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ON PACKING OF SQUARES INTO A RECTANGLE

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Abstract. It is proved in this paper that any system of squares with total area 1 may be packed into a rectangle whose area is less then 1.53.

The following problem is formulated in [7]: Determine the smallest number S such that any system of squares with total area 1 may be (parallelly) packed into a rectangle of area S.

This problem was posed by L. Moser [4]. $S \geq \frac{\sqrt{}}{} \doteq 1.207$ follows from considering two squares of sides x and y, where $x > y$, $x + y = 1$ and $x(x + y)$ is maximal. Novotný [8] proved that any system of three squares with total area 1 may be packed into a rectangle of area 1.227759 (this area is necessary for packing of three squares with sides 0.7297177, 0.5588698 and 0.3939246). The four squares with sides $x = \sqrt{-}$, $x = x = x = \sqrt{-}$ show that $S \geq \frac{\sqrt{}}{} > 1.244$.

Moon and Moser [3] found first results for the upper bound. They proved that

(1) it is possible to pack any system of squares with sides $x \geq x \geq x \geq \cdots$ and with total area 1 into a square of side $a = x + \sqrt{1 - x}$.

A consequence of this is that any system of squares with total area 1 may be packed into a square of area 2.

Meir and Moser [2] extended the result (1) and they proved that

(2) any system of squares with total area V can be packed into a rectangle of size $a \times a$ if $a > x$, $a > x$ and $x + (a - x)(a - x) \geq V$.

Some further results for the upper bound were published by Kleitman and Krieger [1]: Any system of squares with total area 1 can be packed into a rectangle of size $\sqrt{2} \times \sqrt{-}$; its area is $\sqrt{-} \doteq 1.633$. It follows from this result that

(3) any system of squares with total area V can be packed into a rectangle with sides $\sqrt{2V}$ and $\sqrt{\frac{V}{}}$.

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The following theorem improves the upper estimate for S.

Theorem. *Any system of squares with total area* 1 *may be packed into a rectangle whose area is less than* 1.53.

Proof. We denote the squares $Q\ , Q\ , Q\ , \dots$ and their sides $x\ \ge x\ \ge x\ \ge \cdots$. We shall pack the squares in the dependence upon $x\ , x$ as it follows:

I. Let $x\ \ge \sqrt{-}$. By (3) we can pack the squares $Q\ , Q\ , \dots$ into a rectangle P with sides $\sqrt{2(1-x\ )}, \sqrt{\frac{-x_1^2}{}}$ and the whole system can be packed into a rectangle R with sides x and $x\ + \sqrt{2(1-x\ )}$ (Fig.1). The area of R is less than 1.53.

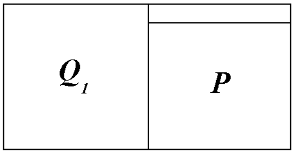


Fig. 1

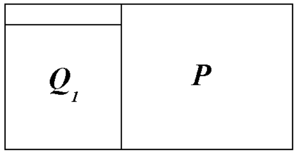


Fig. 2

II. Let $0.645 \le x\ \le \sqrt{-}$. We pack the squares $Q\ , Q\ , \dots$ as in **I** and all squares can be packed into a rectangle R with sides $\sqrt{\frac{-x_1^2}{}}$ and $x\ + \sqrt{2(1-x\ )}$ (Fig.2). The area of R is less than 1.53 for every $x\ \in \langle 0.645, \sqrt{4/7}\rangle$.

III. Let $x\ \le 0.27$. By (1) the squares can be packed into a square R of side $x\ + \sqrt{1-x\ }$; its area $1 + 2x\ \sqrt{1-x\ } < 1.53$ for every $x\ \le 0.27$.

It remains to investigate the domain

$$M = \{[x\ , x\ ];\ 0.27 \le x\ \le 0.645,\ 0 < x\ \le x\ \}.$$

IV. By (3) we can pack the squares $Q\ , Q\ , \dots$ into a rectangle P with sides $\sqrt{2(1-x\ -x\ )}$ and $\sqrt{\frac{-x_1^2-x_2^2}{}}$. All squares can be packed into a rectangle R by Fig. 3 if $x\ + x\ \ge \sqrt{2(1-x\ -x\ )}$, i.e. $3x\ + 2x\ x\ + 3x\ \ge 2$, or by Fig. 4 if $x\ + x\ \le \sqrt{2(1-x\ -x\ )}$.

The area of R from Fig. 3 is

$$f\ (x\ , x\ ) = (x\ + x\ )\left(x\ + \frac{2}{\sqrt{3}}\sqrt{1-x\ -x\ }\right).$$

We have

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{3(1-x\ -x\ )}}\left(2 - 4x\ - 2x\ - 2x\ x\ + (2x\ + x\ )\sqrt{3(1-x\ -x\ )}\right).$$

If we denote $u(x\ , x\ ) = 2 - 4x\ - 2x\ - 2x\ x\ + (2x\ + x\ )\sqrt{3(1-x\ -x\ )}$, then evidently $\frac{\partial u}{\partial x_1} < 0, \frac{\partial u}{\partial x_2} < 0$ in M (Fig. 11). Hence

$u(x\ ,x\ ) \geq u(0.645, 0.645) > 0, \frac{\partial f_1}{\partial x_1} > 0$, thus $f\ (x\ ,x\ ) \leq f\ (0.645, x\ )$ for $[x\ ,x\ ] \in M\ $. We verify easily that $f\ (0.645, x\ ) < 1.53$ for every $x\ \leq 0.645$.

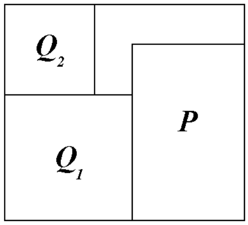


Fig. 3

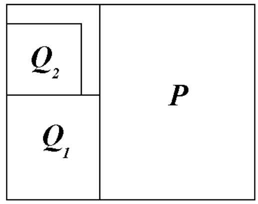


Fig. 4

The area of R from Fig. 4 is

$$f\ (x\ ,x\ ) = \left(x\ + \frac{2}{\sqrt{3}}\sqrt{1 - x\ - x\ }\right)\sqrt{2(1 - x\ - x\ )};$$

$$\frac{\partial f}{\partial x} = \frac{\sqrt{2}}{\sqrt{1 - x\ - x\ }}\left(1 - 2x\ - x\ - \frac{4x}{\sqrt{3}}\sqrt{1 - x\ - x\ }\right) < 0$$

for $[x\ ,x\ ] \in M\ $ (Fig.11). Evidently $\frac{\partial f_2}{\partial x_2} < 0$, too, and since $f\ (Z_i) < 1.53$ for $i \in \{1, 2, 3, 4, 5\}$, we have $f\ (x\ ,x\ ) < 1.53$ for every $[x\ ,x\ ] \in M\ $.

V. We pack the squares $Q\ ,Q\ ,\ldots$ as in **IV**. All squares can be packed into a rectangle R by Fig. 5 if

$$x\ + x\ \geq \sqrt{\frac{4(1 - x\ - x\ )}{3}}, \text{ i.e. } 7x\ + 6x\ x\ + 7x\ \geq 4,$$

or by Fig. 6 if

$$x\ + x\ \leq \sqrt{\frac{4(1 - x\ - x\ )}{3}}.$$

The area of R from Fig. 5 is

$$f\ (x\ ,x\ ) = (x\ + x\ )\left(x\ + \sqrt{2(1 - x\ - x\ )}\right).$$

Since $\frac{\partial f_3}{\partial x_1} > 0, \frac{\partial f_3}{\partial x_2} > 0$ in $M\ $ (Fig. 11) and $f\ (Z_i) < 1.53$ for $i \in \{6, 7, 8, 9\}$, $f\ (x\ ,x\ ) < 1.53$ is fulfilled for every $[x\ ,x\ ] \in M\ $.

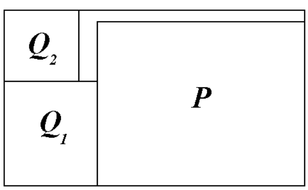


Fig. 5

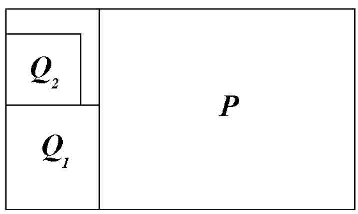


Fig. 6

The area of R from Fig. 6 is

$$f\ (x\ ,x\ ) = \left(x\ + \sqrt{2(1-x\ -x\ )}\right) \frac{2}{\sqrt{3}} \sqrt{1-x\ -x\ }.$$

Since $\frac{\partial f_4}{\partial x_1} < 0$, $\frac{\partial f_4}{\partial x_2} < 0$ in M (Fig. 11) and $f\ (Z_i) < 1.53$ for $i \in \{10, 11, \ldots, 14\}$, we have $f\ (x\ ,x\ ) < 1.53$ for every $[x\ ,x\ ] \in M$.

VI. Let $2x\ \le x$. By (2) we can pack the squares $Q\ ,Q\ ,\ldots$ into a rectangle P with sides a and $x\ +x$ if $x\ +(a-x\ )(x\ +x\ -x\ ) = 1-x\ -x\ -x$ $(\ge 1-x\ -x\ -x\ )$. It is valid for

$$a = \frac{1-x\ -x\ -3x\ +x\ x\ +x\ x}{x\ +x\ -x}.$$

All squares can be packed into a rectangle R by Fig. 7. Its area is

$$f\ (x\ ,x\ ,x\ ) = (x\ +x\ )(x\ +a) = \frac{(x\ +x\ )(1-x\ -3x\ +x\ x\ +x\ x\ )}{x\ +x\ -x}.$$

We have

$$\frac{\partial f}{\partial x} = \frac{(x\ +x\ )(1+2x\ x\ +3x\ -6x\ x\ -6x\ x\ )}{(x\ +x\ -x\ )}.$$

If we denote $v = 1+2x\ x\ +3x\ -6x\ x\ -6x\ x$, then $\frac{\partial v}{\partial x_4} < 0$, thus $v(x\ ,x\ ,x\ ) \ge v(x\ ,x\ ,x\ ) = 1-4x\ x\ -3x\ > 0$ for $[x\ ,x\ ] \in M$ (Fig. 11). Hence $\frac{\partial f_5}{\partial x_4} > 0$ and

$$f\ (x\ ,x\ ,x\ ) \le f\ (x\ ,x\ ,x\ ) = \frac{(x\ +x\ )(1-3x\ +x\ x\ )}{x} = g(x\ ,x\ ).$$

Since

$$\frac{\partial g}{\partial x} = \frac{x\ (x\ +3x\ -1)}{x} < 0 \text{ in } M\ ,$$

$$g(x\ ,x\ ) \le g(2x\ ,x\ ) = \frac{3(1-x\ )}{2} < 1.5,$$

we have $f\ (x\ ,x\ ,x\ ) \le g(x\ ,x\ ) < 1.5$ for every $[x\ ,x\ ] \in M\ ,\ x\ \le x$.

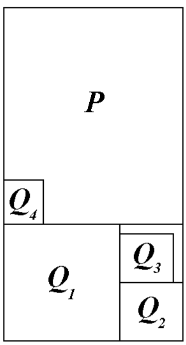


Fig. 7

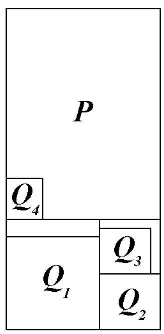


Fig. 8

VII. Let $2x \geq x$. We pack Q , Q , ... as in **VI**. All squares can be packed into a rectangle R by Fig. 8. The area of R is

$$f\ (x\ ,x\ ,x\ ) = (x\ +x\ )(2x\ +a) =$$

$$= \frac{(x\ +x\ )(1 - x\ +x\ -3x\ +x\ x\ +2x\ x\ -x\ x\ )}{x\ +x\ -x};$$

$$\frac{\partial f}{\partial x} = \frac{(x\ +x\ )(1+2x\ x\ +3x\ -6x\ x\ -6x\ x\ )}{(x\ +x\ -x\ )} \geq$$

$$\geq \frac{(x\ +x\ )(1+2x\ x\ +3x\ -6x\ x\ -6x\ )}{(x\ +x\ -x\ )} > 0$$

for $[x\ ,x\ ] \in M$ (Fig. 11), $x \leq x$. Hence $f\ (x\ ,x\ ,x\ ) \leq f\ (x\ ,x\ ,x\ )$. Denoting

$$h(x\ ,x\ ) = f\ (x\ ,x\ ,x\ ) = \frac{(x\ +x\ )(1-x\ -3x\ +3x\ x\ )}{x},$$

we have

$$\frac{\partial h}{\partial x} = 2x\ -2x\ + \frac{x\ (3x\ -1)}{x} < 0,\ \frac{\partial h}{\partial x} = 2x\ + \frac{1-9x}{x} > 0$$

in M . It follows from this that h is maximal in M at some from the points Z , Z . But $h(Z\ ) < 1.53, h(Z\ ) < 1.53$.

VIII. By (2) we can pack Q , Q , ... into a rectangle P with sides $x\ +x$ and a if $x\ +(a-x\ )(x\ +2x\ -x\ ) = 1-x\ -x\ -x$, i.e.

$$a = \frac{1-x\ -x\ -3x\ +x\ x\ +2x\ x}{x\ +2x\ -x}.$$

All squares can be packed into a rectangle R by Fig. 9. Its area is

$$f\ (x\ ,x\ ,x\ ) = \frac{(x\ +2x\ )(1-x\ -3x\ +2x\ x\ +2x\ x\ )}{x\ +2x\ -x}.$$

Since

$$\frac{\partial f}{\partial x} = \frac{(x\ +2x\ )(1+4x\ x\ +3x\ +3x\ -6x\ x\ -12x\ x\ )}{(x\ +2x\ -x\ )} \geq$$

$$\geq \frac{(x\ +2x\ )(1-2x\ x\ -6x\ )}{(x\ +2x\ -x\ )} > 0$$

for $[x\ ,x\ ] \in M$ (Fig. 11), $x \leq x$, we have $f\ (x\ ,x\ ,x\ ) \leq f\ (x\ ,x\ ,x\ )$. If we denote

$$k(x\ ,x\ ) = f\ (x\ ,x\ ,x\ ) = \frac{(x\ +2x\ )(1-2x\ +2x\ x\ )}{x\ +x},$$

then the system

$$\frac{\partial k}{\partial x} = \frac{x\ (6x\ +4x\ x\ +2x\ -1)}{(x\ +x\ )} = 0,$$

$$\frac{\partial k}{\partial x} = \frac{x \ + 2x \ + 4x \ x \ - 10x \ x \ - 8x}{(x \ + x \ )} = 0$$

has no solution in the interior of M . Therefore the function k has a maximum on the boundary of M . An easy calculation shows that this maximum is at the point Z (Fig. 11) and $k(Z \ ) < 1.53$.

Further

$$\frac{\partial f}{\partial x} = \frac{2x \ x \ + 8x \ x \ - 4x \ x \ x \ + 8x \ - 3x \ x \ - 2x \ x \ - x \ + 3x}{(x \ + 2x \ - x \ )} \geq$$

$$\geq \frac{9x \ - x}{(x \ + 2x \ - x \ )} > 0$$

for $[x \ , x \ ] \in M$ (Fig. 11), $x \ \leq x$. It means that $f \ (x \ , x \ , x \ ) \leq f \ (0.42, x \ , x \ )$. If we denote $\varphi(x \ , x \ ) = f \ (0.42, x \ , x \ )$, then

(4) $$\frac{\partial \varphi}{\partial x} = \frac{(0.42 + 2x \ )(1 + 1.68x \ + 3x \ + 3x \ - 2.52x \ - 12x \ x \ )}{(0.42 + 2x \ - x \ )}.$$

For $w(x \ , x \ ) = 1 + 1.68x \ + 3x \ + 3x \ - 2.52x \ - 12x \ x$ and for $x \ \geq 0.35$ we have $w(x \ , x \ ) \leq w(x \ , 0.35) = 3x \ - 2.52x \ + 0.4855 < 0$ for all $x \ \in \langle 0.34, 0.39\rangle$. Similarly, if $x \ \leq 0.34$, then $w(x \ , x \ ) \geq w(x \ , 0.34) = 3x \ - 2.4x \ + 0.49 > 0$ for $x \ \in \langle 0.34, 0.39\rangle$. In consequence of this the function φ has a maximum for $x \ \in \langle 0.34, 0.35\rangle$. We shall estimate $\max_T \varphi(x \ , x \ )$ for $T = \langle 0.34, 0.39\rangle \times$ $\times\langle 0.34, 0, 35\rangle$. It follows from $\frac{\partial w}{\partial x_2} < 0$, $\frac{\partial w}{\partial x_4} < 0$ that $-0.041 = w(0.39, 0.35) \leq$ $\leq w(x \ , x \ ) \leq w(0.34, 0.34) = 0.0208$. Since

$$\frac{0.42 + 2x}{(0.42 + 2x \ - x \ )} \leq \frac{0.42 + 0.78}{(0.42 + 0.33)} < 2.2,$$

we have in regard of (4) $|\frac{\partial \varphi}{\partial x_4}| < 0.1$ in T. Further

$$\frac{\partial \varphi}{\partial x} = (0.148176 + 1.0584x \ + 0.84x \ x \ - 0.84x \ - 8x \ +$$

$$+ 14x \ x \ - 8x \ x \ + 6x \ - 2x \ )/(0.42 + 2x \ - x \ ) \ .$$

Since the function

$t(x \ , x \ ) = 0.148176 + 1.0584x \ + 0.84x \ x \ - 0.84x \ - 8x \ + 14x \ x \ - 8x \ x \ + 6x \ - 2x$

satisfies $\frac{\partial t}{\partial x_2} > 0$, $\frac{\partial t}{\partial x_4} < 0$, we have $-0.01885 = t(0.34, 0.35) \leq t(x \ , x \ ) \leq$ $\leq t(0.39, 0.34) = 0.019828$ and because of $1/(0.42 + 2x \ - x \ ) \ < 1.8$ we get $|\frac{\partial \varphi}{\partial x_2}| < 0.04$ in T.

If $U \subset T$ is a square with side of length 0.01, then for $Y \ , Y \ \in U$ the inequality

(5) $$|\varphi(Y \ ) - \varphi(Y \ )| < 0.0014$$

is satisfied. Since the function φ gets values less than 1.527 at the points $[x \ , 0.34]$ for $x \ \in \{0.34, 0.35, 0.36, 0.37, 0.38\}$, (5) yields $\varphi(x \ , x \ ) < 1.53$ in T.

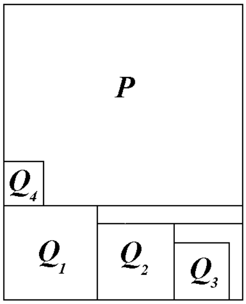


Fig. 9

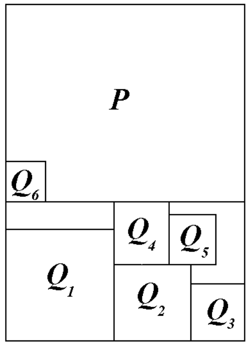


Fig. 10

IX. Let $[x\ , x\ ] \in M\ = \langle 0.42, 0.50\rangle \times \langle 0.29, 0.37\rangle$, $x\ + x\ \geq x\ $. By (2) we can pack $Q\ , Q\ , \ldots$ into a rectangle P with sides a and $x\ + x\ + x\ $ if

$$x\ + (a - x\ )(x\ + x\ + x\ - x\ ) = 1 - x\ - x\ - x\ - x\ - x\ ,$$

i.e

$$a = \frac{1 - x\ - x\ - x\ - x\ - 3x\ + x\ (x\ + x\ + x\ )}{x\ + x\ + x\ - x\ }.$$

All squares can be packed into a rectangle R (Fig. 10) with sides $x\ + x\ + x\ $, $x\ + x\ + a$. Its area is $f\ (x\ , x\ , x\ , x\ , x\ ) = (x\ + x\ + x\ )[(x\ + x\ )(x\ + x\ + + x\ - x\ ) + 1 - x\ - x\ - x\ - x\ - 3x\ + x\ (x\ + x\ + x\ )]/(x\ + x\ + x\ - x\ )$. Evidently $\frac{\partial f_8}{\partial x_4} > 0$, hence

$$f\ (x\ , x\ , x\ , x\ , x\ ) \leq f\ (x\ , x\ , x\ , x\ , x\ ) = m(x\ , x\ , x\ , x\ ) =$$

$$= \frac{(x\ + x\ + x\ )(x\ x\ + 2x\ x\ + x\ x\ + 1 - x\ - x\ - 3x\ + x\ x\ )}{x\ + x\ + x\ - x\ }.$$

Because of $\frac{\partial m}{\partial x_1} = [x\ (2x\ + x\ + 2x\ + x\ x\ + x\ x\ - 1 + 3x\ - 2x\ x\ - x\ x\ - - x\ x\ ) + (x\ + x\ - 2x\ )(x\ + x\ + x\ )(x\ + x\ + x\ - x\ )]/(x\ + x\ + x\ - x\ ) \leq$

$$\leq \frac{x\ (2 \cdot 0.5\ + 3 \cdot 0.37\ + 0.37 - 1 + 3x\ - 1.74x\ ) - 0.1 \cdot 0.71}{(x\ + x\ + x\ - x\ )} =$$

$$= \frac{3x\ - 1.74x\ + 0.2807x\ - 0.05041}{(x\ + x\ + x\ - x\ )} < 0$$

for $x\ \leq 0.37$, m has a maximum in M for $x\ = 0.42$. Similarly, $\frac{\partial m}{\partial x_2} = [(x\ + x\ + x\ )(x\ + 2x\ )(x\ + x\ + x\ - x\ ) - x\ (x\ x\ + 2x\ x\ + x\ x\ + 1 - x\ - - x\ - 3x\ + x\ x\ )]/(x\ + x\ + x\ - x\ ) \geq [(0.71 + x\ )(0.42 + 2x\ ) \cdot 0.71 - - x\ (0.50 \cdot 0.37 + 0.74x\ + 0.50x\ + 1 - 0.42\ - x\ + 0.50x\ )]/(x\ + x\ + x\ - x\ ) =$

$$= \frac{0.211722 + 0.2978x\ - 0.32x\ + x\ }{(x\ + x\ + x\ - x\ )} > 0,$$

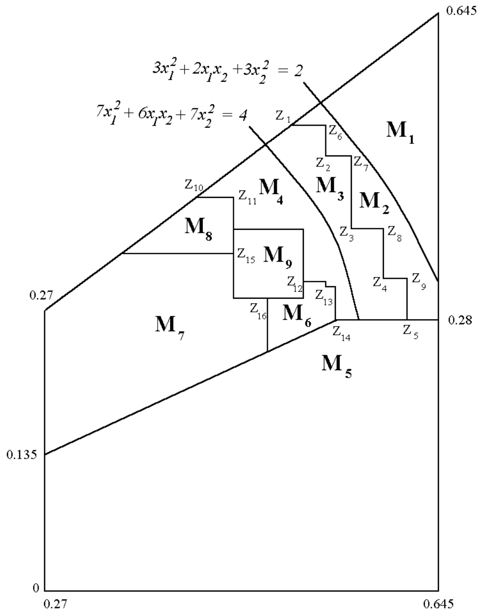
and hence m has a maximum for $x \quad = 0.37$.

Further, on the assumptions $x \quad = 0.42, x \quad = 0.37$, using $x \quad \geq x \quad$,

$$\frac{\partial m}{\partial x} \geq \frac{0.723956 - 0.3318x \quad - 1.58x \quad - 0.979x \quad - 1.16x \quad x \quad + x \quad x \quad + 3x \quad - 0.42x}{(x \quad + x \quad + x \quad - x \quad)}.$$

Since the function $s(x \quad , x \quad) = 0.723956 - 0.3318x \quad - 1.58x \quad - 0.979x - -1.16x \quad x \quad + x \quad x \quad + 3x \quad - 0.42x$ satisfies $\frac{\partial s}{\partial x_3} < 0, \frac{\partial s}{\partial x_6} < 0$, we have $s(x \quad , x \quad) \geq \geq s(0.37, 0.37) > 0$, i.e. $\frac{\partial m}{\partial x_3} > 0$ and hence m has a maximum for $x \quad = 0.37$. We find easily that m is maximal if $x \quad = \frac{\quad - \sqrt{\quad}}{\quad}$ and that the maximal value of f in M is

$$f\left(0.42, 0.37, 0.37, 0.37, \frac{3.48 - \sqrt{6.8349}}{3}\right) < 1.53.$$



$Z \quad = [0.47, 0.47]$	$Z \quad = [0.51, 0.42]$
$Z \quad = [0.55, 0.35]$	$Z \quad = [0.59, 0.31]$
$Z \quad = [0.62, 0.28]$	$Z \quad = [0.51, 0.47]$
$Z \quad = [0.55, 0.42]$	$Z \quad = [0.59, 0.35]$
$Z \quad = [0.62, 0.31]$	$Z \quad = [0.39, 0.39]$
$Z \quad = [0.42, 0.37]$	$Z \quad = [0.50, 0.32]$
$Z \quad = [0.54, 0.31]$	$Z \quad = [0.56, 0.28]$
$Z \quad = [0.42, 0.34]$	$Z \quad = [0.46, 0.29]$

Fig. 11

X. Let $[x \quad , x \quad] \in M \quad , x \quad + x \quad \leq x \quad$. As in **VIII**, the squares can be packed into a rectangle R with area

$$f \quad (x \quad , x \quad , x \quad) = \frac{(x \quad + 2x \quad)(1 - x \quad - 3x \quad + 2x \quad x \quad + 2x \quad x \quad)}{x \quad + 2x \quad - x}.$$

Since

$$\frac{\partial f}{\partial x} \geq \frac{2x \quad x \quad + 4x \quad x \quad + 3x \quad - x}{(x \quad + 2x \quad - x \quad)} > 0,$$

$$\frac{\partial f}{\partial x} \geq (x \ + 2x \ )[1 + 4x \ x \ + 3x \ + 3(x \ - x \ ) \ - 6x \ (x \ - x \ ) -$$

$$-12x \ (x \ - x \ )]/(x \ + 2x \ - x \ ) \ = \frac{(x \ + 2x \ )(1 - 3x \ + 18x \ - 8x \ x \ )}{(x \ + 2x \ - x \ )} > 0$$

for $[x \ , x \ ] \in M \ $, f is maximal for $x \ = 0.5, x \ = 0.5 - x \ $. It is easy to show that $f \ (0.5, x \ , 0.5 - x \ ) \leq f \ (0.5, 0.29, 0.21) < 1.5$ for $x \ \in \langle 0.29, 0.37 \rangle$.

Since the domains $M \ , \ldots, M$ cover M, the proof is completed.

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