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Dispersions for linear differential equations of arbitrary order

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DISPERSIONS FOR LINEAR DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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Dedicated to the memory of Professor Otakar Borůvka

Abstract.

$y'' \quad p \; x \; y$

I. Motivation

For a linear differential equation of the second order in the Jacobi form

$$(p) \qquad y'' + p(x)y = 0, \quad p \in C^0(I), I = (a,b), -\infty \le a < b \le \infty$$

O. Borůvka [2] introduced the notions of a *phase* and the *dispersion* as follows. Consider two linearly independent solutions y_1 and y_2 of (p). A phase of (p) corresponding to the pair y_1, y_2 is a continuous function $\alpha : I \to \mathbb{R}$ satisfying the relation

$$\tan \alpha(x) = y_1(x)/y_2(x)$$

wherever $y_2(x) \neq 0$. The continuity of α implies $\alpha \in C^3(I)$ with $\alpha'(x) \neq 0$ on I, because

$$\alpha'(x) = \frac{c}{y_1^2(x) + y_2^2(x)}, \quad c = \text{const.} \quad \neq 0.$$

Mathematics Subject Classification

Key words and phrases

Moreover, the general solution of (p) can be written in the form

$$y(t;c_1,c_2)=\frac{c_1}{\sqrt{|\alpha'(x)|}}\sin(\alpha(x)+c_2).$$

If the equation (p) is oscillatory for $x\to b_-$, then $\lim_{x\to b_-}|\alpha(x)|=\infty$.

The dispersion φ of (p) is defined [2] as follows. For arbitrary $x_0\in(a,b)$, let y be a nontrivial solution of (p) vanishing at x_0, i.e. $y(x_0)=0$. If there exists a zero of this solution y to the right of x_0, then the first zero of them is denoted by $\varphi(x_0)$. Evidently φ is defined on (a,b) if (p) is oscillatory for $x\to b_-$. Borůvka has shown that $\varphi\in C^3$, $\varphi'(x)>0$ and $\varphi(x)>x$ and the following Abel's functional equation holds:

$$\alpha(\varphi(x))=\alpha(x)+\pi\cdot\operatorname{sign}\,\alpha'$$

wherever φ is defined. This functional equation was intensively studied by B. Choczewski [3], see also M. Kuczma [4], and in connection with the second order differential equations by E. Barvínek [1]. Important connections between distribution of zeros of solutions of oscillatory second-order equations (p) and their asymptotic properties were studied in [5], see also [6]. Here we generalize these results to linear differential equations of the n-th order.

II. Preliminary results

Consider a linear differential equation of the form

$$(P)\qquad y^{(n)}+p_{n-1}(x)y^{(n-1)}+\cdots+p_0(x)y=0\qquad\text{on}\qquad I,$$

I being an open interval of the reals, p_i are real-valued continuous functions defined on I for $i=0,1,\ldots,n-1$, i.e. $p_i\in C^0(I), p_i:I\to\mathbb{R}$.

Take functions $f:J\to\mathbb{R}$ and $h:J\to I$ such that

$$f\in C^n(J), f(t)\neq 0\quad\text{for each}\quad t\in J,\quad\text{and}$$

$$h\in C^n(J), h'(t)\neq 0\quad\text{for each}\quad t\in J,\quad\text{and}\quad h(J)=I.$$

For each solution y of equation (P) the function z defined as

$$(f,h)\qquad z:J\to\mathbb{R},\quad z(t):=f(t)\cdot y(h(t)),\quad t\in J,$$

satisfies again a differential equation of the same form

$$(Q)\qquad z^{(n)}+q_{n-1}(t)z^{(n-1)}+\cdots+q_0(t)z=0\qquad\text{on}\qquad J.$$

Since h is a C^n-diffeomorphism of J onto I, solutions y are transformed into solutions z on their whole intervals of definition. This is why we also speak about a *global* transformation of equation (P) into equation (Q).

Let $\mathbf{y}(x) = (y_1(x), \ldots, y_n(x))^T$ denote an n-tuple of linearly independent solutions of the equation (P) considered as a column vector function or as a curve in n-dimensional Euclidean space $\mathbb{E}_n$ with the independent variable x as the parameter and $y_1(x), \ldots, y_n(x)$ as its coordinate functions; M^T denotes the transpose of the matrix M.

If $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t)^T$ denotes an n-tuple of linearly independent solutions of the equation (Q), then the global transformation (f, h) can be equivalently written as

$$\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(x))$$

or, for an arbitrary regular constant $n \times n$ matrix A ,

$$\mathbf{z}(t) = Af(t) \cdot \mathbf{y}(h(x))$$

expressing only the fact that another n-tuple of linearly independent solutions of the *same* equation (Q) is taken.

To emphasize this situation, let us denote by $(P_{\mathbf{y}})$ and $(Q_{\mathbf{z}})$ the equations (P) and (Q), respectively. Capital P refers to the coefficients p_i of the equation $(P_{\mathbf{y}})$, subscript $\mathbf{y}$ expresses a particular choice of an n-tuple of linearly independent solutions. Similarly for $(Q_{\mathbf{z}})$ and other equations considered here.

Denote by $W[\mathbf{y}](x)$ the Wronski determinant of $\mathbf{y}$, i.e.
$\det(\mathbf{y}(x), \mathbf{y}'(x), \ldots, \mathbf{y}^{(n-1)}(x))$.

The coefficient p_{n-1} in $(P_{\mathbf{y}})$ is given as

$$p_{n-1}(x) = -(\ln |W[\mathbf{y}](x)|)'.$$

We have $p_{n-1} \equiv 0$ exactly when $W[\mathbf{y}](x) = \text{const.} \neq 0$. Since

$$W[f.\mathbf{y}(h)](t) = (f(t))^n.(h'(t))^{\frac{n(n-1)}{2}}.W[\mathbf{y}](h(t)),$$

for the coefficient q_{n-1} in $(Q_{\mathbf{z}})$ we have

$$(1) \qquad q_{n-1}(t) = -n\frac{f'(t)}{f(t)} - \frac{n(n-1)}{2}\frac{h''(t)}{h'(t)} + p_{n-1}(h(t)).h'(t).$$

Namely, if $p_{n-1} \equiv 0$ then $q_{n-1} \equiv 0$ occurs exactly when

$$(2) \qquad f(t) = c.|h'(t)|^{\frac{1-n}{2}}, \quad c = \text{const.} \neq 0.$$

Since the factor f belongs to $C^n(J)$, we have $h \in C^{n+1}(J)$.

III. Notation and basic properties

Let all solutions of an equation $(R_{\mathbf{u}})$ be periodic or half-periodic with a period $d, d > 0$:

$$\mathbf{u}(x+d) = \mathbf{u}(x), \quad \text{or}$$
$$\mathbf{u}(x+d) = -\mathbf{u}(x) \quad \text{on } \mathbb{R}.$$

Then all coefficients r_i of $(R_{\mathbf{u}})$ are periodic, $r_i(x+d) = r_i(x)$ on $\mathbb{R}$.

Lemma 1. *There is no equation* $(R_{\mathbf{u}})$ *of odd order* n *with all half-periodic solutions.*

Proof. Consider $W[\mathbf{u}](x)$ for an equation $(R_{\mathbf{u}})$ and its n-tuple $\mathbf{u}$ of linearly independent solutions. For $\mathbf{u}(x+d) = -\mathbf{u}(x)$ we would have

$$W[\mathbf{u}](x+d) = W[-\mathbf{u}](x) = -W[\mathbf{u}](x),$$

because n is odd. Since $W[\mathbf{u}]$ is nonvanishing and continuous, this is a contradiction. □

Consider an equation $(S_{\mathbf{v}})$ of the same order that can be transformed into the equation $(R_{\mathbf{u}})$:

$$\mathbf{u}(x) = f(x).\mathbf{v}(h(x)),$$

h being a C^n-diffeomorphism of $\mathbb{R}$ onto J, the interval of definition of $(S_{\mathbf{v}})$.

Take $f(x) = ||\mathbf{u}(x)|| := \sqrt{u_1^2(x) + \cdots + u_n^2(x)}$, $||\cdot||$ denoting the Euclidean norm of an n-dimensional vector. Evidently $f \in C^n(\mathbb{R}), f(x) > 0$, and $f(x+d) = f(x)$ on $\mathbb{R}$ and $||\mathbf{v}(t)|| = ||\mathbf{v}(h(x))|| = ||\mathbf{u}(x)||/f(x) = 1$.

Let h be chosen so that

$$r_{n-1}(x) = -n\frac{f'(x)}{f(x)} - \frac{n(n-1)}{2}.\frac{h''(x)}{h'(x)} \quad , \tag{3}$$

i.e. $h(x) := c \cdot \int_{x_0}^{x} [(f(\sigma))^{\frac{2}{1-n}} \cdot \exp\{\frac{-2}{n(n-1)} \cdot \int_{\sigma_0}^{\sigma} r_{n-1}(\tau)d\tau\}]d\sigma + k.$

For $r_{n-1} \in C^{n-2}(\mathbb{R})$, we have $h \in C^n(\mathbb{R}), h'(x) > 0, h(x+d) = h(x)+p$ because of d-periodicity of f and r_{n-1}, $h(\mathbb{R}) = \mathbb{R}$. Select c so that $p = d$. Due to relation (1) with respect to (3) where r_{n-1} stands for q_{n-1} and $s_{n-1} \equiv 0$ for p_{n-1}, we see that

(i) the coefficient s_{n-1} in equation $(S_{\mathbf{v}})$ is identically zero;

(ii) all solutions of $(S_{\mathbf{v}})$ are periodic or half-periodic with the period d, and

(iii) $||\mathbf{v}(t)|| = 1$.

Choose $t_0 \in \mathbb{R}$ arbitrarily. Let v be a nontrivial solution of $(S_{\mathbf{v}})$ with the zero of multiplicity $(n-1)$ at t_0, i.e. satisfying

$$v(t_0) = v'(t_0) = \cdots = v^{(n-2)}(t_0) = 0, v^{(n-1)}(t_0) \neq 0.$$

Up to a constant multiplier, v is determined uniquely.

Lemma 2. *For the above solution* v*, the points* $t_0 + kd, k \in \mathbb{Z}$*, are zeros of multiplicity* $n-1$.

Proof follows from the periodicity or half-periodicity of all solutions of $(S_{\mathbf{v}})$. □

Now suppose that in addition to the above properties of equation $(S_{\mathbf{v}})$, the following one is also satisfied:

(iv) for each $t_0 \in \mathbb{R}$, any solution having a zero of multiplicity $n-1$ at t_0 has the point $t_0 + d$ as its first zero of the same multiplicity to the right of t_0.

Remark 1. Property (iv) implies that d is the smallest positive period for which $\mathbf{v}$ of $(S_\mathbf{v})$ is periodic or half-periodic on the whole $\mathbb{R}$.

Notation 1. Let $\mathcal{S}$ denote the set of all linear differential equations satisfying the properties (i), (ii), (iii), and (iv). Furthermore, let $\mathcal{P}$ be the set of all linear differential equations that can be obtained from equations in $\mathcal{S}$ by all global transformations (f, h).

IV. Dispersions

Definition. Let an equation $(P_\mathbf{y})$ of the order n belong to $\mathcal{P}$. Take an arbitrary x_0 from its interval of definition I and consider a nontrivial solution y having a zero of multiplicity $n-1$ at x_0. Denote by $\varphi(x_0)$ the first zero of the same multiplicity of this solution y to the right of x_0. Call this function φ the *dispersion* of the equation $(P_\mathbf{y})$.

Theorem 1. *Let (f, h) be the transformation that transforms an equation $(P_\mathbf{y})$ of the n-th order from $\mathcal{P}$ into an equation from $\mathcal{S}$. The dispersion φ of $(P_\mathbf{y})$ is well-defined on the whole interval of definition I of this equation and satisfies Abel's functional equation*

$$h(\varphi(x)) = h(x) + d \cdot \operatorname{sign} h' \quad , \qquad x \in I. \tag{4}$$

Futhermore,

$$\varphi \in C^n(I), \quad \varphi(x) > x, \quad \varphi'(x) > 0, \quad \varphi(I) = I, \quad \text{and}$$

$$\lim_{i \to -\infty} \varphi^{[i]}(x_0) = a, \lim_{i \to \infty} \varphi^{[i]}(x_0) = b, \quad \text{for each} \quad x_0 \in I = (a, b),$$

where $\varphi^{[i]}$ denotes the i-th iterate of φ, i.e. $\varphi^{[1]} = \varphi, \varphi^{[i+1]} = \varphi \circ \varphi^{[i]}$.

Proof. Take $x_0 \in I$ arbitrarily, and denote by y a nontrivial solution of $(P_\mathbf{y})$ having x_0 as its zero of multiplicity $n-1$. Writing $y(x) = \mathbf{c}^T \cdot \mathbf{y}(x)$, where $\mathbf{c}$ is a suitable constant vector and '$\cdot$' denotes the dot product, we have

$$\mathbf{c}^T \cdot \mathbf{y}(x_0) = \mathbf{c}^T \cdot \mathbf{y}'(x_0) = \ldots = \mathbf{c}^T \cdot \mathbf{y}^{(n-2)}(x_0) = 0, \quad \mathbf{c}^T \cdot \mathbf{y}^{(n-1)}(x_0) \neq 0.$$

Hence

$$\begin{aligned} &0 = \mathbf{c}^T \cdot f(x_0) \cdot \mathbf{v}(h(x_0)), \\ &0 = \mathbf{c}^T \cdot [f(x_0) \cdot \mathbf{v}'(h(x_0)) \cdot h'(x_0) + f'(x_0) \cdot \mathbf{v}(h(x_0))], \quad \ldots \quad , \\ &0 = \mathbf{c}^T \cdot [f(x_0) \cdot \mathbf{v}^{(n-2)}(h(x_0)) \cdot (h'(x_0))^{n-2} + L(n-3)], \\ &0 \neq \mathbf{c}^T \cdot [f(x_0) \cdot \mathbf{v}^{(n-1)}(h(x_0)) \cdot (h'(x_0))^{n-1} + L(n-2)], \end{aligned} \tag{5}$$

where $L(i)$ is a linear combination of the vectors $\mathbf{v}(h(x_0)), \mathbf{v}'(h(x_0)), \ldots, \mathbf{v}^{(i)}(h(x_0))$ with some scalar functions as coefficients.

Since $f(x_0)$ and $h'(x_0) \neq 0$, the solution $v(t) = \mathbf{c}^T \cdot \mathbf{v}(t)$ of equation $(S_{\mathbf{v}})$ has a zero of multiplicity $n-1$ at $t_0 = h(x_0)$. Due to the property (iv), the first zero of this solution of the same multiplicity to the right of t_0 is $t_0 + d$, to the left of t_0 is $t_0 - d$. Hence

$$h(\varphi(x_0)) = t_0 + d = h(x_0) + d \quad \text{for increasing } h, \text{ and}$$

$$h(\varphi(x_0)) = t_0 - d = h(x_0) - d \quad \text{for decreasing } h.$$

Since $x_0 \in I$ was arbitrary, Abel's equation (4) holds.

Now, h is a C^n-diffeomorphism of I onto $\mathbb{R}$, thus

$$\varphi(x) = h^{-1}(h(x) + d.\mathrm{sign}h') \tag{6}$$

is defined for all $x \in I, \varphi(x) > x$, and $\varphi'(x) > 0$ on I because $h'(\varphi(x)).\varphi'(x) = h'(x)$. Moreover

$$\varphi^{[i]}(x) = h^{-1}(h(x) + i.d.\mathrm{sign}h'),$$

hence

$$\lim_{i\to-\infty} \varphi^{[i]}(x_0) = a \quad \text{and} \quad \lim_{i\to\infty} \varphi^{[i]}(x_0) = b.$$

□

Lemma 3. *For each equation $(P_{\mathbf{y}}) \in \mathcal{P}$ and its dispersion φ the following is true:*

$\mathbf{y}(x_0)$ is parallel to $\mathbf{y}(\varphi(x_0))$, and $\varphi(x_0)$ is the first parameter to the right of x_0 when it happens, i.e.

$$\varphi(x_0) = \min_{x > x_0} \{ \text{rank } (\mathbf{y}(x_0), \mathbf{y}(x)) = 1 \}.$$

Proof follows from the definition of the dispersion, the system (5) and the property (iv). □

V. Asymptotic behaviour

Since the factor f in the global transformation (f, h) is in general independent on the function h, we cannot expect a relation between asymptotic behaviour of solutions (depending on f) and the distribution of their zeros (depending on φ and hence on h). However, for linear differential equations of the n-th order with the vanishing coefficients by the $(n-1)$-st derivative we have the relation (2). Thus let us consider the following class of equations.

Notation 2. A linear differential equation of an order n belongs to the subset $\mathcal{P}_0$ of $\mathcal{P}$ if its coefficient by the $(n-1)$-st derivative is identically zero.

Remark 2. Evidently $\mathcal{S} \subset \mathcal{P}_0$ and $h \in C^{n+1}(I)$. Then, due to (6), the dispersion φ of each equation from $\mathcal{P}_0$ is also in $C^{n+1}(I)$.

Theorem 2. *Let an equation* $(P_{\mathbf{y}})$ *of the n-th order belong to* $\mathcal{P}_0$ *and let* $\varphi : I \to I$ *denote its dispersion. If*

a) $\varphi(x) - x$ *is a nondecreasing function, or*

b) $\varphi(x) - x$ *is a nonincreasing function, or*

c) $\varphi(x) - x = \delta = const. > 0$,

then

a') maxima of absolute values of each solution of $(P_{\mathbf{y}})$ *on consecutive intervals* $[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]$, $i = 0, 1, 2, \dots$, *form a nondecreasing sequence, or*

b') those maxima form a nonincreasing sequence, or

c') each solution of $(P_{\mathbf{y}})$ *is periodic or half-periodic with the period* δ, *respectively.*

Remark 3. Conditions a) - c) mean that the distances between consecutive zeros of multiplicity $n-1$ of each solution of $(P_{\mathbf{y}})$ are a) nondecreasing, b) nonincreasing, c) the same.

Proof of Theorem 2. Since $(P_{\mathbf{y}}) \in \mathcal{P}_0$,

$$\mathbf{y}(x) = |h'(x)|^{\frac{1-n}{2}} \mathbf{v}(h(x)), \quad x \in I, \quad h(I) = \mathbb{R} \tag{7}$$

for some $(S_{\mathbf{v}}) \in \mathcal{S}$. For the dispersion φ of $(P_{\mathbf{y}})$ we have

$$h(\varphi(x)) = h(x) + d \cdot \text{sign } h'$$

and

$$h'(\varphi(x)) \cdot \varphi'(x) = h'(x), \quad x \in I. \tag{8}$$

Each solution y of $(P_{\mathbf{y}})$ can be written as $\mathbf{c}^T \cdot \mathbf{y}(x)$ for a suitable constant vector $\mathbf{c}$. Choose $x_0 \in I$. Let M_i be the maximum of $|y|$ on the interval $[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]$, i.e.

$$M_i := \max_{[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]} |\mathbf{c}^T \cdot \mathbf{y}(x)| =$$

$$\max_{[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]} ||h'(x)|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x))|.$$

Now, due to (8), we have

$$M_{i+1} = \max_{[\varphi^{[i+1]}(x_0), \varphi^{[i+2]}(x_0)]} ||h'(x)|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x))| =$$

$$\max_{[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]} ||h'(\varphi(x))|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(\varphi(x)))| =$$

$$\max_{[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]} \left| |(h'(x)|^{\frac{1-n}{2}} \cdot (\varphi'(x))^{\frac{n-1}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x) \pm d) \right| =$$

$$\max_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]} \{(\varphi'(x))^{\frac{n-1}{2}} \cdot |h'(x)|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x)+d)\} \geq$$

$$\min_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]} (\varphi'(x))^{\frac{n-1}{2}} \cdot \max_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]} |\pm |h'(x)|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x))|\} =$$

$$\min_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]} (\varphi'(x))^{\frac{n-1}{2}} \cdot M_i.$$

For the case a), when $\varphi(x) - x$ is increasing, i.e. $\varphi'(x) \geq 1$ everywhere, we get $M_{i+1} \geq M_i$, hence the consecutive maxima cannot decrease. Analogously in the case when b) $\varphi'(x) \leq 1$, we have

$$M_{i+1} = \max_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]} \{(\varphi'(x))^{\frac{n-1}{2}} | \pm |h'(x)|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x))|\} \leq$$

$$\max_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]} (\varphi'(x))^{\frac{n-1}{2}} \cdot M_i \leq M_i,$$

and the consecutive maxima cannot increase.

For $\varphi(x) - x = \delta > 0$ we have $\varphi(x) = x + \delta$, $\varphi'(x) = 1$, $h(\varphi(x)) = h(x) \pm d$, and also $h'(\varphi(x)) = h'(x)$. Hence

$$\begin{aligned}
\mathbf{y}(\varphi(x)) &= \mathbf{y}(x+\delta) \quad \text{and also} \\
&= |h'(\varphi(x))|^{\frac{1-n}{2}}.\mathbf{v}(h(\varphi(x))) = |h'(x)|^{\frac{1-n}{2}}.\mathbf{v}(h(x) \pm d) \\
&= \pm |h'(x)|^{\frac{1-n}{2}}.\mathbf{v}(h(x)) = \pm \mathbf{y}(x). \square
\end{aligned}$$

VI. Final remarks

The dispersion just introduced for the n-th order linear differential equations is a proper generalization of this notion introduced by O. Borůvka for the second order equations of the Jacobi form. The role of the set of equations $\mathcal{S}$ in his case is played by a single equation

$$v'' + v = 0 \quad \text{on } \mathbb{R}$$

that admits half-periodic solutions with the period π:

$$\mathbf{v}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \qquad \mathbf{v}(t+\pi) = -\mathbf{v}(t).$$

The subset of the second order equations from $\mathcal{P}$ is formed by all both side oscillatory equations in the general form, the set $\mathcal{P}_0$ consists from all both-side oscillatory equations in the Jacobi form, (p). The function h here is the (first) phase and φ coincides with the dispersion introduced by O. Borůvka for equation (p). This φ in fact generalizes the dispersion even for the second order equations because it is defined for equations in the general form: $y'' + p_1(x)y' + p_0(x)y = 0$.

References

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