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CHARACTERIZATIONS OF INNER PRODUCT STRUCTURES INVOLVING THE RADIUS OF THE INSCRIBED OR CIRCUMSCRIBED CIRCUMFERENCE

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Abstract. We define the radius of the inscribed and circumscribed circumferences in a triangle located in a real normed space and we obtain new characterizations of inner product spaces.

1. On the radius of the inscribed circumference in a triangle in a normed space

In an inner product space (i.p.s.) the radius of the inscribed circumference in a triangle of sides x, y and $x - y$ is given by the formula

$$\frac{\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}}{s} \tag{1}$$

where s is the semiperimeter $s = \dfrac{(\|x\|+\|y\|+\|x-y\|)}{2}$.

Let $(E, \|\,\|)$ be a real normed space. If x, y are two independent vectors in $E\backslash\{0\}$ and (see [4]) if $w(x,y) = \dfrac{\|y\|x+\|x\|y}{\|x\|+\|y\|}$ is the bisectrix of x and y, in the triangle of sides x, y and $x-y$, we can consider the bisectrices $w(x,y)$, $w(-y, x-y)$ and $w(-x, y-x)$. It is a straightforward computation to prove that these three lines intersect in a point, i.e. there exist three constants

$$\lambda = \frac{\|x\|+\|y\|}{2s}, \quad \mu = \frac{\|y\|+\|x-y\|}{2s}, \quad \gamma = \frac{\|x\|+\|x-y\|}{2s},$$

in $\mathbb{R}$ such that

$$\lambda w(x,y) = y + \mu w(-y, x-y) = x + \gamma w(-x, y-x)$$

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Now, in order to define the radius of the inscribed circumference, we need a definition of height in a real normed space. To do that, let us consider (see [5]) the functions $\rho'_\pm : E \times E \to \mathbb{R}$ defined by

$$\rho'_\pm(x,y) = \lim_{t\to 0^\pm} \frac{||x+ty||^2 - ||x||^2}{2t}.$$

The mappings $\rho'_\pm$ play a crucial role in characterizing inner product spaces. In fact, when the norm derives from an inner product $(E,\cdot)$, then $\rho'_\pm(x,y) = x \cdot y$, i.e., $\rho'_\pm$ reduce to the given inner product.

We quote here some elementary results concerning the functions $\rho'_\pm$ that we will use in this paper:

(i) $\rho'_\pm(x,x) = ||x||^2$ and $|\rho'_\pm(x,y)| \le ||x||\,||y||$

(ii) $\rho'_+(\alpha x, y) = \rho'_+(x, \alpha y) = \alpha\rho'_+(x,y)$, $\alpha \ge 0$

(iii) $\rho'_+(\alpha x, y) = \rho'_+(x, \alpha y) = \alpha\rho'_-(x,y)$, $\alpha \le 0$

(iv) $\rho'_+(x, \alpha x + y) = \rho'_+(x,y) + \alpha||x||^2$

(v) $\rho'_+(x,y) = \rho'_+(y,x)$ for all x, y in E if and only if E is an inner product space.

In [1] by using the functions $\rho'_\pm$ we introduce the following definition of height over the side $x-y$

$$h(x,y) = y + \frac{\rho'_+(y-x,y)}{||x-y||^2}(x-y) \qquad \text{for all } x,y \text{ in } E \text{ linearly independent,}$$

so we can define "the radius $r(x,y)$ of the inscribed circumference" as the norm of $h\left(\lambda w(x,y), \mu w(-y, x-y)\right)$, i.e.

$$(2) \qquad r(x,y) := \frac{||x||\,||y||}{2s}\left\| \frac{x}{||x||} - \rho'_-\left(\frac{y}{||y||}, \frac{x}{||x||}\right)\frac{y}{||y||}\right\|$$

Observe that $r(x,y)$ is not symmetric in x and y.

We want to find when the expressions (1) and (2) are equal.

Theorem 1.1. *Let $(E, ||\,||)$ be a real normed space with dim $E \ge 2$. Then E is an i.p.s. if and only if for all linearly independent vectors x, y in E*

$$r(x,y) = \frac{\sqrt{s(s-||x||)(s-||y||)(s-||x-y||)}}{s}$$

where $s = \dfrac{||x|| + ||y|| + ||x-y||}{2}$.

Proof. If we assume that $r(x,y)$ defined by (2) can be obtained by means of (1), then we have:

$$r(x,y)^2 = \frac{(s-||x||)(s-||y||)(s-||x-y||)}{s}$$

i.e.

$$\frac{\|x\|^2\|y\|^2}{4s^2}\left\|\frac{x}{\|x\|}-\rho'_-\left(\frac{y}{\|y\|},\frac{x}{\|x\|}\right)\frac{y}{\|y\|}\right\|^2=\frac{(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}{s}$$

or equivalently

$$\left\|\|y\|x-\rho'_-(y,x)\frac{y}{\|y\|}\right\|^2=4s(s-\|x\|)(s-\|y\|)(s-\|x-y\|).$$

Substituting in this equality y by tz, with $t>0$, dividing by t^2 and taking limits when t tends to zero, we obtain

$$\begin{aligned}
\|\|z\|x-\rho'_-(z,x)\frac{z}{\|z\|}\right\|^2 &= \lim_{t\to 0+}\frac{4s(s-\|x\|)(s-\|tz\|)(s-\|x-tz\|)}{t^2}\\
&= \lim_{t\to 0+}\frac{4}{t^2}\frac{\|x\|+t\|z\|+\|x-tz\|}{2}\cdot\frac{t\|z\|+\|x-tz\|-\|x\|}{2}\cdot\\
&\cdot\frac{\|x\|+\|x-tz\|-t\|z\|}{2}\cdot\frac{\|x\|+t\|z\|-\|x-tz\|}{2}\\
&= \lim_{t\to 0+}\frac{(\|x-tz\|+t\|z\|)^2-\|x\|^2}{2t}\frac{\|x\|^2-(\|x-tz\|-t\|z\|)^2}{2t}=\\
&= \lim_{t\to 0+}\left(-\frac{\|x-tz\|^2-\|x\|^2}{2(-t)}+\frac{\|z\|^2}{2}t+\|x-tz\|\,\|z\|\right)\cdot\\
&\cdot\left(\frac{\|x-tz\|^2-\|x\|^2}{2(-t)}-\frac{\|z\|^2}{2}t+\|x-tz\|\,\|z\|\right)\\
&= \left(-\rho'_-(x,z)+\|x\|\,\|z\|\right)\left(\rho'_-(x,z)+\|x\|\,\|z\|\right)\\
&= \|x\|^2\|z\|^2-\rho'_-(x,z)^2,
\end{aligned}$$

and therefore

$$\left\|\|z\|^2x-\rho'_-(z,x)z\right\|^2=\|z\|^2\left(\|x\|^2\|z\|^2-\rho'_-(x,z)^2\right).$$

The substitution $x=z+y$ yields

$$\begin{aligned}
\|z\|^2\left(\|z+y\|^2\|z\|^2-\rho'_-(z+y,z)^2\right) &= \left\|\|z\|^2(z+y)-\rho'_-(z,z+y)z\right\|^2\\
&= \left\|\|z\|^2z+\|z\|^2y-\|z\|^2z-\rho'_-(z,y)z\right\|^2\\
&= \left\|\|z\|^2y-\rho'_-(z,y)z\right\|^2=\\
&= \|z\|^2\left(\|y\|^2\|z\|^2-\rho'_-(y,z)^2\right),
\end{aligned}$$

i.e.,

$$\|y\|^2\|z\|^2-\rho'_-(y,z)^2=\|z+y\|^2\|z\|^2-\rho'_-(z+y,z)^2.$$

Thus for all linearly independent vectors u, v in E with $||u|| = ||v|| = 1$, if we substitute in the last equality $z = u - v$, $y = v$ we obtain:

$$||v||^2 ||u-v||^2 - \rho'_-(v, u-v)^2 = ||u||^2 ||u-v||^2 - \rho'_-(u, u-v)^2,$$

and consequently

$$\left|\rho'_-(v, u-v)\right| = \left|\rho'_-(u, u-v)\right|$$

which in turn implies $\left|\rho'_-(v,u) - ||v||^2\right| = \left|||u||^2 - \rho'_+(u,v)\right|$, i.e.,

$$\left|1 - \rho'_-(v,u)\right| = \left|1 - \rho'_+(u,v)\right|.$$

Since $\rho'_-(v,u) \leq ||u||\,||v|| = 1$ and $\rho'_+(u,v) \leq ||u||\,||v|| = 1$ we deduce

$$\rho'_-(v,u) = \rho'_+(u,v),$$

and interchanging the roles of u and v:

$$\rho'_-(u,v) = \rho'_+(v,u),$$

whence $\rho'_+(v,u) = \rho'_-(u,v) \leq \rho'_+(u,v)$ as well as $\rho'_+(u,v) = \rho'_-(v,u) \leq \rho'_+(v,u)$, i.e., $\rho'_+(u,v) = \rho'_+(v,u)$, whenever $||u|| = ||v|| = 1$, u, v independent vectors and E is an inner product space. $\square$

Note. The value $r(x,y)$ as introduced above is not the unique possible definition of the radius of the inscribed circumference, since we can consider other heights like $h(\lambda w(x,y), \gamma w(-x, y-x))$, $h(\mu w(-y, x-y), \lambda w(x,y))$, $h(\mu w(-y, x-y), \gamma w(-x, y-x))$, $h(\gamma w(-x, y-x), \lambda w(x,y))$ or $h(\gamma w(-x, y-x), \mu w(-y, x-y))$ instead of $h(\lambda w(x,y), \mu w(-y, x-y))$.

2. On the radius of the circumscribed circumference in a triangle in a normed space

If E is an i.p.s. and x, y are two independent vectors in $E \backslash \{0\}$, in the triangle of sides x, y and $x - y$, the center of the circumscribed circumference is the intersection of the perpendicular bisectors, and the radius R is given by the formula

$$\frac{||x||\,||y||\,||x-y||}{4\sqrt{s(s-||x||)(s-||y||)(s-||x-y||)}}$$

where $s = (||x|| + ||y|| + ||x-y||)/2$ is the semiperimeter of the triangle.

In a real normed space $(E, ||\,||)$ we can consider the perpendicular bisector of the side $x - y$ by taking

$$M(x,y) = \left\{ \frac{x+y}{2} + \lambda u \,/\, \lambda \in \mathbb{R} \right\}$$

where u is a vector in E "orthogonal" to $x-y$.

We can assume that u admits the form $u=\alpha x+\beta y$ and having in mind that in an i.p.s. $u\cdot(x-y)=0$ is immediate to prove that

$$\alpha\left(\|x\|^2-x\cdot y\right)=\beta\left(\|y\|^2-x\cdot y\right).$$

Thus in the real normed space E, replacing the inner product $\cdot$ by ρ'_-, we can consider

$$u=\left(\|y\|^2-\rho'_-(x,y)\right)x+\left(\|x\|^2-\rho'_-(y,x)\right)y$$

and

$$M(x,y)=\left\{\frac{x+y}{2}+\lambda\left[(\|y\|^2-\rho'_-(x,y))x+(\|x\|^2-\rho'_-(y,x))y\right]\ \ /\ \lambda\in\mathbb{R}\right\}$$

The next step is to define the radius R. First in the triangle of sides x, y and $x-y$ considering the corresponding three perpendicular bisectors is easy to prove that this three lines intersected in a point, i.e., there exist λ,μ,γ in $\mathbb{R}$ such that

$$\frac{x}{2}+\gamma w=\frac{y}{2}+\mu v=\frac{x+y}{2}+\lambda u$$

where u, v and w are respectively "orthogonal" to $x-y$, y and x.

Then, we define $R(x,y)$ as the norm of $\frac{x}{2}+\gamma w$ and by a straightforward computation we obtain that $R(x,y)$ has the following expression

$$R(x,y)=\frac{\left\|\|y\|^2\left(\|x\|^2-\rho'_-(x,y)\right)x+\|x\|^2\left(\|y\|^2-\rho'_-(y,x)\right)y\right\|}{2\|x\|^2\|y\|^2-\rho'_-(x,y)^2-\rho'_-(y,x)^2}.$$

This definition is only possible if

$$\rho'_-(x,y)^2+\rho'_-(y,x)^2<2\|x\|^2\|y\|^2$$

i.e., $\left|\rho'_-(x,y)\right|<\|x\|\,\|y\|$ or $\left|\rho'_-(y,x)\right|<\|x\|\,\|y\|$. For this reason we will assume that E is strictly convex (in these spaces $|\rho'_-(x,y)|\neq\|x\|\,\|y\|$ for all x and y in E linearly independent (see [7])).

Theorem 2.1. *Let $(E,\|\ \|)$ be a strictly convex real normed space with dim $E\geq 3$. E is an i.p.s., if and only if, for all x, y independent vectors in $E\backslash\{0\}$*

$$R(x,y)=\frac{\|x\|\,\|y\|\,\|x-y\|}{4\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}}$$

where $s=\left(\|x\|+\|y\|+\|x-y\|\right)/2$.

Proof. If we let $y = tz$ with $t > 0$ and we take limit when t tends to zero we obtain

$$\frac{\|x\|^2 \left\|\|z\|^2 x - \rho'_-(z,x)z\right\|}{2\|x\|^2\|z\|^2 - \rho'_-(x,z)^2 - \rho'_-(z,x)^2} = \lim_{t\to 0^+} R(x,tz) =$$

$$= \left(\lim_{t\to 0^+} \frac{\|x\|^2 t^2 \|z\|^2 \|x-tz\|^2}{[\|x-tz\|^2 - (t\|z\| - \|x\|)^2]\left[(\|x\| + t\|z\|)^2 - \|x-tz\|^2\right]}\right)^{1/2} =$$

$$= \|x\|\|z\| \left(\lim_{t\to 0^+} \frac{t\|x-tz\|}{\|x-tz\|^2 - (t\|z\| - \|x\|)^2}\right)^{1/2} \cdot$$

$$\cdot \left(\lim_{t\to 0^+} \frac{t\|x-tz\|}{(\|x\| + t\|z\|)^2 - \|x-tz\|^2}\right)^{1/2}$$

$$= \|x\|\|z\| \sqrt{\frac{\|x\|}{-2\rho'_-(x,z) + 2\|x\|\,\|z\|}} \sqrt{\frac{\|x\|}{2\|x\|\,\|z\| + 2\rho'_-(x,z)}} =$$

$$= \frac{\|z\|\,\|x\|^2}{2\sqrt{\|x\|^2\|z\|^2 - \rho'_-(x,z)^2}}$$

and therefore,

$$\left\| \frac{x}{\|x\|} - \rho'_-\left(\frac{z}{\|z\|}, \frac{x}{\|x\|}\right) \frac{z}{\|z\|} \right\| = \frac{2 - \rho'_-\left(\frac{x}{\|x\|}, \frac{z}{\|z\|}\right)^2 - \rho'_-\left(\frac{z}{\|z\|}, \frac{x}{\|x\|}\right)^2}{2\sqrt{1 - \rho'_-\left(\frac{x}{\|x\|}, \frac{z}{\|z\|}\right)^2}}.$$

If $u = \dfrac{x}{\|x\|}$, $v = \dfrac{z}{\|z\|}$, then for all unitary and independent vectors u and v in E

$$\left\| u - \rho'_-(v,u)v \right\| = \frac{2 - \rho'_-(u,v)^2 - \rho'_-(v,u)^2}{2\sqrt{1 - \rho'_-(u,v)^2}}$$

Now, if $\rho'_-(v,u) = 0$, then,

$$2\sqrt{1 - \rho'_-(u,v)^2} = 2 - \rho'_-(u,v)^2$$

and therefore $\rho'_-(u,v) = 0$ and (see [5], [6]) E is an i.p.s. □

Note. If we define the radius $R(x,y)$ as the norm of $\frac{y}{2} + \mu v$ or as the norm of $\frac{x+y}{2} + \lambda u$ the same expression obtained in the initial definition of $R(x,y)$ appears.

Analogous definitions of $R(x,y)$ can be given by replacing the role of ρ'_- by ρ'_+ or by changing the order of the arguments appearing in ρ'_-. For example if we consider that the radius of the circumscribed circumference is given by

$$\hat{R}(x,y) = \frac{\left\| \|y\|^2 \left(\|x\|^2 - \rho'_+(x,y)\right) x + \|x\|^2 \left(\|y\|^2 - \rho'_+(y,x)\right) y \right\|}{2\left(\|x\|^2\|y\|^2 - \rho'_-(x,y)^2\right)}$$

which is equal to $R(x,y)$ in an i.p.s. then, a strictly convex real normed space E with dim $E \geq 2$ is an i.p.s. if and only if

$$\hat{R}(x,y) = \frac{\|x\|\,\|y\|\,\|x-y\|}{4\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}}.$$

The proof is immediate using the fact that the symmetry of the second member of last expression and the symmetry of the numerator of $\hat{R}(x,y)$ imply $\rho'_-(x,y)^2 = \rho'_-(y,x)^2$ for all x, y in E and E is an i.p.s.

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