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# Singular Eigenvalue Problems for Second Order Linear Ordinary Differential Equations

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**Abstract.** We consider linear differential equations of the form

$$(p(t)x')' + \lambda q(t)x = 0 \quad (p(t) > 0,\ q(t) > 0) \tag{A}$$

on an infinite interval $[a, \infty)$ and study the problem of finding those values of $\lambda$ for which (A) has principal solutions $x_0(t;\lambda)$ vanishing at $t = a$. This problem may well be called a singular eigenvalue problem, since requiring $x_0(t;\lambda)$ to be a principal solution can be considered as a boundary condition at $t = \infty$. Similarly to the regular eigenvalue problems for (A) on compact intervals, we can prove a theorem asserting that there exists a sequence $\{\lambda_n\}$ of eigenvalues such that $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots,\ \lim_{n\to\infty} \lambda_n = \infty$, and the eigenfunction $x_0(t;\lambda_n)$ corresponding to $\lambda = \lambda_n$ has exactly $n$ zeros in $(a,\infty),\ n = 0, 1, 2, \dots$. We also show that a similar situation holds for nonprincipal solutions of (A) under stronger assumptions on $p(t)$ and $q(t)$.

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**Keywords.** Singular eigenvalue problem, Sturm-Liouville equation, zeros of nonoscillatory solutions

## 1 Introduction

We consider the second order linear differential equation

$$(p(t)x')' + \lambda q(t)x = 0, \qquad t \geq a, \tag{A}$$

where $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$, $a \geq 0$, and $\lambda$ is a real parameter. We assume that (A) is nonoscillatory at $t = \infty$ for all $\lambda > 0$ (and hence for all $\lambda \in \mathbb{R}$). It is known [1, Theorem 6.4, p. 355] that there exists a solution $x_0(t;\lambda)$ of (A) which is uniquely determined up to a constant factor by the condition

$$\int^{\infty} \frac{dt}{p(t)(x_0(t;\lambda))^2} = \infty, \tag{1}$$

and that any solution $x_1(t;\lambda)$ of (A) linearly independent of $x_0(t;\lambda)$ has the property that

$$\int^{\infty} \frac{dt}{p(t)(x_1(t;\lambda))^2} < \infty. \tag{2}$$

A solution $x_0(t;\lambda)$ satisfying (1) is called a *principal solution* of (A) (at $t = \infty$), and a solution $x_1(t;\lambda)$ satisfying (2) is called a *nonprincipal solution* of (A) (at $t = \infty$).

We are concerned with the problem of finding principal solutions $x_0(t;\lambda)$ of (A) which satisfy the boundary condition

$$x_0(a;\lambda) = 0. \tag{3}$$

This problem falls within the category of general eigenvalue problems formulated by Hartman [2]. A solution $x_0(t;\lambda)$ of this problem will be said to be a *principal eigenfunction* and the corresponding value of $\lambda$ a *principal eigenvalue*. Our task is to establish the existence of principal eigenvalues and count the number of zeros of the corresponding principal eigenfunctions.

We begin by introducing the notation needed in stating the main results. With regard to the function $p(t)$ the following two cases are possible:
either

$$\int_a^{\infty} \frac{dt}{p(t)} = \infty \tag{4}$$

or

$$\int_a^{\infty} \frac{dt}{p(t)} < \infty. \tag{5}$$

We define the functions $P(t)$ and $\pi(t)$ as follows:

$$P(t) = \int_a^t \frac{ds}{p(s)}, \qquad t \geq a, \text{ in case (4) holds;} \tag{6}$$

$$\pi(t) = \int_t^{\infty} \frac{ds}{p(s)}, \qquad t \geq a, \text{ in case (5) holds.} \tag{7}$$

It is clear that $P(t) \to \infty$ and $\pi(t) \to 0$ as $t \to \infty$. Our fundamental hypotheses on (A) are:

$$\int_a^\infty P(t)q(t)dt < \infty \qquad \text{in case (4) holds;} \tag{8}$$

$$\int_a^\infty \pi(t)q(t)dt < \infty \qquad \text{in case (5) holds.} \tag{9}$$

It is well known that (8) [resp. (9)] implies the existence of solutions $x_0(t;\lambda)$ of (A) satisfying the following boundary condition (10) [resp. (11)] at $t = \infty$:

$$\lim_{t\to\infty} x_0(t;\lambda) = 1, \qquad \lim_{t\to\infty} P(t)p(t)x_0'(t;\lambda) = 0 \qquad \text{in case (4) holds;} \tag{10}$$

$$\lim_{t\to\infty} \frac{x_0(t;\lambda)}{\pi(t)} = 1, \qquad \lim_{t\to\infty} p(t)x_0'(t;\lambda) = -1 \qquad \text{in case (5) holds.} \tag{11}$$

Since this solution $x_0(t;\lambda)$ satisfies (1), we easily find that, under the condition (8) or (9), the requirement that $x_0(t;\lambda)$ be a principal solution of (A) is equivalent to the requirement that $x_0(t;\lambda)$ be a solution of (A) satisfying (10) or (11).

One of the main results of this paper now follows.

**Theorem I.** *Suppose that* (8) *or* (9) *holds. Then, there exists a sequence of principal eigenvalues* $\{\lambda_n\}$*:*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty$$

*such that the corresponding principal eigenfunction* $x_0(t;\lambda_n)$ *satisfying* (10) *or* (11) *has exactly* $n$ *zeros in* $(a,\infty)$, $n = 0, 1, 2, \ldots$.

The proof of this theorem will be given in Section 1. It will be shown that Theorem I follows from the corresponding result for the particular equation $x'' + \lambda q(t)x = 0$.

Let us now turn to the consideration of nonprincipal solutions of the nonoscillatory equation (A). A nonprincipal solution of (A) is by no means unique. It may happen, however, that certain additional restrictions on the functions $p(t), q(t)$ and/or the asymptotic behavior of the solution determine a unique nonprincipal solution $x_1(t;\lambda)$ of (A) for each fixed $\lambda$. If this is the case one could speak of a *nonprincipal eigenvalue problem* for (A) which consists in finding its nonprincipal solutions $x_1(t;\lambda)$ satisfying the boundary condition (3); such a solution $x_1(t;\lambda)$ is termed a *nonprincipal eigenfunction* and the corresponding value of $\lambda$ a *nonprincipal eigenvalue*. This kind of problem has not been studied in the literature.

For example, if we assume (8) or (9), then (A) has a nonprincipal solution, non-unique, $x_1(t;\lambda)$ such that

$$\lim_{t\to\infty} \frac{x_1(t;\lambda)}{P(t)} = 1, \qquad \lim_{t\to\infty} p(t)x_1'(t;\lambda) = 1 \qquad \text{in case (4) holds;} \tag{12}$$

$$\lim_{t\to\infty} x_1(t;\lambda) = 1, \qquad \lim_{t\to\infty} \pi(t)p(t)x_1'(t;\lambda) = 0 \qquad \text{in case (5) holds.} \tag{13}$$

However, if we require that $p(t)$ and $q(t)$ satisfy the more stringent condition

$$\int_a^\infty (P(t))^2 q(t)dt < \infty \qquad \text{in case (4) holds} \tag{14}$$

or

$$\int_a^\infty q(t)dt < \infty \qquad \text{in case (5) holds,} \tag{15}$$

then there exists, for each $\lambda$, a unique nonprincipal solution $x_1(t;\lambda)$ of (A) such that

$$\lim_{t\to\infty}[x_1(t;\lambda) - P(t)] = 0, \qquad \lim_{t\to\infty} P(t)[p(t)x_1'(t;\lambda) - 1] = 0 \qquad \text{in case (4) holds} \tag{16}$$

or

$$\lim_{t\to\infty} \frac{x_1(t;\lambda) - 1}{\pi(t)} = 0, \qquad \lim_{t\to\infty} p(t)x_1'(t;\lambda) = 0 \qquad \text{in case (5) holds.} \tag{17}$$

From these solutions $x_1(t;\lambda)$ one can extract a sequence of nonprincipal eigenfunctions having the prescribed numbers of zeros as is shown by the following theorem which is another main result of this paper.

**Theorem II.** *Suppose that* (14) *or* (15) *holds. Then, there exists a sequence of nonprincipal eigenvalues* $\{\lambda_n\}$*:*

$$0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty$$

*such that the corresponding nonprincipal eigenfunction* $x_1(t;\lambda_n)$ *satisfying* (16) *or* (17) *has exactly* $n$ *zeros in* $(a,\infty)$, $n = 0, 1, 2, \ldots$.

We will prove this theorem in Section 2 by reducing the problem under study to the corresponding problem for the simpler equation $x'' + \lambda q(t)x = 0$. We remark that since (14) and (15) are stronger than (8) and (9), respectively, the hypotheses of Theorem II guarantee the existence of both principal and nonprincipal eigenvalues for the equation (A). Section 3 will be devoted to a discussion of applicability of Theorems I and II to the qualitative study of a certain class of linear elliptic partial differential equations in exterior domains.

## 2 Principal eigenvalue problem

**A**) *A reduced problem.* Consider the particular equation

$$x'' + \lambda q(t)x = 0, \qquad t \ge a, \tag{B}$$

where $q(t)$ is a positive continuous function on $[a,\infty)$ and $\lambda$ is a real parameter. Theorem I specialized to (B) reads as follows.

**Theorem 1.** *Suppose that*

$$\int_a^\infty tq(t)dt < \infty. \tag{18}$$

*Then, there exists a sequence of positive constants* $\{\lambda_n\}$*:*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty \tag{19}$$

*such that, for each* $\lambda = \lambda_n$*, the equation* (B) *possesses a solution* $x_0(t;\lambda)$ *satisfying the boundary conditions*

$$x_0(a;\lambda_n) = 0, \qquad \lim_{t\to\infty} x_0(t;\lambda_n) = 1, \qquad \lim_{t\to\infty} tx_0'(t;\lambda_n) = 0 \tag{20}$$

*and having exactly* $n$ *zeros in* $(a,\infty)$*,* $n = 0, 1, 2, \ldots$*.*

We will show that Theorem I follows from its specialized version: Theorem 1.

First consider the case where $p(t)$ and $q(t)$ satisfy (4) and (8). In this case the change of variables $(t,x) \to (\tau,\xi)$ defined by

$$\tau = P(t), \qquad \xi(\tau;\lambda) = x(t;\lambda) \tag{21}$$

transforms (A) into

$$\ddot{\xi} + \lambda Q(\tau)\xi = 0, \qquad \tau \geq 0, \tag{22}$$

where a dot denotes differentiation with respect to $\tau$ and $Q(\tau) = p(t)q(t)$. Since (22) is of the type (B) and since

$$\int_0^\infty \tau Q(\tau)d\tau = \int_a^\infty P(t)q(t)dt < \infty$$

by (8), it follows from Theorem 1 that there exist a sequence of positive constants $\{\lambda_n\}_{n=0}^\infty$ satisfying (19) and the corresponding sequence of solutions $\{\xi_0(\tau;\lambda_n)\}_{n=0}^\infty$ of (22) such that

$$\xi_0(0;\lambda_n) = 0, \qquad \lim_{\tau\to\infty} \xi_0(\tau;\lambda_n) = 1, \qquad \lim_{\tau\to\infty} \tau\dot{\xi}_0(\tau;\lambda_n) = 0. \tag{23}$$

Define $x_0(t;\lambda_n) = \xi_0(P(t);\lambda_n)$. Then, $x_0(t;\lambda_n)$ is clearly a solution of (A) on $[a,\infty)$, and in view of (21), (23) implies that it satisfies the boundary conditions (3) and (10).

Next suppose that $p(t)$ and $q(t)$ satisfy (5) and (9). Perform the change of variables $(t,x) \to (\tau,\eta)$ given by

$$\tau = \frac{1}{\pi(t)}, \qquad \eta(\tau;\lambda) = \tau x(t;\lambda). \tag{24}$$

The equation (A) then transforms into

$$\ddot{\eta} + \lambda R(\tau)\eta = 0, \qquad \tau \geq \frac{1}{\pi(a)}, \tag{25}$$

where a dot denotes differentiation with respect to $\tau$ and $R(\tau) = p(t)q(t)/\tau^4$. In view of (9) we have

$$\int_{1/\pi(a)}^{\infty} \tau R(\tau)d\tau = \int_a^{\infty} \pi(t)q(t)dt < \infty,$$

and so applying Theorem 1 to (25) we see that there exists a sequence of positive constants $\{\lambda_n\}_{n=0}^{\infty}$ satisfying (19) and the corresponding solutions $\{\eta_0(\tau;\lambda_n)\}_{n=0}^{\infty}$ of (25) such that

$$\eta_0(\frac{1}{\pi(a)};\lambda_n) = 0, \qquad \lim_{\tau\to\infty} \eta_0(\tau;\lambda_n) = 1, \qquad \lim_{\tau\to\infty} \tau\dot{\eta}_0(\tau;\lambda_n) = 0. \tag{26}$$

Define $x_0(t;\lambda_n) = \pi(t)\eta_0(1/\pi(t);\lambda_n)$. As it is easily seen, $x_0(t;\lambda_n)$ is a solution of (A) on $[a,\infty)$ and satisfies the boundary conditions (3) and (11). Thus the proof of Theorem I is reduced to that of Theorem 1.

**B**) *Proof of Theorem* 1. The condition (18) ensures the existence of a unique principal solution $x_0(t;\lambda)$ of (B) such that

$$\lim_{t\to\infty} x_0(t;\lambda) = 1, \qquad \lim_{t\to\infty} tx_0'(t;\lambda) = 0. \tag{27}$$

This $x_0(t;\lambda)$ is characterized as the solution to the integral equation

$$x_0(t;\lambda) = 1 - \lambda\int_t^{\infty}(s-t)q(s)x_0(s;\lambda)ds, \qquad t \geq a, \tag{28}$$

and is subject to the estimate

$$|x_0(t;\lambda)| \leq \exp\left[|\lambda|\int_a^{\infty} sq(s)ds\right] \equiv K(\lambda), \qquad t \geq a. \tag{29}$$

For this fact, see e.g. Hille [3, Theorem 9.1.1 and its proof].

We need only to examine positive values of $\lambda$, since the boundary condition $x_0(a;\lambda) = 0$ is not satisfied for $\lambda \leq 0$.

A simple consequence of (28) and (29) is that $x_0(t;\lambda)$ is positive on $[a,\infty)$ if $\lambda > 0$ is so small that

$$\lambda K(\lambda)\int_a^{\infty} sq(s)ds < 1$$

that is, $x_0(t;\lambda)$ has no zero in $[a,\infty)$ for such small values of $\lambda$.

It can be shown that $x_0(t;\lambda)$ has a zero in $(a,\infty)$ if $\lambda > 0$ is sufficiently large and that the number of zeros of $x_0(t;\lambda)$ in $[a,\infty)$, denoted by $N[x_0(\lambda)]$, tends to

$\infty$ as $\lambda \to \infty$. In fact, let $k \in \mathbb{N}$ be given. Put $q_* = \min\{q(t) : a \le t \le a+\pi\}$ and define $\lambda_k = k^2/q_*$. Then, $\lambda > \lambda_k$ implies $\lambda q(t) > k^2$ on $[a, a+\pi]$. We now compare (B) with the harmonic oscillator $y'' + k^2 y = 0$. Noting that a solution $y(t) = \sin k(t-a)$ of the latter equation has $k+1$ zeros in $[a, a+\pi]$, we conclude from Sturm's comparison theorem that every solution of (B), and hence $x_0(t;\lambda)$, has at least $k$ zeros in $(a, a+\pi)$ provided $\lambda > \lambda_k$. Since $k$ is arbitrary, it follows that $N[x_0(\lambda)] \to \infty$ as $\lambda \to \infty$.

We now make use of the Prüfer transformation:

$$x_0(t;\lambda) = \rho(t;\lambda)\sin\varphi(t;\lambda), \qquad x_0'(t;\lambda) = \rho(t;\lambda)\cos\varphi(t;\lambda), \tag{30}$$

or equivalently,

$$\begin{aligned} \rho(t;\lambda) &= \left[(x_0(t;\lambda))^2 + (x_0'(t;\lambda))^2\right]^{\frac{1}{2}} > 0, \\ \varphi(t;\lambda) &= \arctan\frac{x_0(t;\lambda)}{x_0'(t;\lambda)}. \end{aligned} \tag{31}$$

As it is well-known, $\rho(t;\lambda)$ and $\varphi(t;\lambda)$ are continuously differentiable functions of $t$ on $[a,\infty)$, and $\varphi(t;\lambda)$ satisfies the differential equation

$$\varphi'(t;\lambda) = \cos^2\varphi(t;\lambda) + \lambda q(t)\sin^2\varphi(t;\lambda), \qquad t \ge a. \tag{32}$$

Note that the boundary condition (27) imposed on $x_0(t;\lambda)$ at $t=\infty$ corresponds via (31) to the "terminal" condition for $\varphi(t;\lambda)$:

$$\lim_{t\to\infty}\varphi(t;\lambda) \equiv \frac{\pi}{2} \pmod{\pi}.$$

There is no loss of generality in requiring at the outset that

$$\lim_{t\to\infty}\varphi(t;\lambda) = \frac{\pi}{2}. \tag{33}$$

We will prove that, for each fixed $t \ge a$, $\varphi(t;\lambda)$ is a continuous decreasing function of $\lambda > 0$. From the equation

$$\begin{aligned} x_0(t;\lambda) - x_0(t;\mu) = -\lambda &\int_t^\infty (s-t)q(s)[x_0(s;\lambda) - x_0(s;\mu)]ds \\ &-(\lambda-\mu)\int_t^\infty (s-t)q(s)x_0(s;\mu)ds\,, \end{aligned}$$

which follows from (28), we see with the use of (29) that $u(t) = |x_0(t;\lambda) - x_0(t;\mu)|$ satisfies

$$u(t) \le |\lambda-\mu|K(\mu)\int_a^\infty sq(s)ds + \lambda\int_t^\infty sq(s)u(s)ds, \qquad t \ge a.$$

Using the Gronwall inequality and an easy calculation one may conclude that

$$u(t) \le |\lambda-\mu|K(\lambda)K(\mu)\int_a^\infty sq(s)ds, \qquad t \ge a,$$

which shows that $x_0(t;\lambda)$ is continuous with respect to $\lambda$. The continuity of $x_0'(t;\lambda)$ with respect to $\lambda$ follows from the relation

$$x_0'(t;\lambda) = \lambda \int_t^\infty q(s)x_0(s;\lambda)ds, \qquad t \geq a.$$

Then (31) implies that $\varphi(t;\lambda)$ is continuous with respect to $\lambda$.

The decreasing property of $\varphi(t;\lambda)$ with respect to $\lambda$ is verified by contradiction. Suppose that

$$\varphi(b;\lambda) \geq \varphi(b;\mu) \tag{34}$$

for some $b \in [a,\infty)$ and $\lambda$ and $\mu$ with $\lambda > \mu > 0$. Since the right-hand side of (32) written as $\cos^2\varphi + \lambda q(t)\sin^2\varphi$ is increasing with respect to $\lambda$, the initial inequality (34) implies that

$$\varphi(t;\lambda) > \varphi(t;\mu) \qquad \text{for } t > b,$$

or

$$\arctan\frac{x_0(t;\lambda)}{x_0'(t;\lambda)} > \arctan\frac{x_0(t;\mu)}{x_0'(t;\mu)}, \qquad t > b.$$

Consequently, there exists $c > b$ such that

$$\frac{x_0(t;\lambda)}{x_0'(t;\lambda)} > \frac{x_0(t;\mu)}{x_0'(t;\mu)}, \qquad t \geq c. \tag{35}$$

Put

$$X(t;\lambda,\mu) = x_0(t;\lambda)x_0'(t;\mu) - x_0'(t;\lambda)x_0(t;\mu).$$

Then $X(t;\lambda,\mu) > 0, t \geq c$, by (35), and since

$$X'(t;\lambda,\mu) = (\lambda-\mu)q(t)x_0(t;\lambda)x_0(t;\mu) > 0, \qquad t \geq c,$$

provided $c$ is taken sufficiently large, $X(t;\lambda,\mu)$ tends to a positive constant as $t \to \infty$. But this is impossible, since the boundary condition (27) implies that $X(t;\lambda,\mu) \to 0$ as $t \to \infty$. Therefore, $\varphi(t;\lambda)$ must be a decreasing function of $\lambda > 0$ for each fixed $t \geq a$.

Finally consider the values $\varphi(a;\lambda)$ for $\lambda > 0$. Since $\varphi(t;\lambda)$ is an increasing function of $t$ for fixed $\lambda > 0$, we have $\varphi(a;\lambda) < \pi/2$ (cf. (33)). If $\lambda > 0$ is sufficiently small, then $\varphi(a;\lambda) > 0$, because $x_0(t;\lambda)$ has no zero in $[a,\infty)$ as proven above. On the other hand, the fact that $N[x_0(\lambda)] \to \infty$ as $\lambda \to \infty$ shows that $\varphi(a;\lambda) \to -\infty$ as $\lambda \to \infty$. Since $\varphi(a;\lambda)$ is decreasing with respect to $\lambda > 0$, for every $n \in \mathbb{N} \cup \{0\}$, there exists $\lambda_n > 0$ such that

$$\varphi(a;\lambda_n) = -n\pi, \tag{36}$$

which means that the principal solution $x_0(t;\lambda_n)$ of (B) satisfies the boundary condition $x_0(a;\lambda_n) = 0$ and has exactly $n$ zeros in $(a,\infty)$. It is almost trivial to see that (19) holds for the sequence of principal eigenvalues $\{\lambda_n\}$. This completes the proof of Theorem 1.

*Remark.* It is well-known [6] that the equation (A) is nonoscillatory for all $\lambda > 0$ if and only if

$$\lim_{t\to\infty} P(t)\int_t^\infty q(s)ds = 0 \qquad \text{in case (4) holds;} \tag{37}$$

$$\lim_{t\to\infty} \frac{1}{\pi(t)}\int_t^\infty (\pi(s))^2 q(s)ds = 0 \qquad \text{in case (5) holds.} \tag{38}$$

The condition (8) or (9) required in Theorem I is stronger than (37) or (38), respectively. We conjecture that an analogue of Theorem I will hold under the most general condition (37) or (38).

## 3 Nonprincipal eigenvalue problem

Let us turn to the nonprincipal eigenvalue problem for (A) mentioned in the Introduction. As in the preceding section it can be shown that our main result, Theorem II, follows from the corresponding result for the particular equation (B).

**Theorem 2.** *Suppose that*

$$\int_a^\infty t^2 q(t)dt < \infty. \tag{39}$$

*Then, there exists a sequence of numbers* $\{\lambda_n\}$*:*

$$0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty \tag{40}$$

*such that, for each* $\lambda = \lambda_n$*, the equation* (B) *possesses a solution* $x_1(t;\lambda_n)$ *satisfying the boundary conditions*

$$x_1(a;\lambda_n) = 0, \qquad \lim_{t\to\infty}[x_1(t;\lambda_n) - t] = 0, \qquad \lim_{t\to\infty} t[x_1'(t;\lambda_n) - 1] = 0 \tag{41}$$

*and having exactly* $n$ *zeros in* $(a,\infty)$*,* $n = 0, 1, 2, \ldots$*.*

We will give a proof of this theorem, leaving the reduction of Theorem II to Theorem 2 to the reader.

Because of (39) there exists, for each $\lambda$, a unique nonprincipal solution $x_1(t;\lambda)$ of (B) such that

$$\lim_{t\to\infty}[x_1(t;\lambda) - t] = 0, \qquad \lim_{t\to\infty} t[x_1'(t;\lambda) - 1] = 0. \tag{42}$$

This solution is characterized as the solution to the integral equation

$$x_1(t;\lambda) = t - \lambda\int_t^\infty (s-t)q(s)x_1(s;\lambda)ds, \qquad t \ge a, \tag{43}$$

and this enables us to obtain the estimate

$$|x_1(t;\lambda) - t| \le |\lambda| K(\lambda) \int_a^\infty s^2 q(s)ds \equiv L(\lambda), \qquad t \ge a, \tag{44}$$

where $K(\lambda)$ is the constant defined in (29). For the details the reader is referred to Hille [3, Theorem 9.1.1].

Since

$$\begin{aligned} x_1(t;\lambda) - x_1(t;\mu) = -\lambda \int_t^\infty (s-t)q(s)[x_1(s;\lambda) - x_1(s;\mu)]ds \\ -(\lambda-\mu)\int_t^\infty (s-t)q(s)x_1(s;\mu)ds, \qquad t \ge a, \end{aligned}$$

using (43), we see that the function $u(t) = |x_1(t;\lambda) - x_1(t;\mu)|$ satisfies

$$u(t) \le |\lambda - \mu| \int_a^\infty sq(s)[s + L(\mu)]ds + |\lambda| \int_t^\infty sq(s)u(s)ds, \qquad t \ge a,$$

and hence we have

$$u(t) \le |\lambda - \mu| M(\mu) \exp\left[ |\lambda| \int_a^\infty sq(s)ds \right], \qquad t \ge a,$$

where

$$M(\mu) = \int_a^\infty sq(s)[s + L(\mu)]ds.$$

This shows that $x_1(t;\lambda)$ is a continuous function of $\lambda$ for each fixed $t \ge a$. The continuity of $x_1'(t;\lambda)$ with respect to $\lambda$ follows from the equation

$$x_1'(t;\lambda) = 1 + \lambda \int_t^\infty q(s)x_1(s;\lambda)ds, \qquad t \ge a.$$

Nonnegative values of $\lambda$ [or negative values of $\lambda$] may be excluded from our consideration in the case $a > 0$ [or in the case $a = 0$], since it follows from (A) and (43) that $x_1(t;\lambda)$ is unable to satisfy the boundary condition $x_1(a;\lambda) = 0$ for such values of $\lambda$.

Now we perform the Prüfer transformation:

$$x_1(t;\lambda) = \rho(t;\lambda)\sin\varphi(t;\lambda), \qquad x_1'(t;\lambda) = \rho(t;\lambda)\cos\varphi(t;\lambda), \tag{45}$$

or equivalently

$$\begin{aligned} \rho(t;\lambda) &= \left[(x_1(t;\lambda))^2 + (x_1'(t;\lambda))^2\right]^{\frac{1}{2}} > 0, \\ \varphi(t;\lambda) &= \arctan\frac{x_1(t;\lambda)}{x_1'(t;\lambda)}. \end{aligned} \tag{46}$$

The function $\varphi(t;\lambda)$ satisfies (32), and so it is an increasing function of $t$ for $\lambda > 0$. Also, $\varphi(t;\lambda)$ is continuous with respect to $\lambda$, since so are $x_1(t;\lambda)$ and $x_1'(t;\lambda)$ as stated above.

From (45) and (42) we have $\rho(t;\lambda)/t \to 1$, $\sin\varphi(t;\lambda) \to 1$ and $\cos\varphi(t;\lambda) \to 0$ as $t \to \infty$, which implies that

$$\lim_{t\to\infty} \varphi(t;\lambda) \equiv \frac{\pi}{2} \pmod{\pi}.$$

To fix the idea we suppose that

$$\lim_{t\to\infty} \varphi(t;\lambda) = \frac{\pi}{2}. \tag{47}$$

Proceeding exactly as in the proof of Theorem 1 we can show that the number of zeros of $x_1(t;\lambda)$ in $[a,\infty)$ can be made as large as possible if $\lambda > 0$ is chosen sufficiently large. It follows that $\varphi(a;\lambda) \to -\infty$ as $\lambda \to \infty$.

To examine the values $\varphi(a;\lambda)$ for small $\lambda > 0$, we distinguish the two cases: either $a = 0$ or $a > 0$. Consider the case where $a = 0$. Let $\lambda = 0$. Then, $x_1(t;0) = t$ by inspection. This solution has no zero in $(0,\infty)$. It should be noted that $x_1(t;0)$ itself is a nonprincipal eigenfunction for (B) corresponding to a nonprincipal eigenvalue $\lambda = 0$. Next consider the case where $a > 0$ and claim that $x_1(t;\lambda) > 0$ on $[a,\infty)$ for all sufficiently small $\lambda > 0$. In fact, let $\lambda > 0$ be small enough so that

$$\lambda \int_a^\infty sq(s)[s + L(\lambda)]ds < a,$$

where $L(\lambda)$ is defined in (44). Then, from (43) and (44) we obtain

$$\begin{aligned} x_1(t;\lambda) &\geq a - \lambda \int_a^\infty sq(s)|x_1(s;\lambda)|ds \\ &\geq a - \lambda \int_a^\infty sq(s)[s + L(\lambda)]ds > 0, \qquad t \geq a. \end{aligned}$$

It remains to establish the decreasing property of $\varphi(t;\lambda)$ with respect to $\lambda > 0$. Suppose to the contrary that $\varphi(b;\lambda) \geq \varphi(b;\mu)$ for some $b \in [a,\infty)$ and $\lambda$ and $\mu$ with $\lambda > \mu > 0$. Applying the argument which derived (35) from (34), we see that the function

$$X(t;\lambda,\mu) = x_1(t;\lambda)x_1'(t;\mu) - x_1'(t;\lambda)x_1(t;\mu)$$

and its derivative $X'(t;\lambda,\mu)$ are positive for all sufficiently large $t$. It follows that $X(t;\lambda,\mu)$ tends to a positive constant or $\infty$ as $t \to \infty$. On the other hand, using the relation

$$\begin{aligned} X(t;\lambda,\mu) = {} & [x_1(t;\lambda) - t]x_1'(t;\mu) - [x_1(t;\mu) - t]x_1'(t;\lambda) \\ & -t[x_1'(t;\lambda) - 1] + t[x_1'(t;\mu) - 1] \end{aligned}$$

and (42), we find that $X(t;\lambda,\mu) \to 0$ as $t \to \infty$. This contradiction proves that $\varphi(t;\lambda)$ is a decreasing function of $\lambda > 0$ for each fixed $t \geq a$.

From the above observations we conclude that, for every $n \in \mathbb{N} \cup \{0\}$, there exists $\lambda_n$ such that $\varphi(a; \lambda_n) = -n\pi$, so that $x_1(t; \lambda_n)$ is a desired nonprincipal eigenfunction for (B). It is clear that the sequence of eigenvalues $\{\lambda_n\}$ satisfies (40). Note that $\lambda_0 = 0$ if $a = 0$ and $\lambda_0 > 0$ if $a > 0$. This completes the proof of Theorem 2.

*Example.* As an example of equations to which Theorems 1 and 2 apply we give Halm's equation ([3, p. 357])

$$x'' + \lambda(1 + t^2)^{-2} x = 0, \qquad t \geq 0.$$

## 4 Application to elliptic equations

Our purpose here is to show that Theorems I and II can be applied to a qualitative study of elliptic partial differential equations of the type

$$\Delta u + \lambda c(|x|) u = 0, \qquad x \in E_a, \tag{C}$$

where $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $N \geq 2$, $|x| = \left(\Sigma_{i=1}^N x_i^2\right)^{1/2}$, $\Delta$ is the $N$-dimensional Laplace operator, $E_a = \{x \in \mathbb{R}^N : |x| \geq a\}$, $a > 0$, $c(t)$ is a positive continuous function on $[a, \infty)$, and $\lambda$ is a real parameter. We are interested in the existence of radially symmetric solutions $u(x)$ which satisfy the Dirichlet condition

$$u(x) = 0, \qquad x \in \partial E_a = \{x \in \mathbb{R}^N : |x| = a\}. \tag{48}$$

Radial symmetry of a solution means that it depends only on $|x|$, that is, it is of the form $u(x) = y(|x|)$.

It is easy to see that a radially symmetric function $u(x) = y(|x|)$ is a solution of the exterior Dirichlet problem (C)–(48) if and only if the function $y(t)$ is a solution of the ordinary differential equation

$$(t^{N-1} y')' + \lambda t^{N-1} c(t) y = 0, \qquad t \geq a \tag{49}$$

satisfying

$$y(a) = 0. \tag{50}$$

The equation (49) is a special case of (A) in which

$$p(t) = t^{N-1} \qquad \text{and} \qquad q(t) = t^{N-1} c(t). \tag{51}$$

We note that:

(i) (4) holds if and only if $N = 2$, in which case the function $P(t)$ defined by (6) is

$$P(t) = \log \frac{t}{a}, \qquad t \geq a; \tag{52}$$

(ii) (5) holds if and only if $N \geq 3$, in which case the function $\pi(t)$ defined by (7) is

$$\pi(t) = \frac{t^{2-N}}{N-2}, \qquad t \geq a. \tag{53}$$

Therefore, the conditions (8), (9) reduce to

$$\int^{\infty} t(\log t)c(t)dt < \infty, \tag{54}$$

$$\int^{\infty} tc(t)dt < \infty, \tag{55}$$

and the conditions (14), (15) to

$$\int^{\infty} t(\log t)^2 c(t)dt < \infty, \tag{56}$$

$$\int^{\infty} t^{N-1} c(t)dt < \infty. \tag{57}$$

The next result follows from Theorem I applied to (49)–(50).

**Theorem 3.** (i) *Let $N = 2$ and suppose that* (54) *holds. Then, there exists a sequence of positive numbers $\{\lambda_n\}$:*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty \tag{58}$$

*such that, for each $\lambda = \lambda_n$, the exterior Dirichlet problem* (C)–(48) *possesses a radially symmetric solution $u(x;\lambda_n)$ satisfying*

$$\lim_{|x|\to\infty} u(x;\lambda_n) = 1 \tag{59}$$

*and having exactly $n$ spherical nodes in the interior of $E_a$, $n = 0, 1, 2, \ldots$.*

(ii) *Let $N \geq 3$ and suppose that* (55) *holds. Then, there exists a sequence of positive numbers $\{\lambda_n\}$ with the property* (58) *such that, for each $\lambda = \lambda_n$, the problem* (C)–(48) *possesses a radially symmetric solution $u(x;\lambda_n)$ satisfying*

$$\lim_{|x|\to\infty} |x|^{N-2} u(x;\lambda_n) = 1 \tag{60}$$

*and having exactly $n$ spherical nodes in the interior of $E_a$, $n = 0, 1, 2, \ldots$.*

Theorem II specialized to (49)–(50) yields another result for the exterior Dirichlet problem under consideration.

**Theorem 4.** (i) *Let $N = 2$ and suppose that* (56) *holds. Then, there exists a sequence of positive numbers $\{\lambda_n\}$ with the property* (58) *such that, for each $\lambda = \lambda_n$, the problem* (C)–(48) *possesses a radially symmetric solution $u(x;\lambda_n)$ satisfying*

$$\lim_{|x|\to\infty} \frac{u(x;\lambda_n)}{\log|x|} = 1 \tag{61}$$

*and having exactly* $n$ *spherical nodes in the interior of* $E_a$, $n = 0, 1, 2, \ldots$.

(ii) *Let* $N \geq 3$ *and suppose that* (57) *holds. Then, there exists a sequence of positive numbers* $\{\lambda_n\}$ *with the property* (58) *such that, for each* $\lambda = \lambda_n$*, the problem* (C)–(48) *possesses a radially symmetric solution* $u(x; \lambda_n)$ *satisfying*

$$\lim_{|x| \to \infty} u(x; \lambda_n) = 1 \tag{62}$$

*and having exactly* $n$ *spherical nodes in the interior of* $E_a$, $n = 0, 1, 2, \ldots$.

*Remark.* A related problem for (C) in the entire space $\mathbb{R}^N$ has been studied by Naito [5] and Kabeya [4].

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