\,d\xi_{11}\,,\ \int_{\tau_{12}^0(0)}^{\tau_{12}^0(t)}\|\mathbf{u}(\xi_{12})\|^{\frac{\beta}{2}}\,d\xi_{12}$, respectively, we obtain

$$\left\|(AB)^1(\mathbf{x},\mathbf{u})^T(t)\right\|_{L^\beta} \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{11}^*}\right)^{\frac{2}{\beta}}\left[\int_0^T \left\|\mathbf{x}(\tau_{11}^0(t))\right\|^{\beta} dt\right]^{\frac{1}{\beta}} + \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{12}^*}\right)^{\frac{2}{\beta}}\left[\int_0^T \left\|\mathbf{u}(\tau_{12}^0(t))\right\|^{\beta} dt\right]^{\frac{1}{\beta}}$$

and by the same transformation

$$\left\|(AB)^1(\mathbf{x},\mathbf{u})^T(t)\right\|_{L^\beta} \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{11}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{11}^*}\right)^{\frac{1}{\beta}}\|\mathbf{x}\|_{L^\beta} + \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{12}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{12}^*}\right)^{\frac{1}{\beta}}\|\mathbf{u}\|_{L^\beta}\,. \tag{27}$$

Let k=2. By (13), (14) and (15)

$$\mathbf{x}(0)=-\alpha_{12}\mathbf{S}_2^{-1}\left[\alpha_{21}(AB)^2(\mathbf{x},\mathbf{u})^T(T)+\alpha_{22}(CD)^2(\mathbf{x},\mathbf{u})^T(T)\right]\,.$$

So, by properties of the norm

$$\|\mathbf{x}(0)\|_{L^\beta} \le |\alpha_{12}| T^{\frac{1}{\beta}} \left\|\mathbf{S}_2^{-1}\right\| \left\|\alpha_{21}(AB)^2(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^2(\mathbf{x},\mathbf{u})^T(T)\right\| . \tag{28}$$

From (10) evidently

$$\begin{aligned}(AB)^2(\mathbf{x},\mathbf{u})^T(T) = &\int_0^T \left[\chi_I(\tau_{11}(t))\mathbf{A}(t)\left(\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\,ds\right)\right] dt\\ &+ \int_0^T \left[\chi_I(\tau_{11}(t))\mathbf{A}(t)\left(\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\,ds\right)\right] dt\\ &+ \int_0^T \left[\chi_I(\tau_{12}(t))\mathbf{B}(t)\left(\int_0^{\tau_{12}^0(t)} \chi_I(\tau_{21}(s))\mathbf{C}(s)\mathbf{x}(\tau_{21}^0(s))\,ds\right)\right] dt\\ &+ \int_0^T \left[\chi_I(\tau_{12}(t))\mathbf{B}(t)\left(\int_0^{\tau_{12}^0(t)} \chi_I(\tau_{22}(s))\mathbf{D}(s)\mathbf{u}(\tau_{11}^0(s))\,ds\right)\right] dt .\end{aligned} \tag{29}$$

holds. Clearly, by properties of the norm

$$\begin{aligned}&\left\|\int_0^T \left[\chi_I(\tau_{11}(t))\mathbf{A}(t)\left(\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\,ds\right)\right] dt\right\|\\ &\le \int_0^T \left\|\chi_I(\tau_{11}(t))\mathbf{A}(t)\left(\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\,ds\right)\right\| dt\\ &\le \int_0^T \left[\|\chi_I(\tau_{11}(t))\mathbf{A}(t)\| \left\|\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\,ds\right\|\right] dt\\ &\le \int_0^T \left[\|\chi_I(\tau_{11}(t))\mathbf{A}(t)\| \left(\int_0^{\tau_{11}^0(t)} \left\|\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\right\|\,ds\right)\right] dt\end{aligned} \tag{30}$$

is valid. Hence, by Lemma 1, where $f = \chi_I(\tau_{11})\mathbf{A}$, $g = \mathbf{x}(\tau_{11}^0)$ and $\frac{1}{\alpha^*} + \frac{1}{\beta} = 1$

$$\int_0^{\tau_{11}^0(t)} \left\|\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\right\| ds \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}} \|\mathbf{x}\|_{L^\beta} . \tag{31}$$

Denote

$$\|A\|^1(t) = \int_0^t \|\chi_I(\tau_{11}(s))\mathbf{A}(s)\|\,ds \tag{32}$$

and $\|B\|^1(t)$, $\|C\|^1(t)$, $\|D\|^1(t)$ analogously. By the same way

$$\begin{aligned}\left\|(AB)^2(\mathbf{x},\mathbf{u})^T(T)\right\| \le &\left[\|A\|^1(T)\,\|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}}\right.\\ &\left.+ \|B\|^1(T)\,\|\chi_I(\tau_{21})\mathbf{C}\|_{L^{\alpha^*}}\right] \|\mathbf{x}\|_{L^\beta}\\ &+ \left[\|A\|^1(T)\,\|\chi_I(\tau_{12})\mathbf{B}\|_{L^{\alpha^*}}\right.\\ &\left.+ \|B\|^1(T)\,\|\chi_I(\tau_{22})\mathbf{D}\|_{L^{\alpha^*}}\right] \|\mathbf{u}\|_{L^\beta} .\end{aligned} \tag{33}$$

Analogously

$$\left\|(CD)^2(\mathbf{x},\mathbf{u})^T(T)\right\| \le \left[\|C\|^1(T)\,\|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}} + \|D\|^1(T)\,\|\chi_I(\tau_{21})\mathbf{C}\|_{L^{\alpha^*}}\right]\|\mathbf{x}\|_{L^\beta} + \left[\|C\|^1(T)\,\|\chi_I(\tau_{12})\mathbf{B}\|_{L^{\alpha^*}} + \|D\|^1(T)\,\|\chi_I(\tau_{22})\mathbf{D}\|_{L^{\alpha^*}}\right]\|\mathbf{u}\|_{L^\beta}\ . \tag{34}$$

If we denote

$$\begin{aligned} H_1 &= \Big[\big(|\alpha_{21}|\cdot\|A\|^1(T) + |\alpha_{22}|\cdot\|C\|^1(T)\big)\,\|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}} \\ &\quad + \big(|\alpha_{21}|\cdot\|B\|^1(T) + |\alpha_{22}|\cdot\|D\|^1(T)\big)\,\|\chi_I(\tau_{21})\mathbf{C}\|_{L^{\alpha^*}}\Big] \\ H_2 &= \Big[\big(|\alpha_{21}|\cdot\|A\|^1(T) + |\alpha_{22}|\cdot\|C\|^1(T)\big)\,\|\chi_I(\tau_{12})\mathbf{B}\|_{L^{\alpha^*}} \\ &\quad + \big(|\alpha_{21}|\cdot\|B\|^1(T) + |\alpha_{22}|\cdot\|D\|^1(T)\big)\,\|\chi_I(\tau_{22})\mathbf{D}\|_{L^{\alpha^*}}\Big]\ . \end{aligned} \tag{35}$$

then from (28), (33), (34) and (35)

$$\|\mathbf{x}(0)\|_{L^\beta} \le |\alpha_{12}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_1\,\|\mathbf{x}\|_{L^\beta} + |\alpha_{12}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_2\,\|\mathbf{u}\|_{L^\beta}\ . \tag{36}$$

By (26), (27), (35) and (36)

$$\begin{aligned} \|\mathbf{x}\|_{L^\beta} \le &\left\{\|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{11}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{11}^*}\right)^{\frac{1}{\beta}} + |\alpha_{12}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_1\right\}\|\mathbf{x}\|_{L^\beta} \\ &+ \left\{\|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{12}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{12}^*}\right)^{\frac{1}{\beta}} + |\alpha_{12}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_2\right\}\|\mathbf{u}\|_{L^\beta}\ . \end{aligned} \tag{37}$$

From (7) analogously

$$\begin{aligned} \|\mathbf{u}\|_{L^\beta} \le &\left\{\|\chi_I(\tau_{21})\mathbf{C}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{21}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{21}^*}\right)^{\frac{1}{\beta}} + |\alpha_{11}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_1\right\}\|\mathbf{x}\|_{L^\beta} \\ &+ \left\{\|\chi_I(\tau_{22})\mathbf{D}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{22}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{22}^*}\right)^{\frac{1}{\beta}} + |\alpha_{11}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_2\right\}\|\mathbf{u}\|_{L^\beta}\ . \end{aligned} \tag{38}$$

So, on the whole, by (37) and (38)

$$\begin{pmatrix} \|\mathbf{x}\|_{L^\beta} \\ \|\mathbf{u}\|_{L^\beta} \end{pmatrix} \le \mathbf{M}\begin{pmatrix} \|\mathbf{x}\|_{L^\beta} \\ \|\mathbf{u}\|_{L^\beta} \end{pmatrix}, \tag{39}$$

where

$$\mathbf{M} = \begin{pmatrix} \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}T_{11}^* + U_{21} & \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}T_{12}^* + U_{22} \\ \|\chi_I(\tau_{21})\mathbf{C}\|_{L^\alpha}T_{21}^* + U_{11} & \|\chi_I(\tau_{22})\mathbf{D}\|_{L^\alpha}T_{22}^* + U_{12} \end{pmatrix} \tag{40}$$

and for $i, j = 1, 2$

$$T_{ij}^{*} = \left(\frac{2T}{\pi\tau_{ij}^{*}}\right)^{\frac{2}{\beta}} \left(\frac{1}{\tau_{ij}^{*}}\right)^{\frac{1}{\beta}}, \quad U_{ij} = |\alpha_{1i}| T^{\frac{1}{\beta}} \|\mathbf{S}_2^{-1}\| H_j$$

is valid and from inequality (39) evidently

$$(\mathbf{E}_2 - \mathbf{M}) \begin{pmatrix} \|\mathbf{x}\|_{L^\beta} \\ \|\mathbf{u}\|_{L^\beta} \end{pmatrix} \leq \theta_2 \,.$$

Now, if

$$r(\mathbf{M}) < 1 \tag{41}$$

holds then the matrix $(\mathbf{E}_2 - \mathbf{M})^{-1}$ inverse to the matrix $(\mathbf{E}_2 - \mathbf{M})$ exists and

$$\begin{pmatrix} \|\mathbf{x}\|_{L^\beta} \\ \|\mathbf{u}\|_{L^\beta} \end{pmatrix} \leq (\mathbf{E}_2 - \mathbf{M})^{-1} \theta_2 = \theta_2$$

is valid, so

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} \equiv \theta_{2n}$$

and the problem (6_0), (3_0) has only trivial solution. According to Theorem 1, we obtained, that the condition of invertibility of the matrix (13) for $k = 2$, conditions (23), (24) for $\tau_{ij}^{0}(t)$ and condition (41) are sufficient conditions for existence and uniqueness of the solution of the problem (1), (2), (3). By this result we can formulate the following theorem - sufficient condition for existence and uniqueness of the solution in effective form.

Theorem 3. *Let conditions* (23), (24) *are valid, the matrix* (13) *is invertible for* $k = 2$ *and the matrix* (40) *the condition* (41) *holds, where* $1 \leq \alpha \leq +\infty$, $\frac{1}{\alpha} + \frac{2}{\beta} = 1$, $\frac{1}{\alpha^*} + \frac{1}{\beta} = 1$. *Then the problem* (1), (2), (3) *is uniquely solvable.*

Note 2. Let $\alpha_{11} = \alpha_{21} = 1$ and $\alpha_{12} = \alpha_{22} = 0$. Then we obtain in the both of cases (Theorem 2, Theorem 3) the same results as in [3].

The following corollary for Cauchy-Nicoletti problem we can formulated.

Corollary 3. *Let* $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = \alpha_{21} = 0$, *conditions* (23), (24) *are valid, the matrix*

$$V = -\left[\mathbf{E}_{2n} + D^1(T)\right]$$

is invertible and the matrix

$$\mathbf{M} = \begin{pmatrix} \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha} T_{11}^{*} & \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha} T_{12}^{*} \\ \|\chi_I(\tau_{21})\mathbf{C}\|_{L^\alpha} T_{21}^{*} & \|\chi_I(\tau_{22})\mathbf{D}\|_{L^\alpha} T_{22}^{*} \end{pmatrix} + T^{\frac{1}{\beta}} \|V^{-1}\|$$

$$\begin{pmatrix} 0 & \|C\|^1(T)\|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}} + \|D\|^1(\tau)\|\chi_I(\tau_{21})\mathbf{C}\|_{L^{\alpha^*}} \\ 0 & \|C\|^1(T)\|\chi_I(\tau_{12})\mathbf{B}\|_{L^{\alpha^*}} + \|D\|^1(T)\|\chi_I(\tau_{22})\mathbf{D}\|_{L^{\alpha^*}} \end{pmatrix}^{\top}$$

the condition (41) *holds, where* $1 \le \alpha \le +\infty$, $\frac{1}{\alpha} + \frac{2}{\beta} = 1$, $\frac{1}{\alpha^*} + \frac{1}{\beta} = 1$. *Then the problem* (1), (2), (3) *is uniquely solvable.*

Considering the following differential system on I with delay (retarded argument)

$$(1') \qquad \begin{aligned} \mathbf{x}'(t) &= \mathbf{A}\mathbf{x}(t-\Delta) + \mathbf{B}\mathbf{u}(t-\Delta) + \mathbf{q}_1(t)\,, \\ \mathbf{u}'(t) &= \mathbf{C}\mathbf{x}(t-\Delta) + \mathbf{D}\mathbf{u}(t-\Delta) + \mathbf{q}_2(t)\,, \end{aligned}$$

with boundary conditions (2) and (3), where $\mathbf{q}_1,\ \mathbf{q}_2 \in L(I;\Re^n)$, $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D} \in \Re^{n\times n}$, $\Delta \in \Re_+$, Theorem 3 implies

Corollary 4. *Let the matrix*

$$\mathbf{S}_2 = (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})\mathbf{E}_n + [\alpha_{12}(\alpha_{21}\mathbf{A} + \alpha_{22}\mathbf{C}) - \alpha_{11}(\alpha_{21}\mathbf{B} + \alpha_{22}\mathbf{D})]\max\{0; T-\Delta\}$$

is invertible and the matrix

$$\mathbf{M} = \begin{pmatrix} \|\mathbf{A}\|\left(\frac{2T}{\pi}\right)^{\frac{2}{\beta}} + W_{21} & \|\mathbf{C}\|\left(\frac{2T}{\pi}\right)^{\frac{2}{\beta}} + W_{11} \\ \|\mathbf{B}\|\left(\frac{2T}{\pi}\right)^{\frac{2}{\beta}} + W_{22} & \|\mathbf{D}\|\left(\frac{2T}{\pi}\right)^{\frac{2}{\beta}} + W_{12} \end{pmatrix} (\max\{0; T-\Delta\})^{\frac{\beta-2}{\beta}}$$

where for $i, j = 1, 2$

$$W_{ij} = |\alpha_{1i}| T^{\frac{1}{\beta}} \|\mathbf{S}_2^{-1}\| H_j \max{}^{\frac{2\beta-1}{\beta-2}}\{0; T-\Delta\}$$

and

$$\begin{aligned} H_1 &= [(|\alpha_{21}|\|\mathbf{A}\| + |\alpha_{22}|\|\mathbf{C}\|)\,\|\mathbf{A}\| + (|\alpha_{21}|\|\mathbf{B}\| + |\alpha_{22}|\|\mathbf{D}\|)\,\|\mathbf{C}\|] \\ H_2 &= [(|\alpha_{21}|\|\mathbf{A}\| + |\alpha_{22}|\|\mathbf{C}\|)\,\|\mathbf{B}\| + (|\alpha_{21}|\|\mathbf{B}\| + |\alpha_{22}|\|\mathbf{D}\|)\,\|\mathbf{D}\|] \end{aligned}$$

the condition (41) *holds, where* $2 \le \beta \le +\infty$. *Then the problem* (1'), (2), (3) *is uniquely solvable.*

Proof. The result is obtained from Theorem 3 by directly way. In this special case, Theorem 3 gives using (4) and (5) that

$$\tau_{ij}^0(t) = \begin{cases} 0 & t < \Delta \\ t-\Delta & t \ge \Delta \end{cases}; \qquad t \in I,\ i,j \in \{1,2\}\,,$$

$$\chi_I(\tau_{ij}(t)) = \begin{cases} 1 & t \ge \Delta \\ 0 & t < \Delta \end{cases}; \qquad t \in I,\ i,j \in \{1,2\}\,.$$

From (9) we obtain

$$A^1(t) = \int_0^t [\chi_I(s-\Delta)\mathbf{A}]\, ds = \mathbf{A}\int_0^t \chi_I(s-\Delta)\, ds = \mathbf{A}\max\{0, t-\Delta\}$$

and for $B^1(t),\ C^1(t),\ D^1(t)$ analogously. From (32) we obtain

$$\|A\|^1(t) = \int_0^t \|\chi_I(s-\Delta)\mathbf{A}\|\ ds = \|\mathbf{A}\| \int_0^t \|\chi_I(s-\Delta)\|\ ds = \|\mathbf{A}\| \max\{0, t-\Delta\}$$

and for $\|B\|^1(t),\ \|C\|^1(t),\ \|D\|^1(t)$ analogously. Clearly

$$\begin{aligned}\|\chi_I(s-\Delta)\mathbf{A}\|_{L^\alpha} &= \left(\int_0^T \|\chi_I(s-\Delta)\mathbf{A}\|^\alpha\ ds\right)^{\frac{1}{\alpha}} \\ &= \|\mathbf{A}\| \left(\int_0^T \|\chi_I(s-\Delta)\|\ ds\right)^{\frac{1}{\alpha}} = \|\mathbf{A}\| \left(\max\{0; T-\Delta\}\right)^{\frac{1}{\alpha}}\end{aligned}$$

and for $\|\chi_I(s-\Delta)\mathbf{B}\|_{L^\alpha},\ \|\chi_I(s-\Delta)\mathbf{C}\|_{L^\alpha},\ \|\chi_I(s-\Delta)\mathbf{D}\|_{L^\alpha}$ analogously. The results for α^* we obtain by the same way. So

$$\frac{1}{\alpha} = 1 - \frac{2}{\beta} = \frac{\beta-2}{\beta} \quad \text{and} \quad \frac{1}{\alpha^*} = 1 - \frac{1}{\beta} = \frac{\beta-1}{\beta},$$

if $\beta = +\infty$ we understand, for example, by $\frac{\beta-2}{\beta}$

$$\lim_{\beta\to+\infty} \frac{\beta-2}{\beta}.$$

Evidently, the condition (24) is not valid on segment $\langle 0, \Delta)$. Hence

$$\begin{aligned}&\left(\int_0^T \left(\int_0^t \left\|\chi_I(s-\Delta)\mathbf{A}\mathbf{x}(\tau_{11}^0(s))\right\|\ ds\right)^\beta dt\right)^{\frac{1}{\beta}} \\ &\qquad = \left(\int_\Delta^T \left(\int_\Delta^t \|\chi_I(s-\Delta)\mathbf{A}\mathbf{x}(s-\Delta)\|\ ds\right)^\beta dt\right)^{\frac{1}{\beta}} = (*)\end{aligned}$$

and by Lemma 1, where $f = \chi_I(s-\Delta)\mathbf{A},\ g = \mathbf{x}(s-\Delta)$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we obtain

$$(*) \le \|\chi_I(s-\Delta)\mathbf{A}\|_{L^\alpha} \left[\int_\Delta^T \left(\int_\Delta^t \|\mathbf{x}(s-\Delta)\|^{\frac{\beta}{2}}\ ds\right)^2 dt\right]^{\frac{1}{\beta}} = (**)$$

if $\xi = s-\Delta$, then $d\xi = ds$ and

$$\begin{aligned}(**) &= \|\chi_I(s-\Delta)\mathbf{A}\|_{L^\alpha} \left[\int_\Delta^T \left(\int_0^{t-\Delta} \|\mathbf{x}(\xi)\|^{\frac{\beta}{2}}\ d\xi\right)^2 dt\right]^{\frac{1}{\beta}} \\ &\le \|\chi_I(s-\Delta)\mathbf{A}\|_{L^\alpha} \left[\int_0^T \left(\int_0^t \|\mathbf{x}(\xi)\|^{\frac{\beta}{2}}\ d\xi\right)^2 dt\right]^{\frac{1}{\beta}}.\end{aligned}$$

□

Note 3. The system (1) can be rewrited as follows

$$\mathbf{y}'(t) = \sum_{i=1}^{4} \mathbf{P}_i(t)\mathbf{y}(\tau_i(t)) + \mathbf{q}(t)\,,$$

where

$$\mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}, \quad \mathbf{P}_1(t) = \begin{pmatrix} \mathbf{A}(t) & \Theta_n \\ \boldsymbol{\Theta}_n & \boldsymbol{\Theta}_n \end{pmatrix}, \quad \mathbf{P}_2(t) = \begin{pmatrix} \boldsymbol{\Theta}_n & \mathbf{B}(t) \\ \boldsymbol{\Theta}_n & \boldsymbol{\Theta}_n \end{pmatrix},$$

$$\mathbf{P}_3(t) = \begin{pmatrix} \boldsymbol{\Theta}_n & \boldsymbol{\Theta}_n \\ \mathbf{C}(t) & \boldsymbol{\Theta}_n \end{pmatrix}, \quad \mathbf{P}_4(t) = \begin{pmatrix} \boldsymbol{\Theta}_n & \boldsymbol{\Theta}_n \\ \boldsymbol{\Theta}_n & \mathbf{D}(t) \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix},$$

$$\tau_1(t) = \tau_{11}(t)\,, \quad \tau_2(t) = \tau_{12}(t)\,, \quad \tau_3(t) = \tau_{21}(t)\,, \quad \tau_4(t) = \tau_{22}(t)\,.$$

So, the problem for arbitrary finite integer $N \in \Re_+$, $i = 1, \dots, N$ can be considered.

References

[1] Kiguradze, I., Půža, B., *On boundary value problems for systems of linear functional differential equations*, Czechoslovak Math. J. **47** No. 2 (1997), 233–244.

[2] Kubalčík, J., *On a Two Point Boundary Value Problem for Linear System of ODEs*, in press.

[3] Kubalčík, J., *On one Two Point Boundary Value Problem for Linear System of ODEs with Deviating Arguments*, CDDE 2000 Program, Abstract, Extended abstracts, Masaryk University, Brno 2000, 75–76.

[4] Marshall, A. W., Olkin, I., *Inequalities: Theory of Majorization and Its Applications*, Academic Press, 1979, (in Russian), Mir, Moskva, 1983.

[5] Kiguradze, I., *An initial value problem and boundary value problems for systems of ordinary differential equations. Vol.I: Linear theory*, MU Brno 1997 (in Czech), Macniereba, Tbilisi 1997, 216pp. (in Russian).

[6] Öztürk, I., *On the nonlocal boundary value problem for first order loaded differential equations*, Indian J. Pure Appl. Math. **26**, No. 4 (1995), 309–314.

[7] Jodar, L., *Boundary value problems and Green's operator functions*, Glas. Mat. Ser. III **24**(44), No. 4 (1989), 511–522.

[8] Jodar, L., *An algebraic approach for solving boundary value matrix problem: Existence, uniqueness and closed form solutions*, Rev. Mat. Complut. **1**, No. 1-3 (1988), 145–155.

[9] Kiguradze, I., Půža, B., *On the Vallée-Poussin problem for singular differential equations with deviating arguments*, Arch. Math. (Brno) **33** No. 1-2 (1997), 127–138.

[10] Elsgolc, L. E., Norkin, S. B., *Introduction into theory of differential equations with deviating argument*, Nauka Moskva 1971 (in Russian).

[11] Asmykovich, I. K., Marchenko, V. M., *Spectrum control in systems with delay*, Avtom. Telemekh. No. 7, (1976), 5–14 (in Russian).

[12] Asmykovich, I. K., Marchenko, V. M., *Modal control of multiinput linear systems with delay*, Avtom. Telemekh. No. 1 (1980), 5–10 (in Russian).

Department of Mathematical Analysis, Faculty of Science
Masaryk University, Janáčkovo nám. 2a
662 95 Brno, CZECH REPUBLIC
E-mail: kubalis@math.muni.cz