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SOME PROPERTIES OF THE WEAK SUBALGEBRA LATTICE OF A PARTIAL ALGEBRA OF A FIXED TYPE

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Abstract. We investigate, using results from [9], when a given lattice is isomorphic to the weak subalgebra lattice of a partial algebra of a fixed type. First, we reduce this problem to the question when hyperedges of a hypergraph can be directed to a form of directed hypergraph of a fixed type. Secondly, we show that it is enough to consider some special hypergraphs. Finally, translating these results onto the lattice language, we obtain necessary conditions for our algebraic problem, and also, we completely characterize the weak subalgebra lattice for algebras of some types.

1.

Since the present paper is strongly related to [9], we use the notation and definitions from there. For basic concepts concerning hypergraphs see e.g. [3]; concerning algebras (partial and total) and lattices of subalgebras see e.g. [2], [4], [6] and [7]; concerning lattice theory see e.g. [5] and [7].

It is known that a lattice $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ is isomorphic to the weak subalgebra lattice $\mathbf{S}_w(\mathbf{A})$ of a partial algebra $\mathbf{A}$, or equivalently, to the weak subhypergraph lattice of an algebraic, directed or undirected hypergraph (see Theorem 3.16 in [9]; see also [1]) iff

(W.1) $\mathbf{L}$ is algebraic and distributive,
(W.2) every element of $\mathbf{L}$ is a join of join-irreducible elements,
(W.3) for each non-zero and non-atomic join-irreducible element i, the set of all atoms contained in i is finite and non-empty,
(W.4) the set of all non-zero and non-atomic join-irreducible elements of $\mathbf{L}$ is an antichain with respect to the lattice ordering $\leq_{\mathbf{L}}$.

An element i is join-irreducible iff for each k_1, k_2, $i = k_1 \vee k_2$ implies $i = k_1$ or $i = k_2$.

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In [9] the following result, fundamental for our investigations, is given (Proposition 3.19).

Proposition 1.1. *Let a lattice* $\mathbf{L}$ *satisfy* (W.1)–(W.4), *and* $\langle K, \kappa\rangle$ *be an algebra type. Then there is a partial algebra* $\mathbf{A}$ *of type* $\langle K, \kappa\rangle$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ *iff there is an algebraic hypergraph* $\mathbf{G}$ *of type* $\big(\kappa^{-1}(0), \kappa^{-1}(1), \kappa^{-1}(2), \dots\big)$ *such that* $\mathbf{G}^{**} \simeq \mathbf{U}(\mathbf{L})$.

The first aim of the paper (section 2) is to show that in the above fact such an algebraic hypergraph may be replaced by a directed hypergraph $\mathbf{D}$ of type $\underline{\tau}$, where $\underline{\tau}$ depends, in a simple combinatorial way, on $\big(\kappa^{-1}(0), \kappa^{-1}(1), \kappa^{-1}(2), \dots\big)$. The second aim (section 3) is to prove that it is sufficient to restrict our attention to some special hypergraphs. Next, we will translate hypergraph results onto the lattice language, and thus we will obtain some necessary condition for lattices to be isomorphic to the weak subalgebra lattice of a partial algebra of a fixed type. Further, we will characterize the weak subalgebra lattice for algebras of some types. Although these results are solutions only for some types, it seems, having experiences and ideas from [8], that main hypergraph results of the paper (Theorems 2.2 and 3.3) will be also useful to obtain solutions of our problem for other types. Recall that in [8] we have solved this problem for unary algebras.

Obviously there are lattices satisfying (W.1)–(W.4), being not isomorphic to the weak subalgebra lattice of any partial algebra of a fixed type. For example, take type $\langle K, \kappa\rangle$ consisting of exactly one constant (i.e. $K = \{k\}$ and $\kappa(k) = 0$), and let $\mathbf{L}$ be the family of sets $\big\{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\big\}$ with set-inclusion. Then $\mathbf{L}$ satisfies (W.1)–(W.4), and $\mathbf{L}$ has one atom and two non-zero and non-atomic join-irreducible elements. Thus $\mathbf{U}(\mathbf{L})$ contains one vertex and two hyperedges, so there is not an algebraic hypergraph $\mathbf{G}$ of type $(1, 0, 0, \dots)$ such that $\mathbf{G}^{**} \simeq \mathbf{H}$. Hence, $\mathbf{L} \not\simeq \mathbf{S}_w(\mathbf{A})$ for each partial algebra $\mathbf{A}$ of type $\langle K, \kappa\rangle$.

Let $\mathbf{G}$ be an algebraic hypergraph of type $\underline{\tau}$, and $\mathbf{H} = \mathbf{G}^{**}$. Then obviously for any weak subhypergraph $\mathbf{K}$ of $\mathbf{H}$, there is an algebraic hypergraph $\mathbf{D}$ of type $\underline{\tau}$ such that $\mathbf{D}^{**} = \mathbf{K}$. As in [8] (Propositions 2.8 and 2.9) we can translate this result onto the lattice language. In [8] we have considered only lattices satisfying (W.1), (W.2), (W.4), and such that any non-zero non-atomic join-irreducible element contains one or two atoms. But it does not matter in the proof of Proposition 2.8. Thus for a lattice $\mathbf{L}$ satisfying (W.1)–(W.4), and sets $A \subseteq \mathcal{A}(\mathbf{L})$, $I \subseteq \mathcal{I}(\mathbf{L})$ such that $\mathcal{A}(i) \subseteq A$ for $i \in I$, the complete sublattice $[A \cup I]_{\mathbf{L}}$ generated by the set $\{0\} \cup A \cup I$ consists only of joins of elements from $\{0\} \cup A \cup I$. In particular, $\mathcal{A}\big([A \cup I]_{\mathbf{L}}\big) = A$, $\mathcal{I}\big([A \cup I]_{\mathbf{L}}\big) = I$, and $[A \cup I]_{\mathbf{L}}$ satisfies (W.1)–(W.4). Hence,

Proposition 1.2. *Let* $\langle K, \kappa\rangle$ *be an algebra type, and let* $\mathbf{L}$ *be a lattice isomorphic to the weak subalgebra lattice of a partial algebra of type* $\langle K, \kappa\rangle$. *Then for any sets* $A \subseteq \mathcal{A}(\mathbf{L})$ *and* $I \subseteq \mathcal{I}(\mathbf{L})$ *such that* $\mathcal{A}(i) \subseteq A$ *for all* $i \in I$, *there is a partial algebra* $\mathbf{A}$ *of type* $\langle K, \kappa\rangle$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq [A \cup I]_{\mathbf{L}}$.

Here, $\mathcal{A}(\mathbf{L})$ and $\mathcal{I}(\mathbf{L})$ are the sets of all atoms and of all non-zero and non-atomic join-irreducible elements, respectively. For any element i, $\mathcal{A}(i)$ is the set of all atoms contained in i. Recall also (see Definition 3.17 in [9]) that $V^{\mathbf{U}(\mathbf{L})} = \mathcal{A}(\mathbf{L})$, $E^{\mathbf{U}(\mathbf{L})} = \mathcal{I}(\mathbf{L})$ and $I^{\mathbf{U}(\mathbf{L})}(e) = \mathcal{A}(e)$ for $e \in \mathcal{I}(\mathbf{L})$.

2.

For any sets A and B, let $\mathrm{Sur}\,(A, B)$ be the set of all surjections from A onto B. Further, for $k, m \in \mathbb{N}$, $\mathrm{Sur}\,(m, A) = \mathrm{Sur}\,\big(\{1, \ldots, m\}, A\big)$ and $\mathrm{Sur}_{\,\mathrm{fin}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{Sur}\,(n, A)$ and $\mathrm{Sur}\,(m, k) = \mathrm{Sur}\,\big(\{1, \ldots, m\}, \{1, \ldots, k\}\big)$ and $s(m, k) = \big|\mathrm{Sur}\,(m, k)\big|$ (where $\{1, \ldots, 0\}$ denotes the empty set).

Obviously $\big|\mathrm{Sur}\,(m, A)\big| = \big|\mathrm{Sur}\,(m, |A|)\big| = s\big(m, |A|\big)$, $s(m, k) = 0$ for $m \le k - 1$, $s(0, m) = s(m, 0) = 0$ for $m \ge 1$, $\mathrm{Sur}\,(\emptyset, \emptyset) = \{\emptyset\}$ and $s(0, 0) = 1$, in particular, $\mathrm{Sur}_{\,\mathrm{fin}}(\emptyset) = \{\emptyset\}$.

It is also well-known that for any $k, m \in \mathbb{N}$ with $m \ge k$, $s(m, k) = \sum_{j=0}^{j=k}(-1)^j \cdot \frac{k!}{j!(k-j)!}(k - j)^m$; or equivalently, $s(m, k) = k!S(m, k)$, where $S(m, k)$ are Stirling's numbers of the second kind, i.e. $S(m, m) = 1$ for $m \ge 0$, $S(m, 0) = 0$ for $m \ge 1$, $S(m, k) = S(m - 1, k - 1) + S(m - 1, k)$ for $0 < k < m$.

Definition 2.1. Let $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \ldots)$ be a sequence of cardinal numbers. Then $\mathbf{T}(\underline{\tau}) = \big(T_0(\underline{\tau}), T_1(\underline{\tau}), T_2(\underline{\tau}), \ldots\big)$ is the sequence such that $T_k(\underline{\tau}) = \sum_{m\in\mathbb{N}} \tau_m \cdot s(m, k) = \sum_{m\ge k} \tau_m \cdot s(m, k)$ for any $k \in \mathbb{N}$.

The following facts are immediate (see the notation in [9]; recall that $\mathbb{N}^{\mathbb{N}}_f$ is the family of all sequences of non-negative integers, in which almost all terms are equal zero): $T_0(\underline{\tau}) = \tau_0$ and $T_m(\underline{\tau}) \ge T_{m+1}(\underline{\tau})$ for all $m \in \mathbb{N}\backslash\{0\}$. If $\big|\{i \in \mathbb{N} : \ \tau_i \ge 1\}\big| < \aleph_0$ and $\tau_i \in \mathbb{N}$ for each $i \ge k$, then $T_k(\underline{\tau}) < \aleph_0$. If $\big|\{i \in \mathbb{N} : \ \tau_i \ge 1\}\big| = \aleph_0$ or there is $i \ge k$ such that $\tau_i \ge \aleph_0$, then $T_k(\underline{\tau}) = \sup\{\aleph_0, \tau_k, \tau_{k+1}, \ldots\}$. If $\underline{\tau} \in \mathbb{N}^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}_f$, then $T_m(\underline{\tau}) = \aleph_0$ for all $m \ge 1$. If $\underline{\tau} \in \mathbf{Card}^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}_f$, then $\mathbf{T}(\underline{\tau}) \notin \mathbb{N}^{\mathbb{N}}$.

Theorem 2.2. *Let $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \ldots)$ be a sequence of cardinal numbers, and $\mathbf{D}$ a directed hypergraph. Then there is an algebraic hypergraph $\mathbf{G}$ of type $\underline{\tau}$ such that $\mathbf{G}^* \simeq \mathbf{D}$ iff $\mathbf{D}$ is of type $\mathbf{T}(\underline{\tau})$.*

Proof. $\Longrightarrow$ It is sufficient to show that the directed hypergraph $\mathbf{G}^*$ is of type $\mathbf{T}(\underline{\tau})$.

Take a finite set $V \in P_{\mathrm{fin}}(V^{\mathbf{G}})$. Then

$$(1)\qquad E_s^{\mathbf{G}^*}(V) = \bigcup\big\{E_s^{\mathbf{G}}(\mathbf{v}) : \ \mathbf{v} \in \mathrm{Sur}_{\,\mathrm{fin}}(V)\big\} = \bigcup_{m\in\mathbb{N}} \Big(\bigcup\big\{E_s^{\mathbf{G}}(\mathbf{v}) : \ \mathbf{v} \in \mathrm{Sur}\,(m, V)\big\}\Big).$$

Let $e \in E^{\mathbf{G}}$ and $I_1^{\mathbf{G}}(e) = \langle v_1, \ldots, v_k\rangle$. It is easy to see $e \in E_s^{\mathbf{G}^*}(V)$ iff $\{v_1, \ldots, v_k\} = V$. Moreover, $\langle v_1, \ldots, v_k\rangle$ can be considered as a surjection of $\{1, \ldots, k\}$ onto $\{v_1, \ldots, v_k\}$. These facts imply (1).

Since $\mathbf{G}$ is of type $\underline{\tau}$, $\left|E_s^{\mathbf{G}}(\mathbf{v})\right| = s^{\mathbf{G}}(\mathbf{v}) \leq \tau_m$ for any $\mathbf{v} = \langle v_1, \ldots, v_m\rangle \in V^m$. Hence,

$$\left|\bigcup\left\{E_s^{\mathbf{G}}(\mathbf{v}):\ \mathbf{v} \in \mathrm{Sur}\,(m,V)\right\}\right| \\ = \sum_{\mathbf{v}\in \mathrm{Sur}\,(m,V)} s^{\mathbf{G}}(\mathbf{v}) \leq \tau_m \cdot \left|\mathrm{Sur}\,(m,V)\right| = \tau_m \cdot s\big(m,|V|\big)\,,$$

because $E_s^{\mathbf{G}}(\mathbf{v})$ and $E_s^{\mathbf{G}}(\mathbf{w})$ are disjoint for $\mathbf{v} \neq \mathbf{w}$.

Thus by (1), $s^{\mathbf{G}^*}(V) = \left|E_s^{\mathbf{G}^*}(V)\right| = \left|\bigcup_{m\in\mathbb{N}}\big(\bigcup\{E_s^{\mathbf{G}}(\mathbf{v}):\ \mathbf{v}\in \mathrm{Sur}\,(m,V)\}\big)\right| = \sum_{m\in\mathbb{N}}\left|\bigcup\{E_s^{\mathbf{G}}(\mathbf{v}):\ \mathbf{v}\in\mathrm{Sur}\,(m,V)\}\right| \leq \sum_{m\in\mathbb{N}}\tau_m\cdot s\big(m,|V|\big) = T_{|V|}(\underline{\tau})$.

$\Longleftarrow$ Let $V \in P_{\mathrm{fin}}(V^{\mathbf{D}})$. Then $\left|E_s^{\mathbf{D}}(V)\right| = s^{\mathbf{D}}(V) \leq T_{|V|}(\underline{\tau}) = \sum_{m\in\mathbb{N}}\tau_m\cdot s\big(m,|V|\big)$, so there is a family $\mathcal{F}(V) = \big\{F(V,m)\in P(V^{\mathbf{D}}):\ m\in\mathbb{N}\big\}$ such that

$$F(V,m)\cap F(V,k) = \emptyset \quad \text{for each} \quad m,k\in\mathbb{N}\ \text{ with } m\neq k\,,$$

$$E_s^{\mathbf{D}}(V) = \bigcup_{m\in\mathbb{N}} F(V,m)\,,$$

$$\big|F(V,m)\big| \leq \tau_m\cdot s\big(m,|V|\big) \quad \text{for each} \quad m\in\mathbb{N}\,.$$

Observe, $F(V,m) = \emptyset$ for $m \leq |V|-1$. If $V=\emptyset$, then $F(\emptyset,m)=\emptyset$ for $m\geq 1$, $F(\emptyset,0) = E_s^{\mathbf{D}}(\emptyset) = E^{\mathbf{D}}(0)$ (where $E^{\mathbf{D}}(0)$ is the set of all 0-edges) and $\big|F(\emptyset,0)\big| = \tau_0\cdot s(0,0) = \tau_0$.

Since $s\big(m,|V|\big) = \left|\mathrm{Sur}\,(m,V)\right|$, we obtain by the last condition that for any $m\in\mathbb{N}$, there is a family $\mathcal{F}(V,m) = \big\{F(V,m,\mathbf{v})\in P(V^{\mathbf{D}}):\ \mathbf{v}\in\mathrm{Sur}\,(m,V)\big\}$ such that

$$F(V,m,\mathbf{v})\cap F(V,m,\mathbf{w}) = \emptyset \quad \text{for each} \quad \mathbf{v},\mathbf{w}\in\mathrm{Sur}\,(m,V) \quad \text{with} \quad \mathbf{v}\neq\mathbf{w}\,,$$

$$F(V,m) = \bigcup_{\mathbf{v}\in\mathrm{Sur}\,(m,V)} F(V,m,\mathbf{v})\,,$$

$$\big|F(V,m,\mathbf{v})\big| \leq \tau_m \quad \text{for each} \quad \mathbf{v}\in\mathrm{Sur}\,(m,V)\,.$$

First, for $m\leq |V|-1$, $\mathcal{F}(V,m)$ is the empty family. Secondly, if $V=\emptyset$, then $\mathcal{F}(\emptyset,m)$ is empty for all $m\geq 1$, $\mathcal{F}(\emptyset,0) = \big\{F(\emptyset,0,\emptyset)\big\}$ and $F(\emptyset,0) = F(\emptyset,0,\emptyset)$. Thirdly, if $\tau_m\geq\aleph_0$, then $\big|F(V,m)\big|\leq\tau_m$, so in this case $\mathcal{F}(V,m)$ may consists of exactly one set.

Now, for each $\mathbf{v} = \langle v_1,\ldots,v_m\rangle\in\prod_{\mathrm{fin}}(V^{\mathbf{D}})$, let

$$F(\mathbf{v}) = F\big(\{v_1,\ldots,v_m\},m,\mathbf{v}\big)\,.$$

Then it easily follows from properties of $\mathcal{F}(V)$ and $\mathcal{F}(V,m)$ that

(1) $$\big|F(\mathbf{v})\big|\leq\tau_m \quad \text{for each} \ \ \mathbf{v} = \langle v_1,\ldots,v_m\rangle\in\textstyle\prod_{\mathrm{fin}}(V^{\mathbf{D}})\,.$$

(2) $$E_s^{\mathbf{D}}(V) = \bigcup\big\{F(\mathbf{v}):\ \mathbf{v}\in\mathrm{Sur}_{\,\mathrm{fin}}(V)\big\} \quad \text{for all} \ \ V\in P_{\mathrm{fin}}(V^{\mathbf{D}})\,.$$

Further, for any $\mathbf{v} = \langle v_1,\ldots,v_m\rangle, \mathbf{w} = \langle w_1,\ldots,w_k\rangle\in\prod_{\mathrm{fin}}(V^{\mathbf{D}})$ with $\mathbf{v}\neq\mathbf{w}$,

(3) $$F(\mathbf{v})\cap F(\mathbf{w}) = \emptyset\,.$$

If $\{v_1,\dots,v_m\} = \{w_1,\dots,w_k\}$, then this is implied by the following two facts $F\big(\{v_1,\dots,v_m\},m\big)\cap F\big(\{w_1,\dots,w_k\},k\big)=\emptyset$ if $m\neq k$, and $F\big(\{v_1,\dots,v_m\},m,\mathbf{v}\big)\cap F\big(\{w_1,\dots,w_k\},k,\mathbf{w}\big)=\emptyset$ if $m=k$.

If $\{v_1,\dots,v_m\}\neq\{w_1,\dots,w_k\}$, then $E_s^{\mathbf{D}}\big(\{v_1,\dots,v_m\}\big)\cap E_s^{\mathbf{D}}\big(\{w_1,\dots,w_k\}\big)=\emptyset$. Hence, $F(\mathbf{v})\cap F(\mathbf{w})=\emptyset$, because $F(\mathbf{v})\subseteq E_s^{\mathbf{D}}\big(\{v_1,\dots,v_m\}\big)$ and $F(\mathbf{w})\subseteq E_s^{\mathbf{D}}\big(\{w_1,\dots,w_k\}\big)$.

It is easily shown $E^{\mathbf{D}}=\bigcup\big\{E_s^{\mathbf{D}}(V):\ V\in P_{\mathrm{fin}}(V^{\mathbf{D}})\big\}$. Hence and by (2) we have

$$E^{\mathbf{D}}=\bigcup\big\{F(\mathbf{v}):\ \mathbf{v}\in\textstyle\prod_{\mathrm{fin}}(V^{\mathbf{D}})\big\}\,. \tag{4}$$

For any $\mathbf{v}\in\prod_{\mathrm{fin}}(V^{\mathbf{D}})$, let $I(\mathbf{v})=\big\langle I_1(\mathbf{v}),I_2(\mathbf{v})\big\rangle\colon F(\mathbf{v})\longrightarrow\prod_{\mathrm{fin}}(V^{\mathbf{D}})\times V^{\mathbf{D}}$ be the function such that

$$I_1(\mathbf{v})(e)=\mathbf{v}\quad\text{ and }\quad I_2(\mathbf{v})(e)=I_2^{\mathbf{D}}(e)\quad\text{ for each }\ e\in F(\mathbf{v})\,.$$

Of course, if $F(\mathbf{v})=\emptyset$, then $I(\mathbf{v})$ is the empty function.

Next, let

$$I=\big\langle I_1,I_2\big\rangle=\bigcup\big\{I(\mathbf{v}):\ \mathbf{v}\in\textstyle\prod_{\mathrm{fin}}(V^{\mathbf{D}})\big\}\,.$$

By (3), I is a well-defined function, and by (4), I is defined for all elements of $E^{\mathbf{D}}$. Thus the ordered triple $\mathbf{G}=\langle V^{\mathbf{D}},E^{\mathbf{D}},I\rangle$ is an algebraic hypergraph.

Take $e\in E^{\mathbf{D}}$. Then $e\in F(\mathbf{v})$ for some $\mathbf{v}=\langle v_1,\dots,v_m\rangle\in\prod_{\mathrm{fin}}(V^{\mathbf{D}})$. Thus $I_2^{\mathbf{G}}(e)=I_2(\mathbf{v})(e)=I_2^{\mathbf{D}}(e)$, and $I_1^{\mathbf{G}}(e)=I_1(\mathbf{v})(e)=\mathbf{v}$, and by (2), $I_1^{\mathbf{D}}(e)=\{v_1,\dots,v_m\}$. Hence, $I^{\mathbf{G}^*}(e)=I^{\mathbf{D}}(e)$, so

$$\mathbf{G}^*=\mathbf{D}\,. \tag{5}$$

Finally, we show

$$E_s^{\mathbf{G}}(\mathbf{v})=F(\mathbf{v})\quad\text{ for each }\ \mathbf{v}\in\textstyle\prod_{\mathrm{fin}}(V^{\mathbf{D}})\,. \tag{6}$$

The inclusion $\supseteq$ is trivial. To see the inverse inclusion, take $e\in E_s^{\mathbf{G}}(\mathbf{v})$. By (3), there is exactly one $\mathbf{w}\in\prod_{\mathrm{fin}}(V^{\mathbf{D}})$ such that $e\in F(\mathbf{w})$. Then $\mathbf{v}=I_1^{\mathbf{G}}(e)=I_1(e)=I_1(\mathbf{w})(e)=\mathbf{w}$. Thus $e\in F(\mathbf{v})$.

By (1) and (6) we obtain that for each $\mathbf{v}=\langle v_1,\dots,v_m\rangle\in\prod_{\mathrm{fin}}(V^{\mathbf{D}})$,

$$s^{\mathbf{G}}(\mathbf{v})=\big|F(\mathbf{v})\big|\le\tau_m\,.$$

Thus $\mathbf{G}$ is an algebraic hypergraph of type $\underline{\tau}$. □

The following fact is an immediate consequence of the above theorem.

Corollary 2.3. *Let $\underline{\tau}$ be a (hypergraph) type, and $\mathbf{H}$ a hypergraph. Then there is an algebraic hypergraph $\mathbf{G}$ of type $\underline{\tau}$ such that $\mathbf{G}^{**}\simeq\mathbf{H}$ iff there is a directed hypergraph $\mathbf{D}$ of type $\mathbf{T}(\underline{\tau})$ such that $\mathbf{D}^*\simeq\mathbf{H}$.*

Hence and by Proposition 1.1 we obtain that our algebraic problem can be reduced in the following way.

Corollary 2.4. *Let a lattice* $\mathbf{L}$ *satisfy* (W.1)–(W.4)*, and* $\langle K, \kappa \rangle$ *be an algebra type. Then there is a partial algebra* $\mathbf{A}$ *of type* $\langle K, \kappa \rangle$ *such that* $\mathbf{L} \simeq \mathbf{S}_w(\mathbf{A})$ *iff there is a directed hypergraph* $\mathbf{D}$ *of type* $\mathbf{T}(K, \kappa)$ *such that* $\mathbf{D}^* \simeq \mathbf{U}(\mathbf{L})$*, where* $\mathbf{T}(K, \kappa) = \mathbf{T}\big(\kappa^{-1}(0), \kappa^{-1}(1), \kappa^{-1}(2), \dots\big)$.

3.

In the previous section we have reduced our algebraic problem to the question when hyperedges of a hypergraph can be directed to a form of directed hypergraph of a fixed type. Observe also that each hypergraph represents some lattice satisfying (W.1)–(W.4). (To see it take a hypergraph $\mathbf{H}$, and let $\mathbf{L}$ be the weak subhypergraph lattice of $\mathbf{H}$. Then $\mathbf{U}(\mathbf{L}) \simeq \mathbf{H}$, see Theorem 3.18 in [9].) Now we show that it is sufficient to investigate some special hypergraphs.

Let $\mathbf{D}$ be a directed hypergraph and $k \in \mathbb{N}$. Then $\mathbf{D}$ is said to be a directed k-hypergraph iff $\mathbf{D}$ contains only k-edges, i.e. $E^{\mathbf{D}} = E^{\mathbf{D}}(k)$ (or equivalently, for each $e \in E^{\mathbf{D}}$, $\big|I_1^{\mathbf{D}}(e)\big| = k$). Obviously a directed 1-hypergraph can be easily identified with usual directed graph. Moreover, for a directed k-hypergraph, each of its weak subhypergraphs is also a directed k-hypergraph.

Let $\mathbf{H}$ be a hypergraph and $k \in \mathbb{N}$. Then $\mathbf{H}$ is said to be a $(k, k+1)$-hypergraph iff for any $e \in V^{\mathbf{H}}$, $I^{\mathbf{H}}(e)$ has k or $k+1$ vertices. $\mathbf{H}$ is a k-hypergraph iff for any $e \in E^{\mathbf{H}}$, $\big|I^{\mathbf{H}}(e)\big| = k$. First, 0-hypergraphs are just discrete hypergraphs. Secondly, k-hypergraphs and $k+1$-hypergraphs are, in particular, $(k, k+1)$-hypergraphs. Thirdly, for a $(k, k+1)$-hypergraph (k-hypergraph), each of its weak subhypergraphs is also a $(k, k+1)$-hypergraph (k-hypergraph). Of course, a $(1, 2)$-hypergraph is a graph.

Obviously if $\mathbf{D}$ is a directed k-hypergraph, then $\mathbf{D}^*$ is a $(k, k+1)$-hypergraph. Conversely, for any $(k, k+1)$-hypergraph $\mathbf{H}$, we can construct (by the axiom of choice) a directed k-hypergraph $\mathbf{D}$ such that $\mathbf{D}^* = \mathbf{H}$ (for each hyperedge it is enough to take one of its endpoints as the final vertex).

Since for any directed k-hypergraph $\mathbf{D}$, $s^{\mathbf{D}}(V) = 0$ for all $V \notin P_k(V)$, we can a little modify, in this case, the concept of type of directed hypergraph.

Definition 3.1. Let η be a cardinal number, and $\mathbf{D}$ a directed k-hypergraph. Then $\mathbf{D}$ is said to be of k-type η iff $s^{\mathbf{D}}(V) \leq \eta$ for each $V \in P_k(V^{\mathbf{D}})$.

Remark 3.1. (a) Let $\mathbf{H}$ be a k-hypergraph and η a cardinal number. Then there is a directed k-hypergraph $\mathbf{D}$ of k-type η such that $\mathbf{D}^* \simeq \mathbf{H}$ iff $\big|\{e \in E^{\mathbf{H}} : I^{\mathbf{H}}(e) = V\}\big| \leq \eta$ for $V \in P_k(V^{\mathbf{H}})$.

(b) Let $\mathbf{H}$ be a $(0, 1)$-hypergraph and η a cardinal number. Then there is a directed 0-hypergraph $\mathbf{D}$ of 0-type η such that $\mathbf{D}^* \simeq \mathbf{H}$ iff $|E^{\mathbf{H}}| \leq \eta$.

Proof. It is easily shown that for any directed k-hypergraph $\mathbf{D}$, if $\mathbf{D}^*$ is a k-hypergraph, then $\{e \in E^{\mathbf{D}^*} : I^{\mathbf{D}^*}(e) = V\} = E_s^{\mathbf{D}}(V)$ for any $V \in P_k(V^{\mathbf{D}})$. This fact implies (a).

(b) $\Longrightarrow$ Of course, we can assume $\mathbf{D}^* = \mathbf{H}$. Then $E^{\mathbf{H}} = E^{\mathbf{D}} = E_s^{\mathbf{D}}(\emptyset)$ and $\big|E_s^{\mathbf{D}}(\emptyset)\big| = s^{\mathbf{D}}(\emptyset) \leq \eta$.

$\Longleftarrow$ $\mathbf{H}$ can be regarded as a directed 0-hypergraph $\mathbf{D}$. Then $E_s^{\mathbf{D}}(\emptyset) = E^{\mathbf{H}}$, so $s^{\mathbf{D}}(\emptyset) = |E^{\mathbf{H}}| \leq \eta$. $\square$

Let $\mathbf{D}$ be a directed hypergraph and $k \in \mathbb{N}$. Then $\mathbf{D}(k) \leq_w \mathbf{D}$ is the weak subhypergraph of $\mathbf{D}$ consisting of all vertices of $\mathbf{D}$ and all k-edges, i.e. $V^{\mathbf{D}(k)} = V^{\mathbf{D}}$ and $E^{\mathbf{D}(k)} = E^{\mathbf{D}}(k)$. Obviously $\mathbf{D}(k)$ is a directed k-hypergraph. If $\mathbf{D}$ is a directed k-hypergraph, then $\mathbf{D}(k) = \mathbf{D}$, and for $l \neq k$, $\mathbf{D}(l)$ is a discrete hypergraph. It is also easy to see

$$\mathbf{D} = \bigvee_{m\in\mathbb{N}} \mathbf{D}(m) \quad \text{and} \quad s^{\mathbf{D}}(V) = s^{\mathbf{D}(k)}(V) \quad \text{for each} \quad V \in P_k(V^{\mathbf{D}})\,.$$

Further, $\mathbf{D}$ is a directed hypergraph of type $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \dots)$ iff for each $m \in \mathbb{N}$, $\mathbf{D}(m)$ is a directed m-hypergraph of m-type τ_m. If $\mathbf{D}$ is a directed k-hypergraph, then $\mathbf{D}$ is of type $\underline{\tau}$ iff $\mathbf{D}$ is of k-type τ_k.

By these facts and Theorem 2.2 we deduce

Proposition 3.2. *Let $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \dots)$ be a sequence of cardinal numbers, and $\mathbf{D}$ a directed hypergraph. Then the following conditions are equivalent:*

(a) *There is an algebraic hypergraph $\mathbf{G}$ of type $\underline{\tau}$ such that $\mathbf{G}^* \simeq \mathbf{D}$.*
(b) *For each $m \in \mathbb{N}$, $\mathbf{D}(m)$ is a directed m-hypergraph of m-type $T_m(\underline{\tau})$.*
(c) *For each $m \in \mathbb{N}$, there is an algebraic hypergraph $\mathbf{G}_m$ of type $\underline{\tau}$ such that $\mathbf{G}_m^* \simeq \mathbf{D}(m)$.*

Note that (a) $\Longleftrightarrow$ (c) may be proved independently. First, the implication $\Longrightarrow$ is trivial, because $\mathbf{D}(m)$ is a weak subhypergraph of $\mathbf{D}$. Secondly, for each $k \in \mathbb{N}$, take an algebraic hypergraph $\mathbf{G}_k$ of type $\underline{\tau}$ such that $\mathbf{G}_k^* \simeq \mathbf{D}(k)$. We can assume that $\mathbf{G}_k^* = \mathbf{D}(k)$ (see the next proof). Now it is sufficient to verify that the triple $\left\langle V^{\mathbf{D}}, E^{\mathbf{D}}, \bigcup_{k\in\mathbb{N}} I^{\mathbf{G}_k} \right\rangle$ is the desired algebraic hypergraph.

Now we prove the main result of this section.

Theorem 3.3. *Let $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \dots)$ be a sequence of cardinal numbers, and $\mathbf{H}$ be a hypergraph. Then the following conditions are equivalent:*

(a) *There is a directed hypergraph $\mathbf{D}$ of type $\underline{\tau}$ such that $\mathbf{D}^* \simeq \mathbf{H}$.*
(b) *There is a family $\{\mathbf{H}_k\}_{k\in\mathbb{N}}$ of weak subhypergraphs of $\mathbf{H}$ such that*
  (b.1) *For each $k \in \mathbb{N}$, $\mathbf{H}_k$ is a $(k, k+1)$-hypergraph.*
  (b.2) $E^{\mathbf{H}} = \bigcup\limits_{k\in\mathbb{N}} E^{\mathbf{H}_k}$.
  (b.3) *For any $k \in \mathbb{N}$, there is a directed k-hypergraph $\mathbf{D}_k$ of k-type τ_k such that $\mathbf{D}_k^* \simeq \mathbf{H}_k$.*

The analogous result for algebraic hypergraphs can also be formulated. More precisely, a directed hypergraph may be replaced by an algebraic hypergraph. Then, in (b), $\underline{\tau}$ should be replaced by $\mathbf{T}(\underline{\tau})$.

This theorem and Corollary 2.4 reduce our algebraic problem to the question about the orientation of hyperedges of a $(k, k+1)$-hypergraph to a form of directed k-hypergraph of a fixed k-type.

Proof. (a) $\Longrightarrow$ (b) Let $\mathbf{D}$ be a directed hypergraph of type $\underline{\tau}$ such that $\mathbf{D}^* \simeq \mathbf{H}$. We can assume $\mathbf{D}^* = \mathbf{H}$. It is sufficient to transport the structure of directed hypergraph onto $\mathbf{H}$ by any hypergraph isomorphism. For each $k \in \mathbb{N}$, let

$$\mathbf{H}_k = \mathbf{D}(k)^* .$$

First, $\mathbf{H}_k$ is a $(k, k+1)$-hypergraph. Secondly, $\mathbf{D}(k)$ is of k-type τ_k. Thirdly, (see Theorem 2.10 in [9]) * preserves weak subhypergraphs, so

$$\mathbf{H}_k = \mathbf{D}(k)^* \leq_w \mathbf{D}^* = \mathbf{H} .$$

Moreover,

$$V^{\mathbf{H}_k} = V^{\mathbf{D}(k)} = V^{\mathbf{D}} = V^{\mathbf{H}} \quad \text{and} \quad \bigcup_{k\in\mathbb{N}} E^{\mathbf{H}_k} = \bigcup_{k\in\mathbb{N}} E^{\mathbf{D}(k)} = E^{\mathbf{D}} = E^{\mathbf{H}} ,$$

because a hyperedge e of $\mathbf{D}$ belongs to $\mathbf{D}(m)$, where $m = \left|I_1^{\mathbf{D}}(e)\right|$.

Thus the family $\{\mathbf{H}_k\}_{k\in\mathbb{N}}$ satisfies (b).

(b) $\Longrightarrow$ (a) Obviously if we add to each $\mathbf{H}_k$ all vertices of $\mathbf{H}$ outside $\mathbf{H}_k$, then the family $\{\mathbf{H}_k\}_{k\in\mathbb{N}}$ also satisfies (b.1)–(b.3). Thus we can assume that $V^{\mathbf{H}_k} = V^{\mathbf{H}}$ for any $k \in \mathbb{N}$.

Now for each $k \in \mathbb{N}$, let $\mathbf{M}_k \leq_w \mathbf{H}_k$ be the weak subhypergraph such that

$$V^{\mathbf{M}_k} = V^{\mathbf{H}_k} \quad \text{and} \quad E^{\mathbf{M}_k} = E^{\mathbf{H}_k} \setminus \bigcup_{j=0}^{j=k-1} E^{\mathbf{H}_j}$$

(of course, $E^{\mathbf{M}_0} = E^{\mathbf{H}_0}$).

Then first, $E^{\mathbf{M}_k} \cap E^{\mathbf{M}_l} = \emptyset$ for $k, l \in \mathbb{N}$ with $k \neq l$. Secondly, by (b.2),

$$\bigcup_{k\in\mathbb{N}} E^{\mathbf{M}_k} = \bigcup_{k\in\mathbb{N}} E^{\mathbf{H}_k} = E^{\mathbf{H}} .$$

Thirdly, $\mathbf{M}_k$ is a $(k, k+1)$-hypergraph. Further, by (b.3), there is a directed k-hypergraph $\mathbf{D}_k$ of k-type τ_k such that $\mathbf{D}_k^* \simeq \mathbf{M}_k$, because $\mathbf{M}_k$ is a weak subhypergraph of $\mathbf{H}_k$. As above we can assume $\mathbf{D}_k^* = \mathbf{M}_k$.

Now take the directed hypergraph $\mathbf{D}$ such that

$$V^{\mathbf{D}} = V^{\mathbf{H}} , \quad E^{\mathbf{D}} = E^{\mathbf{H}} \quad \text{and} \quad I^{\mathbf{D}} = \bigcup_{k\in\mathbb{N}} I^{\mathbf{D}_k} .$$

$I^{\mathbf{D}}$ is well-defined, because $\bigcup_{k\in\mathbb{N}} E^{\mathbf{D}_k} = \bigcup_{k\in\mathbb{N}} E^{\mathbf{M}_k} = E^{\mathbf{H}}$ and $E^{\mathbf{D}_k} \cap E^{\mathbf{D}_l} = E^{\mathbf{M}_k} \cap E^{\mathbf{M}_l} = \emptyset$ for all $k, l \in \mathbb{N}$ with $k \neq l$.

Let $e \in E^{\mathbf{H}} = E^{\mathbf{D}}$ be a hyperedge. Then there is exactly one $m \in \mathbb{N}$ such that $e \in E^{\mathbf{D}_m} = E^{\mathbf{M}_m}$. Hence, $I^{\mathbf{D}^*}(e) = I_1^{\mathbf{D}}(e) \cup \left\{I_2^{\mathbf{D}}(e)\right\} = I_1^{\mathbf{D}_m}(e) \cup \left\{I_2^{\mathbf{D}_m}(e)\right\} = I^{\mathbf{D}_m^*}(e) = I^{\mathbf{M}_m}(e) = I^{\mathbf{H}}(e)$. Thus

$$\mathbf{D}^* = \mathbf{H} .$$

It is easy to see that for any $k \in \mathbb{N}$, $\mathbf{D}_k \leq_w \mathbf{D}$, and also $E^{\mathbf{D}}(k) = E^{\mathbf{D}_k}$, $V^{\mathbf{D}_k} = V^{\mathbf{D}} = V^{\mathbf{D}(k)}$. These facts imply $\mathbf{D}(k) = \mathbf{D}_k$ for $k \in \mathbb{N}$. Hence we infer that $\mathbf{D}$ is of type $\underline{\tau}$, because $\mathbf{D}_k$ is of k-type τ_k. □

Take the directed hypergraph $\mathbf{D}$ with two vertices v_1, v_2, and three hyperedges e_1, e_2, e_3 such that $I^{\mathbf{D}}(e_1) = \langle \emptyset, v_1 \rangle$, $I^{\mathbf{D}} = \langle \{v_2\}, v_2 \rangle$, $I^{\mathbf{D}}(e_3) = \langle \{v_1\}, v_2 \rangle$. Then $\mathbf{H} = \mathbf{D}^*$ satisfies (b) of Theorem 3.3 for $(1, 1, 0, 0, \ldots)$, because $\mathbf{D}$ is of this type.

On the other hand, take two weak subhypergraphs $\mathbf{H}'$ and $\mathbf{H}''$ of $\mathbf{H}$ such that $\mathbf{H}'$ consists of all vertices and e_1, e_2; and $\mathbf{H}''$ consists of all vertices and e_3. In other words, $\mathbf{H}'$ contains all hyperedges of $\mathbf{H}$ with exactly one endpoint, and $\mathbf{H}''$ contains all hyperedges of $\mathbf{H}$ with exactly two endpoints. By Remark 3.1(b), there is not a directed 0-hypergraph $\mathbf{C}$ of 0-type 1 such that $\mathbf{C}^* \simeq \mathbf{H}'$. Hence, $\{\mathbf{H}', \mathbf{H}''\}$ does not satisfy (b) of Theorem 3.3. Thus, unfortunately, it is sometimes difficult to find a suitable family.

4.

Now we give one necessary (but not sufficient) and one sufficient (but not necessary) condition for a hypergraph $\mathbf{H}$ to exist a directed hypergraph $\mathbf{D}$ of a fixed type $\underline{\tau}$ such that $\mathbf{D}^* \simeq \mathbf{H}$. Hence we obtain necessary and sufficient conditions, but only for some (hypergraph) types. Of course, using Theorem 2.2, and also Corollary 2.3, analogous results for algebraic hypergraphs can be formulated.

For a hypergraph $\mathbf{H}$ and $V \in P_{\text{fin}}(V^{\mathbf{H}})$, $E_h^{\mathbf{H}}(V) = \{e \in E^{\mathbf{H}} :\ I^{\mathbf{H}}(e) = V\}$ and $h^{\mathbf{H}}(V) = |E_h^{\mathbf{H}}(V)|$.

Theorem 4.1. *Let $\mathbf{D}$ be a directed hypergraph of type $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \ldots)$. Then for each $V \in P_{\text{fin}}(V^{\mathbf{D}})$,*

$$h^{\mathbf{D}^*}(V) \leq |V| \cdot \tau_{|V|-1} + \tau_{|V|}\,.$$

If $\tau_{|V|-1} \geq \aleph_0$ or $\tau_{|V|} \geq \aleph_0$, then $h^{\mathbf{D}^*}(V) \leq \max\{\tau_{|V|-1}, \tau_{|V|}\}$.

Proof. Obviously the second part follows from the first.

Take a finite and non-empty subset $V \subseteq V^{\mathbf{D}}$ and let $k = |V| - 1$. First,

$$E_h^{\mathbf{D}^*}(V) = E_h^{\mathbf{D}(k)^*}(V) \cup E_h^{\mathbf{D}(k+1)^*}(V)\,. \tag{1}$$

The inclusion $\supseteq$ is trivial, because $\mathbf{D}(k)^*$ and $\mathbf{D}(k+1)^*$ are weak subhypergraphs of $\mathbf{D}^*$. On the other hand, take $e \in E_h^{\mathbf{D}^*}(V)$. Then $I_1^{\mathbf{D}}(e) \cup \{I_2^{\mathbf{D}}(e)\} = V$, so e is a k-edge or a $k+1$-edge. Hence, e belongs to $\mathbf{D}(k)$ or $\mathbf{D}(k+1)$. Thus $e \in E_h^{\mathbf{D}(k)^*}(V)$ or $e \in E_h^{\mathbf{D}(k+1)^*}(V)$.

Secondly, it is easy to see

$$\begin{aligned} E_h^{\mathbf{D}(k)^*}(V) &= \{e \in E^{\mathbf{D}(k)} :\ I_1^{\mathbf{D}(k)}(e) \cup \{I_2^{\mathbf{D}(k)}(e)\} = V\} \\ &\subseteq \{e \in E^{\mathbf{D}(k)} :\ I_1^{\mathbf{D}(k)}(e) \in P_k(V)\} = \bigcup \{E_s^{\mathbf{D}(k)}(W) :\ W \in P_k(V)\}\,, \end{aligned}$$

because $\mathbf{D}(k)$ contains only k-edges.

Analogously,

$$\begin{aligned} E_h^{\mathbf{D}(k+1)^*}(V) &= \{e \in E^{\mathbf{D}(k+1)} :\ I_1^{\mathbf{D}(k+1)}(e) \cup \{I_2^{\mathbf{D}(k+1)}(e)\} = V\} \\ &\subseteq \{e \in E^{\mathbf{D}(k+1)} :\ I_1^{\mathbf{D}(k+1)}(e) \in P_{k+1}(V)\} = E_s^{\mathbf{D}(k+1)}(V)\,, \end{aligned}$$

because $\mathbf{D}(k+1)$ contains only $k+1$-edges, and V has exactly $k+1$ vertices, i.e. $P_{k+1}(V) = \{V\}$.

Moreover, $\mathbf{D}(k)$ is of k-type τ_k and $\mathbf{D}(k+1)$ is of $k+1$-type τ_{k+1}, because $\mathbf{D}$ is of type $\underline{\tau}$. Thus by the above two facts we obtain

$$h^{\mathbf{D}(k)^*}(V) \leq \sum_{W \in P_k(V)} s^{\mathbf{D}(k)}(W) \leq \big|P_k(V)\big| \cdot \tau_k = (k+1) \cdot \tau_k = |V| \cdot \tau_{|V|-1}$$

and

$$h^{\mathbf{D}(k+1)^*}(V) \leq \big|E_s^{\mathbf{D}(k+1)}(V)\big| = s^{\mathbf{D}(k+1)}(V) \leq \tau_{|V|} .$$

Hence and by (1) we have $h^{\mathbf{D}^*}(V) \leq h^{\mathbf{D}(k)^*}(V) + h^{\mathbf{D}(k+1)^*}(V) \leq |V| \cdot \tau_{|V|-1} + \tau_{|V|}$. □

Let $\underline{\tau}$ be a sequence of cardinal numbers and $k \in \mathbb{N}$.

Take a family of pairwise disjoint sets $\{E_i\}_{i=0}^{i=k+1}$ such that $|E_i| = \tau_k$ for $i = 0, \ldots, k$ and $|E_{k+1}| = \tau_{k+1}$. Next, let $\mathbf{D}$ be the directed hypergraph with $V^{\mathbf{D}} = \{v_0, \ldots, v_k\}$, $E^{\mathbf{D}} = E_0 \cup E_1 \cup \ldots \cup E_k \cup E_{k+1}$, and $I^{\mathbf{D}}(e) = \big\langle V^{\mathbf{D}} \setminus \{v_i\}, v_i \big\rangle$ for $e \in E_i$ and $0 \leq i \leq k$, $I^{\mathbf{D}}(e) = \langle V^{\mathbf{D}}, v_0 \rangle$ for $e \in E_{k+1}$. Then $s^{\mathbf{D}}(W) = \tau_k$ for $W \in P_k(V^{\mathbf{D}})$, $s^{\mathbf{D}}(V^{\mathbf{D}}) = \tau_{k+1}$ and $s^{\mathbf{D}}(W) = 0$ for $W \notin P_k(V^{\mathbf{D}}) \cup P_{k+1}(V^{\mathbf{D}})$. Hence, $\mathbf{D}$ is of type $\underline{\tau}$. Further, $h^{\mathbf{D}^*}(V^{\mathbf{D}}) = |E^{\mathbf{D}}| = |E_0| + |E_1| + \ldots + |E_k| + |E_{k+1}| = (k+1) \cdot \tau_k + \tau_{k+1}$, since $E_0, \ldots, E_{k+1}$ are pairwise disjoint. Thus a stronger inequality than in Theorem 4.1 does not hold.

Having Theorem 4.1 we can also construct a hypergraph $\mathbf{H}$ such that $\mathbf{H}$ has only $(k+1)$-edges, and $\mathbf{D}^* \not\simeq \mathbf{H}$ for any directed hypergraph $\mathbf{D}$ of type $\underline{\tau}$. More precisely, let $\mathbf{H}$ be the hypergraph such that $|V^{\mathbf{H}}| = k+1$, $|E^{\mathbf{H}}| > \max\{\aleph_0, \tau_k, \tau_{k+1}\}$ and $I^{\mathbf{H}}(e) = V^{\mathbf{H}}$ for $e \in E^{\mathbf{H}}$. Then $h^{\mathbf{H}}(V^{\mathbf{H}}) = |E^{\mathbf{H}}| > (k+1) \cdot \tau_k + \tau_{k+1}$. Now it is remained to use Theorem 4.1.

Unfortunately, the condition in Theorem 4.1 is not sufficient. Let $\underline{\tau} = (0, 1, 0, 0, \ldots)$ and let $\mathbf{H}$ be the usual graph with $V^{\mathbf{H}} = \{v_1, v_2\}$, $E^{\mathbf{H}} = \{e_1, e_2, e_3\}$ and $I^{\mathbf{H}}(e_1) = I^{\mathbf{H}}(e_2) = \{v_1, v_2\}$, $I^{\mathbf{H}}(e_3) = \{v_2\}$. Then $h^{\mathbf{H}}\big(\{v_1, v_2\}\big) = 2$, $h^{\mathbf{H}}\big(\{v_1\}\big) = 0$ and $h^{\mathbf{H}}\big(\{v_2\}\big) = 1$. Thus $\mathbf{H}$ and $\underline{\tau}$ satisfy the inequality of Theorem 4.1. But, there is not a directed hypergraph $\mathbf{D}$ of type $\underline{\tau}$ such that $\mathbf{D}^* \simeq \mathbf{H}$ (note that $\mathbf{D}$ would have to be a functional directed graph).

Proposition 4.2. *Let a sequence $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \ldots)$ and a hypergraph $\mathbf{H}$ be such that $h^{\mathbf{H}}(V) \leq \tau_{|V|}$ for each $V \in P_{\mathrm{fin}}(V^{\mathbf{H}})$. Then there is a directed hypergraph $\mathbf{D}$ of type $\underline{\tau}$ such that $\mathbf{D}^* \simeq \mathbf{H}$.*

Proof. For any $k \in \mathbb{N}$, let $\mathbf{H}_k$ be the weak subhypergraph of $\mathbf{H}$ consisting of all vertices and all hyperedges with exactly k endpoints. Obviously the family $\{\mathbf{H}_k\}_{k \in \mathbb{N}}$ satisfies (b.1) and (b.2) of Theorem 3.3.

Further, for any $k \in \mathbb{N}$, there is a directed k-hypergraph $\mathbf{D}_k$ of k-type τ_k such that $\mathbf{D}_k \simeq \mathbf{H}_k$. It follows from Remark 3.1(a), and the equality $h^{\mathbf{H}_k}(V) = h^{\mathbf{H}}(V)$ for any $V \in P_k(V^{\mathbf{H}})$ and $k \in \mathbb{N}$. Hence and by Theorem 3.3, there is the desired hypergraph $\mathbf{D}$. □

Let $\underline{\tau} = (0, 1, 0, \ldots)$ and $\mathbf{H}$ be the graph with two vertices v_1, v_2 and one hyperedge e between them. Then $h^{\mathbf{H}}\big(\{v_1, v_2\}\big) = 1 > \tau_2$, but obviously there is a directed hypergraph $\mathbf{D}$ of type $\underline{\tau}$ (i.e. a functional directed graph) such that $\mathbf{D}^* = \mathbf{H}$. Thus the condition in Proposition 4.2 is not necessary.

By Theorem 4.1 and Proposition 4.2 we immediate obtain the following result.

Proposition 4.3. *Let* $\mathbf{H}$ *be a hypergraph, and let* $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \dots)$ *be a sequence such that*

$(*)$ $\tau_k \leq \tau_{k+1}$ *for each* $k \in \mathbb{N}$,
$(**)$ *there is* $m \in \mathbb{N}$ *such that* $\tau_m = 0$ *for* $k \leq m-1$ *and* $\tau_k \geq \aleph_0$ *for* $k \geq m+1$.

Then there is a directed hypergraph $\mathbf{D}$ *of type* $\underline{\tau}$ *such that* $\mathbf{D}^* \simeq \mathbf{H}$ *iff* $h^{\mathbf{H}}(V) \leq \tau_{|V|}$ *for* $V \in P_{\text{fin}}(V^{\mathbf{H}})$.

If $\underline{\tau}$ does not satisfy $(*)$ or $(**)$, then there is $k \in \mathbb{N}$ such that $\tau_k > \tau_{k+1}$, or $\tau_k, \tau_{k+1} < \aleph_0$ and $\tau_k \geq 1$. Hence, there is $k \in \mathbb{N}$ such that $(k+1)\cdot\tau_k + \tau_{k+1} > \tau_{k+1}$. Thus by the first example under Theorem 4.1, these assumptions cannot be weaker. More formally, this example implies that if $\underline{\tau}$ does not satisfy $(*)$ or $(**)$, then the implication $\Longrightarrow$ is not true. Hence also, the sufficient condition in Proposition 4.2 is not necessary for any type $\underline{\tau}$ which does not satisfy $(*)$ or $(**)$.

Take a sequence $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \dots)$ of cardinal numbers. Then by Definition 2.1, $T_k(\underline{\tau}) \geq T_{k+1}(\underline{\tau})$ for any $k \in \mathbb{N} \setminus \{0\}$. Thus $\mathbf{T}(\underline{\tau})$ satisfies $(*)$ and $(**)$ of Proposition 4.3 iff

$(*')$ $T_0(\underline{\tau}) \leq T_1(\underline{\tau})$,
$(**')$ $T_k(\underline{\tau}) = T_{k+1}(\underline{\tau}) \geq \aleph_0$ for each $k \in \mathbb{N} \setminus \{0\}$.

Hence and by Theorem 2.2, if $\mathbf{T}(\underline{\tau})$ satisfies $(*')$ and $(**')$, then for a hypergraph $\mathbf{H}$, there is an algebraic hypergraph $\mathbf{G}$ of type $\underline{\tau}$ such that $\mathbf{G}^{**} \simeq \mathbf{H}$ iff $h^{\mathbf{H}}(V) \leq T_{|V|}(\underline{\tau})$ for all $V \in P_{\text{fin}}(V^{\mathbf{H}})$.

If $\mathbf{T}(\underline{\tau})$ satisfies $(*')$ and $(**')$, then we have two cases (c.1): $T_k(\underline{\tau}) > \aleph_0$ for each $k \in \mathbb{N} \setminus \{0\}$, or (c.2): $T_k(\underline{\tau}) = \aleph_0$ for each $k \in \mathbb{N} \setminus \{0\}$. Moreover, it is well-known that if $T_k(\underline{\tau}) = \sum_{m \geq k} \tau_m \cdot s(m,k) > \aleph_0$, then $T_k(\underline{\tau}) = \sup\{\tau_k, \tau_{k+1}, \dots\}$. Thus $\mathbf{T}(\underline{\tau})$ satisfies $(*')$ and $(**')$ iff one from the following two facts hold (the first for the case (c.1); and the second for (c.2))

(h.1) $\sup\{\tau_m : m \in \mathbb{N}\} > \aleph_0$ and $\tau_k \leq \sup\{\tau_m : m \geq k+1\}$ for each $k \in \mathbb{N}$;
then $T_k(\underline{\tau}) = \sup\{\tau_m : m \geq k\} = \sup\{\tau_m : m \in \mathbb{N}\}$ for $k \in \mathbb{N} \setminus \{0\}$.
(h.2) $\left|\{m \in \mathbb{N} : \tau_m \neq 0\}\right| = \aleph_0$ and $\sup\{\tau_m : m \in \mathbb{N}\} \leq \aleph_0$;
then $T_k(\underline{\tau}) = \aleph_0$ for $k \in \mathbb{N} \setminus \{0\}$.

Thus we obtain

Proposition 4.4. *Let* $\mathbf{H}$ *be a hypergraph, and let* $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \dots)$ *be a sequence such that*

$$\sup\{\tau_m : m \in \mathbb{N}\} > \aleph_0 \quad \textit{and} \quad \tau_k \leq \sup\{\tau_m : m \geq k+1\} \quad \textit{for each} \quad k \in \mathbb{N}$$

or

$$\left|\{m \in \mathbb{N} : \tau_m \neq 0\}\right| = \aleph_0 \ \textit{and} \ \sup\{\tau_m : m \in \mathbb{N}\} \leq \aleph_0 .$$

Then there is an algebraic hypergraph $\mathbf{G}$ *of type* $\underline{\tau}$ *such that* $\mathbf{G}^{**} \simeq \mathbf{H}$ *iff* $h^{\mathbf{H}}(V) \leq \sup\{\tau_m : m \in \mathbb{N}\}$ *for any* $V \in P_{\text{fin}}(V^{\mathbf{H}})$.

Corollary 4.5. *Let* $\mathbf{H}$ *be a hypergraph, and let a sequence of non-negative integers* $\underline{\tau}$ *contain infinitely many non-zero elements (i.e.* $\underline{\tau} \in \mathbb{N}^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}_f$*). Then there is an* *algebraic hypergraph* $\mathbf{G}$ *of type* $\underline{\tau}$ *such that* $\mathbf{G}^{**} \simeq \mathbf{H}$ *iff* $h^{\mathbf{H}}(V) \le \sup\{\tau_m : m \in \mathbb{N}\}$ *for any* $V \in P_{\text{fin}}(V^{\mathbf{H}})$.

Unfortunately, for each type $\underline{\tau} \in \mathbb{N}_f^{\mathbb{N}}$, $\mathbf{T}(\underline{\tau})$ does not satisfy $(*')$, $(**')$. There are also sequences $\underline{\tau} \notin \mathbb{N}_f^{\mathbb{N}}$ such that $\mathbf{T}(\underline{\tau})$ does not satisfy $(*')$, $(**')$; for instance, $\underline{\tau} = (0, \aleph_0, 0, 0, \dots)$, then $\mathbf{T}(\underline{\tau}) = \underline{\tau}$.

5.

Now we translate hypergraph results onto the lattice language, and thus we obtain one necessary (but not sufficient) condition and one sufficient (but not necessary) condition for lattices that are isomorphic to the weak subalgebra lattice of a partial algebra of a fixed type. Next, we completely characterize the weak subalgebra lattice of a partial algebra of a fixed type, but only for special types.

By Corollary 2.4, Theorem 4.1 and the definition of $\mathbf{U}(\mathbf{L})$ (see Definition 3.17 in [9]) we obtain

Theorem 5.1. *Let* $\mathbf{L}$ *be a lattice isomorphic to the weak subalgebra lattice of a partial algebra of type* $\langle K, \kappa\rangle$. *Then for* $B \in P_{\text{fin}}\big(\mathcal{A}(\mathbf{L})\big)$,

$$\big|\{i \in \mathcal{I}(\mathbf{L}) : \mathcal{A}(i) = B\}\big| \le |B| \cdot T_{|B|-1}(K, \kappa) + T_{|B|}(K, \kappa)\,.$$

If $T_{|B|-1}(K, \kappa) \ge \aleph_0$ *or* $T_{|B|}(K, \kappa) \ge \aleph_0$, *then*

$$\big|\{i \in \mathcal{I}(\mathbf{L}) : \mathcal{A}(i) = B\}\big| \le \max\big\{T_{|B|-1}(K, \kappa), T_{|B|}(K, \kappa)\big\}\,.$$

Let $\langle K, \kappa\rangle$ be an arbitrary algebra type and $k \in \mathbb{N}$.

Take pairwise disjoint sets $V, E_0, E_1, \dots, E_k, E_{k+1}$ such that $|V| = k + 1$ and $|E_i| = T_k(K, \kappa)$ for $i = 0, \dots, k$, and $|E_{k+1}| = T_{k+1}(K, \kappa)$. Let $\mathbf{L} = \langle L, \le_{\mathbf{L}}\rangle$ be the sublattice of the powerset lattice of $V \cup E_0 \cup \ldots \cup E_{k+1}$ such that $L = \big\{B \in P(V \cup E_0 \cup \ldots \cup E_{k+1}) : B \subseteq V \text{ or } V \subseteq B\big\}$. Then $\mathcal{A}(\mathbf{L}) = \big\{\{v\} : v \in V\big\}$ and $\mathcal{I}(\mathbf{L}) = \big\{V \cup \{e\} : e \in E_0 \cup E_1 \cup \ldots \cup E_{k+1}\big\}$. Thus $\mathbf{L}$ satisfies (W.1)–(W.4), and $\mathbf{U}(\mathbf{L})$ is isomorphic to the hypergraph $\mathbf{D}^*$ from the first example under Theorem 4.1 (where we replace $\underline{\tau}$ by $\mathbf{T}(K, \kappa)$). Hence and by Corollary 2.4, there is $\mathbf{A} \in \mathcal{PA}lg(K, \kappa)$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$. Further, $\big|\big\{i \in \mathcal{I}(\mathbf{L}) : \mathcal{A}(i) = \mathcal{A}(\mathbf{L})\big\}\big| = |E_0 \cup \ldots \cup E_k \cup E_{k+1}| = (k+1) \cdot T_k(K, \kappa) + T_{k+1}(K, \kappa)$, since $E_0, E_1, \dots, E_{k+1}$ are pairwise disjoint. Thus a stronger inequality than in Theorem 5.1 does not hold.

We can also construct a lattice $\mathbf{L}$ such that $\mathbf{L}$ satisfies (W.1)–(W.4), and each non-zero and non-atomic join-irreducible element of $\mathbf{L}$ contains $k + 1$ atoms, and $\mathbf{L} \not\simeq \mathbf{S}_w(\mathbf{A})$ for all partial algebra $\mathbf{A}$ of type $\langle K, \kappa\rangle$. Take disjoint sets V, W such that $|V| = k + 1$ and $|W| > \max\big\{\aleph_0, T_k(K, \kappa), T_{k+1}(K, \kappa)\big\}$. Let $\mathbf{L} = \langle L, \le_{\mathbf{L}}\rangle$ be the sublattice of the powerset lattice of $V \cup W$ such that $L = \big\{B \in P(V \cup W) : B \subseteq V \text{ or } V \subseteq B\big\}$. Then $\mathcal{A}(\mathbf{L}) = \big\{\{v\} : v \in V\big\}$, $\mathcal{I}(\mathbf{L}) = \big\{V \cup \{w\} : w \in W\big\}$, $\mathcal{A}(i) = \mathcal{A}(\mathbf{L})$ for $i \in \mathcal{I}(\mathbf{L})$. Hence, $\mathbf{L}$ satisfies (W.1)–(W.4), and $\big|\{i \in \mathcal{I}(\mathbf{L}) : \mathcal{A}(i) = \mathcal{A}(\mathbf{L})\}\big| = \big|\mathcal{I}(\mathbf{L})\big| = |W| > \big|\mathcal{A}(\mathbf{L})\big| \cdot T_k(K, \kappa) + T_{k+1}(K, \kappa)$. Thus by Theorem 5.1, there is not $\mathbf{A} \in \mathcal{PA}lg(K, \kappa)$ such that $\mathbf{L} \simeq \mathbf{S}_w(\mathbf{A})$.

Let $\langle K, \kappa\rangle$ be a monounary type (i.e. $K = \{k\}$ and $\kappa(k) = 1$) and let $\mathbf{L}$ be the sublattice of the powerset lattice of $\{0, 1, 2, 3, 4\}$ generated by $\big\{\{0\}, \{1\}, \{0, 1, 2\},$

$\{0,1,3\},\{1,4\}\big\}$. Then $\mathcal{A}(\mathbf{L}) = \big\{\{0\},\{1\}\big\}$ and $\mathcal{I}(\mathbf{L}) = \big\{\{0,1,2\},\{0,1,3\},\{1,4\}\big\}$. Hence, $\mathbf{L}$ satisfies (W.1)–(W.4), and it easily follows that $\mathbf{L}$ and $\mathbf{T}(K,\kappa)$ satisfy the inequality of Theorem 5.1, because $\mathbf{T}(K,\kappa) = (0,1,0,0,\ldots)$. Hence also, $\mathbf{U}(\mathbf{L})$ is isomorphic to the graph $\mathbf{H}$ from the example before Proposition 4.2 (where $\underline{\tau}$ should be replaced by $\mathbf{T}(K,\kappa)$). Thus by Corollary 2.4, $\mathbf{L} \not\simeq \mathbf{S}_w(\mathbf{A})$ for each partial algebra $\mathbf{A}$ of type $\langle K,\kappa\rangle$. Thus the condition in Theorem 5.1 is not sufficient in general.

By Corollary 2.4 and Proposition 4.2 we also obtain

Proposition 5.2. *Let $\langle K,\kappa\rangle$ be any type, and let a lattice $\mathbf{L}$ satisfy* (W.1)–(W.4), *and $\big|\{i \in \mathcal{I}(\mathbf{L}) :\ \mathcal{A}(i) = B\}\big| \le T_{|B|}(K,\kappa)$ for each $B \in P_{\text{fin}}\big(\mathcal{A}(\mathbf{L})\big)$. Then there is $\mathbf{A} \in \mathcal{PA}lg(K,\kappa)$ such that $\mathbf{L} \simeq \mathbf{S}_w(\mathbf{A})$.*

Let $\langle K,\kappa\rangle$ be a monounary type and $\mathbf{L}$ be the lattice of subsets $\big\{\emptyset,\{0\},\{1\},$ $\{0,1\},\{0,1,2\}\big\}$ with set-inclusion. Then $\mathcal{A}(\mathbf{L}) = \big\{\{0\},\{1\}\big\}$ and $\mathcal{I}(\mathbf{L}) = \big\{\{0,1,2\}\big\}$. Hence, $\mathbf{L}$ satisfies (W.1)–(W.4) and $\big|\big\{i \in \mathcal{I}(\mathbf{L}) :\ \mathcal{A}(i) = \{\{0\},\{1\}\}\big\}\big| = \big|\big\{\{0,1,2\}\big\}\big|$ $= 1 > 0 = T_2(K,\kappa)$. Hence also, $\mathbf{U}(\mathbf{L})$ is isomorphic to the hypergraph $\mathbf{H}$ from the example before Proposition 4.3. (where $\underline{\tau}$ should be replaced by $\mathbf{T}(K,\kappa)$). Thus by Corollary 2.4, there is $\mathbf{A} \in \mathcal{PA}lg(K,\kappa)$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$. This example shows that the condition in Proposition 5.2 is not necessary.

By Corollary 2.4, Proposition 4.3, if $\langle K,\kappa\rangle$ is a type such that $\mathbf{T}(K,\kappa)$ satisfies $(*')$ and $(**')$, then for any lattice $\mathbf{L}$ satisfying (W.1)–(W.4), there is a partial algebra $\mathbf{A}$ of type $\langle K,\kappa\rangle$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ iff $\big|\{i \in \mathcal{I}(\mathbf{L}) :\ \mathcal{A}(i) = B\}\big| \le$ $T_{|B|}(K,\kappa)$ for $B \in P_{\text{fin}}\big(\mathcal{A}(\mathbf{L})\big)$.

By (h.1) and (h.2) (because $K = \bigcup_{m\in\mathbb{N}} \kappa^{-1}(m)$, and obviously for pairwise disjoint sets $B_0, B_1, B_2, \ldots$, if $\sup\big\{|B_0|,|B_1|,\ldots\big\} \ge \aleph_0$, then $\sup\big\{|B_0|,|B_1|,\ldots\big\} =$ $\big|\bigcup_{n\in\mathbb{N}} B_n\big|$) we obtain that for any type $\langle K,\kappa\rangle$, $\mathbf{T}(K,\kappa)$ satisfies $(*')$ and $(**')$ iff one from the following two conditions hold:

(a.1) $|K| > \aleph_0$ and $\big|\kappa^{-1}(k)\big| \le \big|\kappa^{-1}(\{m \in \mathbb{N} :\ m \ge k+1\}\big|$ for each $k \in \mathbb{N}$,
(a.2) $|K| = \aleph_0$ and $\big|\{m \in \mathbb{N} :\ \kappa^{-1}(m) \ne \emptyset\}\big| = \aleph_0$.

In both cases $T_k(K,\kappa) = |K|$ for any $k \in \mathbb{N}\setminus\{0\}$.

Thus we obtain

Proposition 5.3. *Let $\mathbf{L}$ be a lattice satisfying* (W.1)–(W.4), *and let $\langle K,\kappa\rangle$ be a type such that*

$$|K| > \aleph_0 \quad \textit{and} \quad \big|\kappa^{-1}(k)\big| \le \big|\kappa^{-1}\big(\{m \in \mathbb{N} :\ m \ge k+1\}\big)\big| \quad \textit{for each } k \in \mathbb{N}$$

or

$$|K| = \aleph_0 \quad \textit{and} \quad \big|\{m \in \mathbb{N} :\ \kappa^{-1}(m) \ne \emptyset\}\big| = \aleph_0\,.$$

Then there is $\mathbf{A} \in \mathcal{PA}lg(K,\kappa)$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ iff $\big|\big\{i \in \mathcal{I}(\mathbf{L}) : \mathcal{A}(i) = B\big\}\big| \le$ $|K|$ for $B \in P_{\text{fin}}\big(\mathcal{A}(\mathbf{L})\big)$.

Corollary 5.4. *Let $\mathbf{L}$ be a lattice satisfying* (W.1)–(W.4), *and let $\langle K,\kappa\rangle$ be a type such that*

$$|K| = \aleph_0 \quad \textit{and} \quad \big|\kappa^{-1}(i)\big| < \aleph_0 \quad \textit{for each} \quad i \in \mathbb{N}\,.$$

Then there is $\mathbf{A} \in \mathcal{P}\mathcal{A}lg(K,\kappa)$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ *iff* $\left|\left\{i \in \mathcal{I}(\mathbf{L}) : \mathcal{A}(i) = B\right\}\right| \leq |K|$ *for* $B \in P_{\text{fin}}\big(\mathcal{A}(\mathbf{L})\big)$.

The above results give only partial solutions of our algebraic problem, because many algebra types do not satisfy $(*')$ and $(**')$. More precisely, for any finite type $\langle K,\kappa\rangle$, $\mathbf{T}(K,\kappa)$ does not satisfy these conditions (see Definition 2.1). Moreover, there are also infinite types $\langle K,\kappa\rangle$ such that $\mathbf{T}(K,\kappa)$ does not satisfy these two conditions; for instance, take a countable unary type $\langle K,\kappa\rangle$, then $\mathbf{T}(K,\kappa) = (0,\aleph_0,0,0,\ldots)$.

Finally, we characterize the weak subalgebra lattice for nullary types $\langle K,\kappa\rangle$ (i.e. $\kappa(K) \subseteq \{0\}$).

Proposition 5.5. *Let* $\mathbf{L}$ *be a lattice satisfying* (W.1)–(W.4), *and* $\langle K,\kappa\rangle$ *a nullary type. Then there is* $\mathbf{A} \in \mathcal{P}\mathcal{A}lg(K,\kappa)$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ *iff* $\left|\mathcal{I}(\mathbf{L})\right| \leq |K|$ *and* $\left|\mathcal{A}(i)\right| = 1$ *for each* $i \in \mathcal{I}(\mathbf{L})$.

Proof. By Corollary 2.4, there is $\mathbf{A} \in \mathcal{P}\mathcal{A}lg(K,\kappa)$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ iff $\mathbf{L}$ is a lattice satisfying (W.1)–(W.4), and there is a directed hypergraph $\mathbf{D}$ of type $\mathbf{T}(K,\kappa)$ such that $\mathbf{D}^* \simeq \mathbf{U}(\mathbf{L})$.

Since $T_0(K,\kappa) = |K|$ and $T_i(K,\kappa) = 0$ for $i \geq 1$, there is a directed hypergraph $\mathbf{D}$ of type $\mathbf{T}(K,\kappa)$ such that $\mathbf{D}^* \simeq \mathbf{U}(\mathbf{L})$ iff $\mathbf{U}(\mathbf{L})$ is a 1-hypergraph, and there is a directed 0-hypergraph $\mathbf{D}$ of 0-type $|K|$ such that $\mathbf{D}^* \simeq \mathbf{U}(\mathbf{L})$. Hence and by Remark 3.1(b), there is $\mathbf{A} \in \mathcal{P}\mathcal{A}lg(K,\kappa)$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ iff $\mathbf{L}$ satisfies (W.1)–(W.4), $\mathbf{U}(\mathbf{L})$ is a 1-hypergraph, and $\left|\mathcal{I}(\mathbf{L})\right| = \left|E^{\mathbf{U}(\mathbf{L})}\right| \leq |K|$. Further, $\mathbf{U}(\mathbf{L})$ is a 1-hypergraph iff $\left|\mathcal{A}(i)\right| = 1$ for $i \in \mathcal{I}(\mathbf{L})$. □

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