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POISSON COHOMOLOGY AND CANONICAL HOMOLOGY OF POISSON MANIFOLDS

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ABSTRACT. In this paper we present recent results concerning the Lichnerowicz-Poisson cohomology and the canonical homology of Poisson manifolds.

1. INTRODUCTION

Poisson manifolds are a subject of active research in Mathematics and Physics [3, 18, 27]. The notion of Poisson manifold was introduced by Lichnerowicz [22] as a manifold endowed with a Poisson bracket. We remit the reader to a paper by Weinstein [28] for some interesting historical remarks.

Lichnerowiz proved that a Poisson bracket on M is equivalent to give a skew-symmetric contravariant tensor field of second order G on M satisfying that the Schouten-Nijenhuis bracket of G with itself identically vanishes, that is, $[G, G] = 0$. G is called the Poisson tensor. A differential operator σ on the contravariant Grassmann algebra of M may be defined by $\sigma(P) = -[P, G]$. The integrability condition for G (i.e., $[G, G] = 0$) implies that $\sigma^2 = 0$ and the corresponding cohomology is the so-called Lichnerowicz-Poisson cohomology of the Poisson manifold M.

On the other hand, Koszul [19] has introduced an operator δ on the Grassman algebra of differential forms defined by $\delta = [i(G), d]$, where $i(G)$ is the contraction

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by the Poisson tensor G and d is the exterior derivative. Since $\delta^2 = 0$, it defines a homology on M which is called the canonical homology of the Poisson manifold. Taking into account that $d\delta + \delta d = 0$, Brylinski [7] has introduced a canonical double complex and studied the degeneracy of the first spectral sequence associated with it.

The aim of this paper is to present an account of recent results on the Poisson-Lichnerowicz cohomology and the canonical homology of any Poisson manifold.

The first part of the paper is devoted to study the Lichnerowicz-Poisson cohomology of a Poisson manifold. It is very hard to compute it although there exists a Mayer-Vietoris sequence. Nevertheless Poisson cohomology is not directly related with the topology of the manifold as in the case of the de Rham cohomology. For symplectic manifolds, the Lichnerowicz-Poisson cohomology is isomorphic to the de Rham cohomology, but it is not the case for non-symplectic Poisson manifolds. An example is presented in Section 6. In this paper we also introduce the so-called coeffective Lichnerowicz-Poisson cohomology. Since $\sigma(G) = 0$, we have $\sigma(P \wedge G) = \sigma(P) \wedge G$ and we obtain a subcomplex of the complex of contravariant tensor fields. Its cohomology is called coeffective. Again, for symplectic manifolds it is isomorphic to the coeffective cohomology introduced by Bouché [6]. Moreover, the cohomology class of G induces a truncated Lichnerowicz-Poisson cohomology. Both cohomologies are not isomorphic. In fact, the authors in [1, 9, 12, 13] have shown examples of compact symplectic manifolds and compact almost cosymplectic manifolds for which the coeffective cohomology is not isomorphic to the truncated Lichnerowicz-Poisson cohomology. In these cases we also have proved a Nomizu's theorem for the coeffective cohomology; an open question is the possibility to extend it to arbitrary Poisson manifolds of constant rank.

The second part of this paper concerns with the canonical homology of a Poisson manifold. We establish its relationship with the Lichnerowicz-Poisson cohomology. In [2] Bhaskara and Viswanath have defined a natural pairing between the Lichnerowicz-Poisson cohomology and the canonical homology of a Poisson manifold M. We prove in Section 10 that for symplectic manifolds this pairing is nothing but the Poincaré duality.

In [7] Brylinski introduced the notion of harmonic forms on a Poisson manifold as follows: α is harmonic if $d\alpha = 0$ and $\delta\alpha = 0$. He proposed the following *Problem A: to give conditions to ensure that any de Rham cohomology class has a representative which is harmonic.* He proved that, for some symplectic manifolds, any de Rham cohomology class has a representative which is symplectically harmonic and conjectured that this is true for any symplectic manifold. In [11] we have constructed a counterexample. Independently, Mathieu [23] has proved that a symplectic manifold satisfies the Brylinski conjecture if and only if it verifies the Lefschetz theorem. As a corollary, if a symplectic manifold satisfies the Brylinski conjecture then its odd Betti numbers are even. So, many counterexamples can now be given.

Brylinski also studied in [7] the first spectral sequence associated with the canonical double complex and proved that, if the manifold is symplectic, it degenerates at the first term. He proposed the following *Problem B: to find conditions on a*

Poisson manifold to ensure that the first spectral sequence degenerates at the first term. We have studied the second spectral sequence and proved that it degenerates for arbitrary Poisson manifolds [14]. As a consequence, we find the Brylinski's result. However, the first spectral sequence does not degenerate for non-symplectic Poisson manifolds. To prove this fact, we study the canonical homology of an almost cosymplectic manifold and prove the finiteness of their homology groups. We also prove a Nomizu's theorem for the case of almost cosymplectic nilmanifolds, that is, the canonical homology may be computed at the Lie algebra level. By using these results we find a compact almost cosymplectic 5-dimensional nilmanifold for which the first spectral sequence does not degenerate at the first term.

Finally, in Section 17, we show that the previous Problems A and B have independent answers.

PART I: LICHNEROWICZ-POISSON COHOMOLOGY OF POISSON MANIFOLDS

2. Schouten-Nijenhuis bracket

Let M be C^∞ manifold of dimension m and denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields and by $\mathfrak{F}(M)$ the algebra of C^∞ functions on M.

There are two Grassmann algebras on M:

$$\begin{cases} \textbf{Covariant Grassmann algebra} \quad (\Lambda(M) = \bigoplus_{p=0}^{m} \Lambda^p(M), \wedge) \\ \\ \textbf{Contravariant Grassmann algebra} \quad (\mathcal{V}(M) = \bigoplus_{p=0}^{m} \mathcal{V}^p(M), \wedge) \end{cases}$$

$\Lambda(M)$ is endowed with the exterior derivative d and $\mathcal{V}(M)$ with the Schouten-Nijenhuis bracket $[\, ,\,]$. We recall that the Schouten-Nijenhuis bracket is defined as the unique $\mathbb{R}$-bilinear operator of local type

$$[\, ,\,] : \mathcal{V}^p(M) \times \mathcal{V}^q(M) \longrightarrow \mathcal{V}^{p+q-1}(M)$$

such that

$$[X_1 \wedge \cdots \wedge X_p, Q] = \sum_{i=1}^{q} (-1)^{i+1} X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge [X_i, Q] \, ,$$

for all $X_1, \ldots, X_p \in \mathfrak{X}(M)$ and for all $Q \in \mathcal{V}^q(M)$, where the hat denotes missing arguments. Here $[X, Q] = \mathcal{L}_X Q$ is the usual Lie derivative of a tensor field with respect to a vector field X.

The Schouten-Nijenhuis bracket satisfies the following properties:

$$[P, Q] = (-1)^{pq}[Q, P] \, ,$$
$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq+q} Q \wedge [P, R] \, ,$$
$$(-1)^{p(r-1)}[P, [Q, R]] + (-1)^{q(p-1)}[Q, [R, P]] + (-1)^{r(q-1)}[R, [P, Q]] = 0 \, .$$

3. Poisson manifolds

Definition 3.1. A Poisson bracket $\{\,,\,\}$ on a manifold M is a bilinear mapping

$$\{\,,\,\} : \mathfrak{F}(M) \times \mathfrak{F}(M) \longrightarrow \mathfrak{F}(M)$$

satisfying the following properties:

(1) (skew-symmetry) $\{f, g\} = -\{g, f\}$,

(2) (Leibniz rule) $\{f, gh\} = \{f, g\}h + g\{f, h\}$,

(3) (Jacobi's identity) $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$,

for $f, g, h \in \mathfrak{F}(M)$.

The pair $(M, \{\,,\,\})$ will be called a Poisson manifold. Next, if there is no danger of confusion, we shall write M instead of $(M, \{\,,\,\})$.

Notice that (1) is equivalent to that the Poisson bracket $\{f, f\}$ vanishes, for any $f \in \mathfrak{F}(M)$; it expresses a conservation law of energy. Also, the Leibniz rule implies that the mapping $f \rightsquigarrow X_f(g) = \{f, g\}$ defines a vector field X_f on M which will be called a Hamiltonian vector field with energy f. Finally, the Jacobi's identity can be equivalently written as $X_f\{g, h\} = \{X_f(g), h\} + \{g, X_f(h)\}$, which says that X_f is a derivation law of $\mathfrak{F}(M)$.

In [22], A. Lichnerowicz gave a more compact definition of a Poisson manifold. Define a skewsymmetric (2,0) type tensor field G by $G(df, dg) = \{f, g\}$. Then $[G, G] = 0$. Conversely, let G be an skewsymmetric (2,0) type tensor field on M and define a bracket of functions $\{f, g\} = G(df, dg)$. Then $\{\,,\,\}$ satisfies Jacobi's identity iff $[G, G] = 0$. G will be called the Poisson tensor. The rank of G will be called the rank of the Poisson manifold.

The local structure of a Poisson manifold was elucidated by A. Weinstein (see [28]). We have

Theorem 3.1. *Let M be a Poisson manifold of dimension m, with Poisson bracket $\{\,,\,\}$. Let x be a point of M where the rank of the Poisson structure is $2r$. Then, there exist local coordinates $\{q^1, \cdots, q^r, p_1, \cdots, p_r, z^1, \cdots, z^{m-2r}\}$ around x such that*

$$\{q^i, q^j\} = 0, \quad \{q^i, p_j\} = \delta_i^j, \quad \{q^i, z^a\} = 0,$$

$$\{p_i, p_j\} = 0, \quad \{p_i, z_a\} = 0,$$

for all $1 \leq i, j \leq r$, $1 \leq a \leq m - 2r$. Furthermore, the Poisson bracket $\{z^a, z^b\}$ is a function only of the local coordinates $z^1, \cdots, z^{m-2r}$ and vanishes at x. If the rank of the Poisson structure is constant and equal to $2r$, the z-coordinates satisfy

$$\{z^a, z^b\} = 0,$$

for all $1 \leq a, b \leq m - 2r$. (Coordinates $\{q^1, \cdots, q^r, p_1, \cdots, p_r, z^1, \cdots, z^{m-2r}\}$ are called Darboux coordinates.)

Suppose that G has constant rank $2r$. Then

$$\begin{cases} G &= \sum_{i=1}^{r} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \\ X_f &= \sum_{i=1}^{r} \left\{ \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} \right\}, \\ \{f, g\} &= \sum_{i=1}^{r} \left\{ \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right\}. \end{cases}$$

4. Calculus on Poisson manifolds

Let $(M, \{\ ,\ \})$ be a Poisson manifold with Poisson tensor G.
There exists a Poisson bracket of 1-forms

$$\{\ ,\ \} : \Lambda^1(M) \times \Lambda^1(M) \longrightarrow \Lambda^1(M)$$

such that $\{df, dg\} = d\{f, g\}$ (see K. H. Bhaskara, K. Viswanath [3]).

Define a mapping $\mathcal{I} : \Lambda^1(M) \longrightarrow \mathcal{V}^1(M)$ as follows:

$$\mathcal{I}(\alpha)(\beta) = G(\alpha, \beta), \forall \alpha, \beta \in \Lambda^1(M)\ .$$

Hence we can extend $\mathcal{I}$ to a mapping

$$\mathcal{I} : \Lambda^p(M) \longrightarrow \mathcal{V}^p(M)$$

by putting:

- $\mathcal{I}(f) = f$, for any $f \in \mathfrak{F}(M)$;
- $\mathcal{I}(\alpha)(\alpha_1, \ldots, \alpha_p) = (-1)^p \alpha(\mathcal{I}(\alpha_1), \ldots, \mathcal{I}(\alpha_p))$, for $\alpha \in \Lambda^p(M)$ and $\alpha_1, \ldots, \alpha_p \in \Lambda^1(M)$.

Proposition 4.1. *We have:*

- $\mathcal{I}(\alpha \wedge \beta) = \mathcal{I}(\alpha) \wedge \mathcal{I}(\beta)$
- $\mathcal{I}(df) = X_f$
- $\mathcal{I} : \Lambda^1(M) \longrightarrow \mathcal{V}^1(M)$ *is a Lie algebra homomorphism, i.e.,*

$$\mathcal{I}(\{\alpha, \beta\}) = [\mathcal{I}(\alpha), \mathcal{I}(\beta)], \forall \alpha, \beta \in \Lambda^1(M)\ .$$

5. The Lichnerowicz-Poisson cohomology

We define the contravariant exterior derivative $\sigma : \mathcal{V}^p(M) \longrightarrow \mathcal{V}^{p+1}(M)$ by

$$\sigma(P) = -[G, P]\ .$$

Alternatively, we can express σ in a more explicit way:

$$
\begin{aligned}
(\sigma P)(\alpha_0, \alpha_1, \ldots, \alpha_p) &= \sum_{i=0}^{p} (-1)^i (\mathcal{I}\alpha_i)(P(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_p)) \\
&+ \sum_{i<j=0}^{p} (-1)^{i+j} P(\{\alpha_i, \alpha_j\}, \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_p) ,
\end{aligned}
$$

where $\alpha_i \in \Lambda^1(M)$, and the hat denotes missing arguments.

Proposition 5.1. *We have*

$$
\begin{cases}
\sigma(P \wedge Q) = \sigma(P) \wedge Q + (-1)^p P \wedge \sigma(Q) \quad (P \in \mathcal{V}^p(M)) , \\
\sigma^2 = 0 , \\
\sigma(G) = 0 , \\
\sigma \mathcal{I} = -\mathcal{I}\, d .
\end{cases}
$$

Thus, σ defines a cohomology on M which is called the Lichnerowicz-Poisson (LP for simplicity) cohomology of the Poisson manifold M. The p-th LP-cohomology group is then defined by

$$
H^p_{LP}(M) = \frac{\ker\{\sigma : \mathcal{V}^p(M) \longrightarrow \mathcal{V}^{p+1}(M)\}}{Im\,\{\sigma : \mathcal{V}^{p-1}(M) \longrightarrow \mathcal{V}^p(M)\}} .
$$

Notice that G defines a cohomology class $[G] \in H^2_{LP}(M)$.

Since $\sigma \mathcal{I} = -\mathcal{I}\, d$, we have induced homomorphisms in cohomology

$$
\mathcal{I} : H^p_{DR}(M) \longrightarrow H^p_{LP}(M)
$$

6. Symplectic manifolds

Suppose that $(M, \{\ ,\ \})$ has maximal rank, say $m = 2n$. In this case, $\mathcal{I} : \Lambda^2(M) \longrightarrow \mathcal{V}^2(M)$ is an isomorphism. If we put

$$
\omega = -\mathcal{I}^{-1}(G) ,
$$

then we deduce

$$
\begin{cases}
d\omega = 0 , \\
\text{rank } \omega = 2n , \\
-\omega(X_f, X_g) = \{f, g\} , \\
i_{X_f}\ \omega = df .
\end{cases}
$$

Thus, (M, ω) is a symplectic manifold.

Conversely, if (M, ω) is a symplectic manifold, we define a Poisson bracket by $\{f, g\} = -\omega(X_f, X_g)$, where $i_{X_f}\ \omega = df$ and $i_{X_g}\ \omega = dg$.

Theorem 6.1. *For a symplectic manifold (M, ω) we have*

$$
H^p_{DR}(M) \cong H^p_{LP}(M)
$$

Proof. In fact, $\mathcal{I}$ induces isomorphisms in cohomology. □

However, the result does not hold for non symplectic Poisson manifolds as the next example shows.

Example 6.1. (The manifold $M(k,n)$) Let $M(k,n)$ be the completely solvable four-dimensional manifold defined by the 1-forms $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ such that

$$\begin{cases} d\alpha_1 = -k\alpha_1 \wedge \alpha_3 \\ d\alpha_2 = k\alpha_2 \wedge \alpha_3 \\ d\alpha_3 = 0 \\ d\alpha_4 = n\alpha_1 \wedge \alpha_2 \end{cases}$$

where k is a real number such that $e^k + e^{-k}$ is an integer different from 2, and n a non-zero integer number. $M(k,n)$ is a compact quotient of a completely solvable Lie group $G(k,n)$ with Lie algebra $\mathfrak{g}(k,n)$.

Let $\{X_1, X_2, X_3, X_4\}$ be the dual basis of vector fields. We have

$$\begin{cases} [X_1, X_2] = -nX_4 \\ [X_1, X_3] = kX_1 \\ [X_2, X_3] = -kX_2 \end{cases}$$

all the other brackets being zero.

Consider the Poisson tensor G given by

$$G = X_3 \wedge X_4 \ .$$

G is obtained from a left invariant Poisson structure on $G(k,n)$. (We are using the same notation for the basis of left invariant 1-forms on $G(k,n)$ and for the basis of 1-forms induced on $M(k,n)$; as well for the basis of left invariant vector fields on $G(k,n)$ and the basis of vector fields induced on $M(k,n)$). We compute the Lichnerowicz-Poisson cohomology groups at the Lie algebra level. They are

$$\begin{aligned} H^0_{LP}(\mathfrak{g}(k,n)) &= 0 \ , \\ H^1_{LP}(\mathfrak{g}(k,n)) &= \{[X_3], [X_4]\} \ , \\ H^2_{LP}(\mathfrak{g}(k,n)) &= \{[X_1 \wedge X_2], [X_3 \wedge X_4]\} \ , \\ H^3_{LP}(\mathfrak{g}(k,n)) &= \{[X_1 \wedge X_2 \wedge X_3], [X_1 \wedge X_2 \wedge X_4]\} \ , \\ H^4_{LP}(\mathfrak{g}(k,n)) &= \{[X_1 \wedge X_2 \wedge X_3 \wedge X_4]\} \ . \end{aligned}$$

A direct computation shows that $[X_1 \wedge X_2]$ and $[X_3 \wedge X_4]$ are not zero in $H^2_{LP}(M(k,n))$. Since $H^2_{DR}(M(k,n)) = 0$ (see [10]) we conclude that

$$H^2_{LP}(M(k,n)) \ncong H^2_{DR}(M(k,n)) \ .$$

7. The coeffective Lichnerowicz-Poisson cohomology

Since $\sigma(G)=0$, $H^2_{LP}(M)$ has a distinguished element $[G]$. Hence we can define the truncated LP-cohomology groups as follows:

$$\widetilde{H}^p_{LP}(M)=\{[P]\in H^p_{LP}(M)|[P]\wedge[G]=0\}\ .$$

Moreover, since $\sigma(P\wedge G)=\sigma(P)\wedge G$ we obtain a differential subcomplex of the complex $(\mathcal{V}(M),\sigma)$:

$$\mathcal{A}^p_{LP}(M)=\{P\in\mathcal{V}^p(M)|P\wedge G=0\}$$

Its cohomology will be called the coeffective cohomology of M, and its p-th group is defined to be

$$H^p(\mathcal{A}_{LP}(M))=\frac{\ker\{\sigma:\mathcal{A}^p(M)\longrightarrow\mathcal{A}^{p+1}(M)\}}{Im\,\{\sigma:\mathcal{A}^{p-1}(M)\longrightarrow\mathcal{A}^p(M)\}}\ .$$

In a natural way, it arises the following question: *Are $H^p(\mathcal{A}_{LP}(M))$ and $\widetilde{H}^p_{LP}(M)$ the same groups, unless isomorphism?*

Next, we shall establish some partial results.

If G has constant rank $2r$, then

$$\mathcal{A}^p_{LP}(M)=0\ ,p\leq r-1$$

Therefore we get

$$H^p(\mathcal{A}_{LP}(M))=0\ ,\ p\leq r-1\ .$$

Now, let (M,ω) be a symplectic manifold. We define a coeffective cohomology with respect to the symplectic form ω by declaring that a p-form α is coeffective if $\alpha\wedge\omega=0$. Since ω is closed we obtain a differential subcomplex $(\mathcal{A}(M),d)$ of the de Rham complex, where

$$\mathcal{A}^p(M)=\{\alpha\in\Lambda^p(M)|\alpha\wedge\omega=0\}\ .$$

The corresponding coeffective cohomology groups are denoted by $H^p(\mathcal{A}(M))$, for every integer p and were introduced by T. Bouché [6].

On the other hand, we have the truncated de Rham cohomology $\widetilde{H}^*_{DR}(M)$ by the cohomology class $[\omega]$ of ω. Using the isomorphism $\mathcal{I}$ we deduce that

$$\widetilde{H}^p_{DR}(M)\cong\widetilde{H}^p_{LP}(M)\ ,\ H^p(\mathcal{A}(M))\cong H^p(\mathcal{A}_{LP}(M))\ .$$

The next theorem follows from Bouché [6]:

Theorem 7.1. *If M is a compact Kähler manifold of dimension $2n$, we have*

$$H^p(\mathcal{A}_{LP}(M))\cong\widetilde{H}^p_{LP}(M)\ ,$$

for $p\neq n$.

In fact, if M is a compact Köhler manifold we have

$$H^p(\mathcal{A}(M))\cong\widetilde{H}^p(M)\ ,$$

Bouché conjectured that his result holds for arbitrary symplectic manifolds. However, a counterexample was constructed in [1]. Actually, a method to compute the coeffective cohomology was presented in [12, 13].

First of all, we recall the Nomizu's theorem which permits us to compute the de Rham cohomology of compact nilmanifolds.

Theorem 7.2. *(Nomizu's theorem [24]) Let $\mathcal{G}$ be a connected nilpotent Lie group with discrete subgroup Γ such that the space of right cosets $M = \Gamma\backslash\mathcal{G}$ is compact. Then there is an isomorphism of cohomology groups*

$$H^*(\mathfrak{g}^*) \cong H^*_{DR}(M) ,$$

where $H^*(\mathfrak{g}^*)$ denotes the Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ and $H^*_{DR}(M)$ denotes the de Rham cohomology of M.

Hattori [17] has extended this result for completely solvable manifolds.

In [13], we have proved a Nomizu's theorem for the coeffective cohomology of symplectic manifolds.

Theorem 7.3. *Let $\mathcal{G}$ be a connected nilpotent Lie group endowed with an invariant symplectic form ω^* and with a discrete subgroup Γ such that the space of right cosets $M = \Gamma\backslash\mathcal{G}$ is compact. Then there is an isomorphism of cohomology groups*

$$H^p(\mathcal{A}(\mathfrak{g}^*)) \cong H^p(\mathcal{A}(M)) ,$$

for all $p \geq n+1$, $\dim\mathcal{G} = 2n$, where $H^p(\mathcal{A}(\mathfrak{g}^*))$ is the coeffective cohomology with respect to ω^* and $H^p(\mathcal{A}(M))$ is the coeffective cohomology defined by the projected symplectic form ω on M.

The result still holds for completely solvable manifolds.

Example 7.1. (The manifold R^6) Let R^6 be a 6-dimensional compact nilmanifold defined by the 1-forms $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ such that

$$\left\{ \begin{array}{lcl} d\alpha_i & = & 0 \ , \ 1 \leq i \leq 3 \ , \\ d\alpha_4 & = & -\alpha_1 \wedge \alpha_2 \ , \\ d\alpha_5 & = & -\alpha_1 \wedge \alpha_3 \ , \\ d\alpha_6 & = & -\alpha_1 \wedge \alpha_4 \ . \end{array} \right.$$

We write $\alpha_{ij} = \alpha_i \wedge \alpha_j$, $\alpha_{ijk} = \alpha_i \wedge \alpha_j \wedge \alpha_k$, and so forth.

Using Nomizu's theorem we obtain:

$$\begin{aligned}
H^0_{DR}(R^6) &= \{1\},\\
H^1_{DR}(R^6) &= \{[\alpha_1],[\alpha_2],[\alpha_3]\},\\
H^2_{DR}(R^6) &= \{[\alpha_{15}],[\alpha_{16}],[\alpha_{23}],[\alpha_{24}],[\alpha_{35}],[\alpha_{25}+\alpha_{34}]\},\\
H^3_{DR}(R^6) &= \{[\alpha_{135}],[\alpha_{145}],[\alpha_{146}],[\alpha_{156}],[\alpha_{234}],[\alpha_{235}],[\alpha_{246}],\\
&\quad [\alpha_{236}+\alpha_{245}]\},\\
H^4_{DR}(R^6) &= \{[\alpha_{1246}],[\alpha_{1256}],[\alpha_{1356}],[\alpha_{1456}],[\alpha_{2345}],[\alpha_{2346}]\},\\
H^5_{DR}(R^6) &= \{[\alpha_{12456}],[\alpha_{13456}],[\alpha_{23456}]\},\\
H^6_{DR}(R^6) &= \{[\alpha_{123456}]\}.
\end{aligned}$$

Since $b_1(R^6)=3$, we deduce that R^6 does not admit Kähler structures. However R^6 admits symplectic structures, for instance,

$$\omega = \alpha_{15}+\alpha_{16}+\alpha_{25}+\alpha_{34}+\alpha_{13}\ .$$

By a direct computation, we obtain

$$\widetilde{H}^4_{DR}(R^6) = \{[\alpha_{1246}],[\alpha_{1356}],[\alpha_{1456}],[\alpha_{1256}+\alpha_{2346}],[\alpha_{2345}+\alpha_{2346}]\}\ ,$$

and, from Nomizu's theorem for the coeffective cohomology (see [13]) we have

$$H^4(\mathcal{A}(R^6)) = \{\{\alpha_{1245}\},\{\alpha_{1246}\},\{\alpha_{1356}\},\{\alpha_{1456}\},\{\alpha_{1256}-\alpha_{2345}\},\{\alpha_{1256}+\alpha_{2346}\}\}\ .$$

Thus,

$$\widetilde{H}^4(R^6) \not\cong H^4(\mathcal{A}(R^6))$$

Remark 7.1. The coeffective cohomology for almost cosymplectic manifolds was introduced in [9]. Moreover, in [13], a Nomizu's type theorem for this cohomology was proved.

To end this section we propose the following open problems:

1.- To obtain a Nomizu's theorem for the Lichnerowicz-Poisson cohomology. This would mean the following:

Let $\mathcal{G}$ be a connected nilpotent Lie group endowed with an invariant Poisson structure G^* (of course, of constant rank $2r$) and with a discrete subgroup Γ such that the space of right cosets $M=\Gamma\backslash\mathcal{G}$ is compact. Then there is an isomorphism of cohomology groups

$$H^*_{LP}(\mathfrak{g}) \cong H^*_{LP}(M)\ ,$$

where $H^*_{LP}(\mathfrak{g})$ denotes the LP-cohomology of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ and $H^*_{LP}(M)$ denotes the de LP-cohomology of M.

2.- The same problem for the coeffective LP-cohomology.

PART II: CANONICAL HOMOLOGY OF POISSON MANIFOLDS

8. Canonical homology

Let M be a Poisson manifold with Poisson tensor G and Poisson bracket $\{\,,\,\}$. J.L. Koszul [19] introduced the differential operator

$$\delta : \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M) \ ,$$

defined by

$$\delta = [i(G), d] = i(G) \circ d - d \circ i(G) \ ,$$

where $i(G)$ denotes the contraction by G, and d is the exterior derivative.

Alternatively, J.L. Brylinski [7] gave the following explicit expression for δ:

$$\delta(f_0\ df_1 \wedge \cdots \wedge df_k) = \sum_{1 \le i \le k} (-1)^{i+1}\ \{f_0, f_i\}\ df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k$$

$$(1) \qquad + \sum_{1 \le i < j \le k} (-1)^{i+j}\ f_0\ d\{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k \ .$$

The operator δ satisfies (see [19, 7])

$$\delta^2 = 0$$

and so, we obtain the canonical complex

$$\cdots \longrightarrow \Lambda^{k+1}(M) \stackrel{\delta}{\longrightarrow} \Lambda^k(M) \stackrel{\delta}{\longrightarrow} \Lambda^{k-1}(M) \longrightarrow \cdots$$

whose homology groups $H_*^{can}(M)$ are given by

$$H_k^{can}(M) = \frac{\ker\{\delta : \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M)\}}{Im\,\{\delta : \Lambda^{k+1}(M) \longrightarrow \Lambda^k(M)\}} \ .$$

$H_*^{can}(M)$ is called the canonical homology of M.

9. Brylinski conjecture

Brylinski, in [7], stated the following problem.

Problem A: *Give conditions on a compact Poisson manifold M which ensure that any cohomology class in $H^k_{DR}(M)$ has a representative α such that $d\alpha = 0$ and $\delta\alpha = 0$, (i.e., α is harmonic for the Poisson structure of M).*

Suppose that M is symplectic. Then there is a natural pairing

$$\Lambda^k(G) : \Lambda^k(T^*M) \times \Lambda^k(T^*M) \longrightarrow \mathfrak{F}(M)$$

induced by the Poisson tensor G. In fact, $\Lambda^k(G)$ is defined as follows:

$$\Lambda^k(G)(\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) = det\,(G(\alpha_i, \beta_j)) \ .$$

Denote by $v_M = \frac{\omega^n}{n!}$ the symplectic volume form on M. The symplectic star operator is defined in [7] as follows:

$$\begin{aligned} \star : \Lambda^k(M) &\longrightarrow \Lambda^{2n-k}(M) \\ \beta \wedge (\star\alpha) &= \Lambda^k G(\beta, \alpha) \cdot v_M \ , \end{aligned}$$

for $\alpha, \beta \in \Lambda^k(M)$.

Proposition 9.1. [7] *We have*

$$\begin{aligned} \star(\star\alpha) &= \alpha \\ \delta\alpha &= (-1)^{k+1} \star d \star (\alpha) \ , \end{aligned}$$

for $\alpha \in \Lambda^k(M)$.

Now, we can relate the canonical homology with the de Rham cohomology of the symplectic manifold M.

Theorem 9.1. [7] *Let M be a compact symplectic manifold of dimension $2n$. Then, the operator $\star$ establishes an isomorphism of the canonical homology group $H_k^{can}(M)$ with the de Rham cohomology group $H_{DR}^{2n-k}(M)$:*

$$H_k^{can}(M) \cong H_{DR}^{2n-k}(M) \ .$$

Let $M = \Gamma\backslash\mathcal{G}$ be a compact symplectic nilmanifold, whose symplectic form comes from the projection of a left invariant symplectic form on $\mathcal{G}$. Then, Theorem 9.1 and Nomizu's theorem imply that there is a natural isomorphism

$$H_i^{can}(\mathfrak{g}^*) \cong H_i^{can}(M) \ ,$$

where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$.

Because $d\delta + \delta d = 0$, the symplectic Laplacian operator $\Delta = d\delta + \delta d$ vanishes. However, we still give the following definition.

Definition 9.1. A k-form ν such that $d\nu = 0$ and $\delta\nu = 0$ will be called symplectically harmonic.

Brylinski made the following conjecture:

If M is a compact symplectic manifold, any de Rham cohomology class in $H_{DR}^*(M)$ has a symplectically harmonic representative.

He obtained the following evidences for the conjecture:

(1) The conjecture is true if $M = \mathbb{R}^{2n}/\Gamma$, where Γ is a discrete subgroup and $\mathbb{R}^{2n}$ is endowed with the standard symplectic structure.

(2) Every cotangent bundle T^*N satisfies the conjecture.

(3) Every compact Kähler manifold satisfies the conjecture.

However, the assertion fails for arbitrary compact symplectic manifolds.

Example 9.1. (The Kodaira-Thurston manifold) The Heisenberg group H is the connected, simply connected and nilpotent Lie group of dimension 3 of the form

$$H = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \Big/ \quad x_1, x_2, x_3 \in \mathbb{R} \right\} .$$

A standard computation shows that a basis for the left invariant 1-forms on H is given by $\{dx_1, dx_2, dx_3 - x_1 dx_2\}$. Now, we take the compact quotient $\Gamma\backslash H$, where Γ is the uniform subgroup of H consisting of those matrices whose entries are integers. Thus, $\Gamma\backslash H$ is a 3-dimensional compact nilmanifold; and the 1-forms dx_1, dx_2, $dx_3 - x_1 dx_2$ all descend to 1-forms α_1, α_2, α_3 on $\Gamma\backslash H$.

The Kodaira-Thurston manifold KT is

$$KT = (\Gamma\backslash H) \times S^1$$

Denote by α_4 the canonical 1-form on S^1. Then, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis for the 1-forms on KT such that

$$d\alpha_1 = d\alpha_2 = d\alpha_4 = 0, \quad d\alpha_3 = -\alpha_1 \wedge \alpha_2 . \tag{2}$$

Nomizu's theorem permits us to compute the de Rham cohomology groups of KT. They are:

$$\begin{cases} H^0_{DR}(KT) & = & \{1\} , \\ H^1_{DR}(KT) & = & \{[\alpha_1], [\alpha_2], [\alpha_4]\} , \\ H^2_{DR}(KT) & = & \{[\alpha_1 \wedge \alpha_3], [\alpha_1 \wedge \alpha_4], [\alpha_2 \wedge \alpha_3], [\alpha_2 \wedge \alpha_4]\} , \\ H^3_{DR}(KT) & = & \{[\alpha_1 \wedge \alpha_2 \wedge \alpha_3], [\alpha_1 \wedge \alpha_3 \wedge \alpha_4], [\alpha_2 \wedge \alpha_3 \wedge \alpha_4]\} , \\ H^4_{DR}(KT) & = & \{[\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4]\} . \end{cases}$$

Define a symplectic form

$$\omega = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4$$

with Poisson tensor

$$G = X_3 \wedge X_1 + X_4 \wedge X_2 ,$$

where $\{X_1, X_2, X_3, X_4\}$ is the dual basis of vector fields.

Therefore, we have

$$\delta(\alpha_3 \wedge \alpha_4) = \alpha_1 , \quad \delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4) = \alpha_1 \wedge \alpha_2 ,$$

and $\delta(\beta) = 0$ for the another left invariant forms β.

Theorem 9.2. [11] *The cohomology class $[\alpha_2 \wedge \alpha_3 \wedge \alpha_4]$ in $H^3_{DR}(KT)$ does not admit a representative ν such that $d\nu = \delta\nu = 0$.*

Proof. For a symplectic manifold (M,ω) we define $L : \Lambda^k(M) \longrightarrow \Lambda^{k+2}(M)$ by $L(\alpha) = \alpha \wedge \omega$. By using the identities

$$\begin{aligned} &L \circ d = d \circ L \\ &i(G) = - \star L \star \ , \\ &[L, \delta] = -d \ , \\ &\star(\beta) = -(n-1)! \ L^{n-1}(\beta) \ , \ \beta \in \Lambda^1(M) \ , \end{aligned}$$

the theorem is proved by a long by straightforward computation. □

Remark 9.1. O. Mathieu [23] have obtained a characterization of the space of the cohomology classes which contain a harmonic representative. To do this, he used a classification result for representations of the Lie superalgebra $sl(2) \times \mathbb{C}^2$. As a consequence he obtain the following characterization:

Theorem 9.3. *(Mathieu) A compact symplectic manifold (M^{2n},ω) satisfies the Brylinski conjecture if and only if for any $k \leq n$ the cup product*

$$[\omega]^k : H^{n-k}_{DR}(M) \longrightarrow H^{n+k}_{DR}(M)$$

is an isomorphism, or, in other words, if and only if M satisfies the strong Lefschetz theorem.

Thus, any compact symplectic manifold which does not verify the strong Lefschetz theorem gives a counterexample of Brylinski conjecture.

Corollary 9.1. *(Mathieu) The odd Betti numbers of a manifold satisfying the conjecture are even.*

Since $b_1(KT) = 3$, Theorem 9.2 follows from Mathieu's corollary.

10. Duality between canonical homology and Lichnerowicz-Poisson cohomology

In [2], Bhaskara and Viswanath have defined a duality between the canonical homology and the Lichnerowicz-Poisson cohomology of any Poisson manifold M. First of all, they defined a natural pairing

$$\Lambda^p(M) \times \mathcal{V}^p(M) \longrightarrow F(M)$$

by putting

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_p, X_1 \wedge \cdots \wedge X_p \rangle = det(\alpha_i(X_j)) \ .$$

Moreover, they defined the following operation. If $P \in \mathcal{V}^{p-1}(M)$ and $\gamma \in \Lambda^p(M)$, then $i(P)\gamma \in \Lambda^1(M)$ is given by

$$(i(P)\gamma)(X) = \langle \gamma, P \wedge X \rangle \ , \ \forall X \in \mathfrak{X}(M) \ .$$

So, they obtained the following formula

$$\langle \gamma, \sigma(P) \rangle - \langle \delta\gamma, P \rangle = (-1)^p \delta(i(P)\gamma) \ .$$

from which it follows that $\langle\ ,\ \rangle$ induces a natural pairing

$$H_p^{can}(M) \times H_{LP}^p(M) \longrightarrow H_0^{can}(M)$$

by putting

$$\langle \gamma, [P] \rangle = \langle \gamma, P \rangle \ .$$

Now, suppose that M is a compact symplectic manifold of dimension $2n$. Taking into account that

$$\begin{aligned} &H_p^{can}(M) \cong H_{DR}^{2n-p}(M) \ , \ [\gamma] \rightsquigarrow [\star\gamma] \ , \\ &H_{LP}^p(M) \cong H_{DR}^p(M) \ , \ [P] \rightsquigarrow [\mathcal{I}^{-1}(P)] \ , \\ &H_0^{can}(M) \cong H_{DR}^{2n}(M) \ , \ [\langle\gamma, P\rangle] \rightsquigarrow [\star\langle\gamma, P\rangle] \ , \end{aligned}$$

we deduce that the pairing is non-singular. In fact, by integrating over M we obtain the well-known duality of Poincaré, which can be now stated as follows:

$$H_p^{can}(M) \cong H_{LP}^p(M) \ .$$

11. The double periodic complex

Since

$$d\delta + \delta d = 0 \ ,$$

Brylinski [7] introduced the canonical double complex

$$\mathcal{C}_{*,*}(M) \ ,$$

$$\mathcal{C}_{p,q}(M) = \Lambda^{q-p}(M) \ , \ \forall p, q \geq 0 \ ,$$

together with differentials

- $d : \mathcal{C}_{p,q}(M) \longrightarrow \mathcal{C}_{p-1,q}(M)$ (the horizontal differential of degree -1),
- $\delta : \mathcal{C}_{p,q}(M) \longrightarrow \mathcal{C}_{p,q-1}(M)$ (the vertical differential of degree -1).

The periodic double complex is defined by

$$(\mathcal{C}_{*,*}^{per}(M), d, \delta) \ ,$$

$$\mathcal{C}_{p,q}^{per}(M) = \Lambda^{q-p}(M) \ , \ (p, q \in \mathbb{Z}) \ .$$

Thus, (see [5, 16]) there are two spectral sequences associated with this periodic double complex which will be studied in the forthcoming sections.

12. The second spectral sequence

Let $'\delta_r$ be the differential of bidegree $(r-1, -r)$, so that the groups $'E_{p,q}^{r+1}(M)$ are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow {}'E_{p-r+1,q+r}^r(M) \xrightarrow{'\delta_r} {}'E_{p,q}^r(M) \xrightarrow{'\delta_r} {}'E_{p+r-1,q-r}^r(M) \longrightarrow \cdots$$

Observe that a differential form $\beta \in \mathcal{C}^{per}_{p,q}(M)$ lives to $'E^r_{p,q}(M)$ if it satisfies

$$\begin{cases} d\beta &= 0, \\ \delta\beta &= d\beta_1, \\ \delta\beta_1 &= d\beta_2, \\ &\vdots \\ \delta\beta_{r-3} &= d\beta_{r-2}, \\ \delta\beta_{r-2} &= d\beta_{r-1}, \end{cases}$$

for some differential forms $\beta_1, \cdots, \beta_{r-1}$. In such a case, denote by $'[\beta]_r$ the homology class defined by β in $'E^r_{p,q}(M)$. The differential $'\delta_r$ on $'E^r_{p,q}(M)$ is given by

$$'\delta_r\, '[\beta]_r = '[\delta\beta_{r-1}]_r\ .$$

Now, for $r = 1$ the groups $'E^1_{p,q}(M)$ of the second spectral sequence are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow \mathcal{C}^{per}_{p+1,q}(M) \xrightarrow{d} \mathcal{C}^{per}_{p,q}(M) \xrightarrow{d} \mathcal{C}^{per}_{p-1,q}(M) \longrightarrow \cdots$$

Thus we obtain

$$'E^1_{p,q}(M) \cong H^{q-p}_{DR}(M)\ .$$

For $r = 2$ the groups $'E^2_{p,q}(M)$ of the second spectral sequence are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow H^{q-p+1}_{DR}(M) \xrightarrow{\delta} H^{q-p}_{DR}(M) \xrightarrow{\delta} H^{q-p-1}_{DR}(M) \longrightarrow \cdots$$

Next, we shall study the degeneracy of the second spectral sequence. First of all, we need the following lemma.

Lemma 12.1. [14] *Let $(M, \{\, ,\, \})$ be a Poisson manifold with Poisson tensor G. We have*

$$ki(G)di(G)^{k-1} = i(G)^k d + (k-1)di(G)^k\ ,\ \forall k \in N$$

As a direct consequence, we get

Theorem 12.1. [14] *The second spectral sequence of the double complex $\mathcal{C}^{per}_{p,q}(M)$ degenerates at $'E^1(M)$, that is, $'E^1(M) \cong {'E^\infty}(M)$.*

Proof. In fact, after some manipulations by using the above Lemma 12.1, we deduce that $\beta_{r-1} = d(\gamma)$, for some $(p+q-1)$-form γ. So $'\delta_r\, '[\beta]_r = '[\delta\beta_{r-1}]_r = 0$.

13. The first spectral sequence

Denote by δ_r the differential of bidegree $(-r, r-1)$, so that the groups $E^{r+1}_{p,q}(M)$

are isomorphic to the homology groups of the following sequence

$$\cdots \longrightarrow E^r_{p+r,q-r+1}(M) \xrightarrow{\delta_r} E^r_{p,q}(M) \xrightarrow{\delta_r} E^r_{p-r,q+r-1}(M) \longrightarrow \cdots$$

Note that a differential form $\alpha \in \mathcal{C}^{per}_{p,q}(M)$ lives to $E^r_{p,q}(M)$ if it satisfies

$$\begin{cases} \delta\alpha &= 0, \\ d\alpha &= \delta\alpha_1, \\ d\alpha_1 &= \delta\alpha_2, \\ & \vdots \\ d\alpha_{r-3} &= \delta\alpha_{r-2}, \\ d\alpha_{r-2} &= \delta\alpha_{r-1}, \end{cases}$$

for some differential forms $\alpha_1, \cdots, \alpha_{r-1}$. Denote by $[\alpha]_r$ the homology class defined by α in $E^r_{p,q}(M)$. The differential operator δ_r is given by

$$\delta_r[\alpha]_r = [d\alpha_{r-1}]_r \ .$$

In particular, for $r = 1$ the groups $E^1_{p,q}(M)$ of the first spectral sequence are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow \mathcal{C}^{per}_{p,q+1}(M) \xrightarrow{\delta} \mathcal{C}^{per}_{p,q}(M) \xrightarrow{\delta} \mathcal{C}^{per}_{p,q-1}(M) \longrightarrow \cdots$$

Thus, we have

$$E^1_{p,q}(M) \cong H^{can}_{q-p}(M) \ .$$

For $r = 2$, the groups $E^2_{p,q}(M)$ are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow H^{can}_{q-p-1}(M) \xrightarrow{d} H^{can}_{q-p}(M) \xrightarrow{d} H^{can}_{q-p+1}(M) \longrightarrow \cdots$$

In [7], Brylinski has proposed the following problem:

Problem B: *Give conditions on a compact Poisson manifold M which ensure the degeneracy at E^1 of the first spectral sequence of M.*

In fact, he proved the following result.

Theorem 13.1. *([7]) For any compact symplectic manifold M, the first spectral sequence of the double complex $\mathcal{C}^{per}_{p,q}(M)$ degenerates at $E^1(M)$.*

By using the symplectic star operator, we have

Theorem 13.2. [14] *For all $r \geq 0$, the homomorphism*

$$f_r : E^r_{p,q}(M) \longrightarrow {}'E^r_{q,2n+p}(M)$$

given by

$$f_r[\alpha]_r = {}'[\star\alpha]_r$$

is an isomorphism of homology groups. Moreover, f_r commutes with the differential, that is,

$$(f_r \circ \delta_r)[\alpha]_r = (-1)^{q-p+1}({}'\delta_r \circ f_r)[\alpha]_r,$$

for all $[\alpha]_r \in E^r_{p,q}(M)$.

Proof. The theorem follows by using the symplectic star operator and Theorem 12.1. □

Now, as a consequence of Theorem 12.1 and Theorem 13.2, we obtain Theorem 13.1.

14. Almost cosymplectic manifolds

An important class of odd dimensional Poisson manifolds are almost cosymplectic manifolds.

Definition 14.1. An almost contact metric structure $(\phi, \mathcal{R}, \eta, g)$ on a $(2n+1)$-dimensional manifold M consists of:

- a tensor field ϕ of type $(1,1)$:
- a vector field $\mathcal{R}$;
- a 1-form η;
- a Riemannian metric g on M;

such that

$$\phi^2 = -I + \eta \otimes \mathcal{R},$$
$$\eta(\mathcal{R}) = 1,$$
$$g(\phi(X), \phi(Y)) = g(X,Y) - \eta(X)\eta(Y) \ , \ \forall X, Y \in \mathfrak{X}(M) \ .$$

M is then called an almost contact metric manifold.

Define the fundamental 2-form Φ by

$$\Phi(X,Y) = g(\phi(X), Y) \ , \ \forall X, Y \in \mathfrak{X}(M) \ .$$

Hence, we have

$$\Phi^n \wedge \eta \neq 0 \ ,$$

which defines a volume form

$$v_M = \frac{1}{n!}(\Phi^n \wedge \eta) \ .$$

Definition 14.2. M is called almost cosymplectic if Φ and η are closed.

Next, we shall introduce a Poisson structure on an almost cosymplectic manifold.

Definition 14.3. Let M be an almost cosymplectic manifold. For each $f \in \mathfrak{F}(M)$ the Hamiltonian vector field X_f of f is the vector field on M defined by

$$\begin{cases} i_{X_f}\,\Phi &= df - \mathcal{R}(f)\eta, \\ i_{X_f}\,\eta &= 0 \end{cases}$$

Moreover, there are local coordinates $\{q^1, \cdots, q^n, p_1, \cdots, p_n, z\}$ in a neighborhood of every point, such that

$$\begin{cases} \Phi &= \sum_{i=1}^{n} dp_i \wedge dq^i , \\ \eta &= dz , \quad \mathcal{R} = \frac{\partial}{\partial z} , \\ X_f &= \sum_{i=1}^{n} \left\{ \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} \right\} . \end{cases}$$

(see [4, 8, 20]).

Let M be an almost cosymplectic manifold, with fundamental 2-form Φ. Define the mapping $\{\, , \,\} : \mathfrak{F}(M) \times \mathfrak{F}(M) \longrightarrow \mathfrak{F}(M)$ by

$$\{f, g\} = -\Phi(X_f, X_g) ,$$

for $f, g \in \mathfrak{F}(M)$, where X_f and X_g are the Hamiltonian vector fields of f and g, respectively. Then, $\{\, , \,\}$ is a Poisson bracket on M. So, M is a Poisson manifold with Poisson tensor G given by

$$G = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} ,$$

and the Poisson bracket satisfies

$$\{f, g\} = \sum_{i=1}^{n} \left\{ \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right\} .$$

Remark 14.1. Let M be a $(2n+1)$-dimensional manifold endowed with a closed 2-form Φ and a closed 1-form η such that $\Phi^n \wedge \eta \neq 0$. Then there exists an almost contact metric structure $(\phi, \mathcal{R}, \eta, g)$ on M such that Φ is the fundamental form (see [4] for the details).

15. The canonical homology of almost cosymplectic manifolds

Let $(M, \phi, \mathcal{R}, \eta, g)$ be an almost cosymplectic manifold, with fundamental 2-form Φ and Poisson tensor G. Denote by δ the Koszul differential of M.

Define the subspaces

$$\begin{cases} \Lambda^k_{\mathcal{R}}(M) &= \{\alpha \in \Lambda^k(M) \mid i(\mathcal{R})\alpha = 0\}, \\ \Lambda^k_{\eta}(M) &= \{\alpha \in \Lambda^k(M) \mid \eta \wedge \alpha = 0\}. \end{cases}$$

We obtain the following decomposition

$$\Lambda^k(M) = \Lambda^k_{\mathcal{R}}(M) \oplus \Lambda^k_{\eta}(M) . \tag{3}$$

In fact, if $\alpha \in \Lambda^k(M)$ then

$$\alpha = (\alpha - \eta \wedge i(\mathcal{R})\alpha) + \eta \wedge i(\mathcal{R})\alpha ,$$

where

$$\alpha - \eta \wedge i(\mathcal{R})\alpha \in \Lambda^k_{\mathcal{R}}(M) \quad , \quad \eta \wedge i(\mathcal{R})\alpha \in \Lambda^k_{\eta}(M) \; .$$

Proposition 15.1. *δ preserves the above decomposition, i.e, we have*
i) $\delta(\Lambda^k_{\mathcal{R}}(M)) \subset \Lambda^{k-1}_{\mathcal{R}}(M)$;
ii) $\delta(\Lambda^k_{\eta}(M)) \subset \Lambda^{k-1}_{\eta}(M)$.

So, we introduce the differential complexes

$$\cdots \longrightarrow \Lambda^{k+1}_{\mathcal{R}}(M) \xrightarrow{\delta} \Lambda^k_{\mathcal{R}}(M) \xrightarrow{\delta} \Lambda^{k-1}_{\mathcal{R}}(M) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \Lambda^{k+1}_{\eta}(M) \xrightarrow{\delta} \Lambda^k_{\eta}(M) \xrightarrow{\delta} \Lambda^{k-1}_{\eta}(M) \longrightarrow \cdots$$

and their corresponding homology groups

$$\hat{H}{}^{can}_k(M) = \frac{Ker\{\delta : \Lambda^k_{\mathcal{R}}(M) \longrightarrow \Lambda^{k-1}_{\mathcal{R}}(M)\}}{\delta(\Lambda^{k+1}_{\mathcal{R}}(M))},$$
$$\overset{\vee}{H}{}^{can}_k(M) = \frac{Ker\{\delta : \Lambda^k_{\eta}(M) \longrightarrow \Lambda^{k-1}_{\eta}(M)\}}{\delta(\Lambda^{k+1}_{\eta}(M))}.$$

In order to prove that the canonical homology groups $H^{can}_k(M)$ have finite dimension, we show in [14] that the groups $\check{H}^{can}_k(M)$ and $\hat{H}^{can}_{k-1}(M)$ are isomorphic. Then, from (3) and Proposition 15.1, we deduce

$$\begin{aligned} H^{can}_k(M) &\cong \hat{H}{}^{can}_k(M) \oplus \overset{\vee}{H}{}^{can}_k(M) \\ &\cong \hat{H}{}^{can}_k(M) \oplus \hat{H}{}^{can}_{k-1}(M), \end{aligned}$$

for any $k \geq 1$.

Moreover, in [14] we have proved the following

Proposition 15.2. [14] *For any compact almost cosymplectic manifold M of dimension $(2n+1)$, the homology group $\hat{H}{}^{can}_{2n-k}(M)$ has finite dimension. Therefore, the canonical homology group $H^{can}_k(M)$ has also finite dimension.*

16. The canonical homology of compact almost cosymplectic nilmanifolds

In this section, we shall prove an approximation to Nomizu's theorem for the canonical homology of compact almost cosymplectic nilmanifolds.

Let $M = \Gamma \backslash \mathcal{G}$ be a compact almost contact metric nilmanifold of dimension $(2n+1)$. This means that:

- $\mathcal{G}$ is a connected, simply-connected and nilpotent Lie group of dimension $(2n+1)$;
- Γ is a discrete subgroup of $\mathcal{G}$ such that the quotient space $\Gamma \backslash \mathcal{G}$ is compact;
- There is a left invariant almost contact metric structure $(\phi, \mathcal{R}, \eta, g)$ on $\mathcal{G}$.

We also denote by $(\phi, \mathcal{R}, \eta, g)$ the induced almost contact metric structure on M; and by Φ the fundamental 2-form on $\mathcal{G}$ and M. Then, $\mathcal{G}$ and M are Poisson manifolds. Denote by δ the Koszul differential of $\mathcal{G}$ and M.

Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{h} = \{X \in \mathfrak{g} | \eta(X) = 0\} .$$

So, $\mathfrak{h}$ is a nilpotent Lie algebra of dimension $2n$. Integrate $\mathfrak{h}$ to obtain a Lie subgroup $\mathcal{H}$ of $\mathcal{G}$, i.e., $\mathcal{H}$ is a connected, simply-connected, nilpotent Lie group whose Lie algebra is $\mathfrak{h}$. Moreover, $\tilde{\Gamma} = \Gamma \cap \mathcal{H}$ is a discrete subgroup of $\mathcal{H}$ such that the quotient space $N = \tilde{\Gamma} \backslash \mathcal{H}$ is a compact nilmanifold. Notice that N is in fact a submanifold of M, for which the canonical inclusion will be denoted by $j : N \longrightarrow M$. Moreover, $\mathfrak{h}$ is endowed with a symplectic form obtained by the restriction of Φ. Thus, N inherits a symplectic form also denoted by Φ which is the projection onto N of the left invariant symplectic form on $\mathcal{H}$. In fact, N is a symplectic leaf of the symplectic foliation on M.

But we know that there are canonical isomorphisms

$$\hat{H}_q^{can}(\mathfrak{g}^*) \cong H_q^{can}(\mathfrak{h}^*) \cong H_q^{can}(N) .$$

Now, denote by $\lambda : \hat{H}_q^{can}(\mathfrak{g}^*) \longrightarrow \hat{H}_q^{can}(M)$ the homomorphism which maps the homology class of a left invariant form α on $\mathcal{G}$ into the homology class of the projected form on M, namely $\lambda[\alpha] = [\alpha]$. Suppose that $\lambda[\alpha] = 0$. Since $\alpha \in \Lambda^q_{\mathcal{R}}(\mathfrak{g})$ we have that $\alpha \in \Lambda^q(\mathfrak{h})$. Because $\lambda[\alpha] = 0$, we deduce that there exists $\beta \in \Lambda^{q+1}_{\mathcal{R}}(M)$ such that $\alpha = \delta\beta$. From (1), we deduce that $\beta = \beta_0 + \beta_1$, where $\beta_0 \in \Lambda^{q+1}(N)$ and the components of β_1 in Darboux coordinates are linear on the z's. Thus, $\alpha = \delta\beta = \delta\beta_0$, and, therefore, $[\alpha]$ is the zero class in $\hat{H}_q^{can}(\mathfrak{g}^*)$.

This proves

Theorem 16.1. *Let $\mathcal{G}$ be a connected nilpotent Lie group endowed with an invariant almost cosymplectic structure $(\phi, \mathcal{R}, \eta, g)$ and with a discrete subgroup Γ such that the space of right cosets $M = \Gamma \backslash \mathcal{G}$ is compact. Then there is an injective homomorphism of homology groups from $H_q^{can}(\mathfrak{g}^*)$ into $H_q^{can}(M)$, for all $q \geq 0$, where we consider on M the projected almost cosymplectic structure.*

Example 16.1. (The manifold M^5) We exhibit an example of a compact almost cosymplectic nilmanifold M^5 for which the first spectral sequence $\{E^r(M^5)\}$ is such that $E^1(M^5) \ncong E^2(M^5)$.

Let $\mathcal{G}$ be the 5-dimensional connected, simply-connected and nilpotent Lie group, defined by the left invariant 1-forms $\{\alpha_1, \cdots, \alpha_5\}$ such that

$$\begin{cases} d\alpha_1 = d\alpha_2 = d\alpha_5 = 0 , \\ d\alpha_3 = \alpha_2 \wedge \alpha_5 , \\ d\alpha_4 = \alpha_1 \wedge \alpha_2. \end{cases}$$

These structure equations can be integrated explicitly; in fact, $\mathcal{G}$ can be realized as the nilpotent Lie group

$$\mathcal{G} = \left\{ \left(\begin{array}{cccccc} 1 & x_1 & x_2 & x_5 & x_3 & x_4 \\ 0 & 1 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 1 & 0 & -x_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \middle| \quad x_i \in \mathbb{R} \right\}.$$

We take Γ to be the subgroup of $\mathcal{G}$ consisting of those matrices whose entries are integers. Define $M^5 = \Gamma\backslash\mathcal{G}$.

Let $\{X_1, \cdots, X_5\}$ be the basis dual to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Consider the metric g on M^5 defined by

$$g = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2.$$

Then $\{X_1, \cdots, X_5\}$ is an orthonormal frame with respect to g on M^5.

Define a tensor field ϕ of type $(1,1)$ over M^5 by

$$\begin{aligned} \phi(X_1) &= X_4, \quad \phi(X_2) = X_3, \quad \phi(X_5) = 0, \\ \phi(X_4) &= -X_1, \quad \phi(X_3) = -X_2, \end{aligned}$$

and consider $\mathcal{R} = X_5$, $\eta = \alpha_5$. Then $(\phi, \mathcal{R}, \eta, g)$ is an almost contact metric structure on M^5 whose fundamental 2-form Φ is

$$\Phi = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3.$$

The compact nilmanifold $M^5 = \Gamma\backslash\mathcal{G}$ with the almost contact metric structure $(\phi, \mathcal{R}, \eta, g)$ is an almost cosymplectic nilmanifold whose structure arises from a left invariant almost contact metric structure on $\mathcal{G}$.

The Poisson tensor G is given by

$$G = -X_1 \wedge X_4 - X_2 \wedge X_3 \ ,$$

and we have

$$\delta(\alpha_3 \wedge \alpha_4) = \alpha_1, \quad \delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4) = -\alpha_1 \wedge \alpha_2 \ ,$$

$$\delta(\alpha_3 \wedge \alpha_4 \wedge \eta) = \alpha_1 \wedge \eta, \quad \delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta) = -\alpha_1 \wedge \alpha_2 \wedge \eta \ ,$$

and $\delta(\beta) = 0$ for the another left invariant forms β.

Denote by $\mathfrak{g}$ the Lie algebra of $\mathcal{G}$. Then, we have

$$\begin{aligned} H_0^{can}(\mathfrak{g}^*) &= \{1\}, \\ H_1^{can}(\mathfrak{g}^*) &= \{\{\alpha_2\}, \{\alpha_3\}, \{\alpha_4\}, \{\eta\}\}, \\ H_2^{can}(\mathfrak{g}^*) &= \{\{\alpha_1 \wedge \alpha_3\}, \{\alpha_1 \wedge \alpha_4\}, \{\alpha_2 \wedge \eta\}, \{\alpha_2 \wedge \alpha_3\}, \\ &\quad \{\alpha_2 \wedge \alpha_4\}, \{\alpha_3 \wedge \eta\}, \{\alpha_4 \wedge \eta\}\}, \end{aligned}$$

$$\begin{aligned}
H_3^{can}(\mathfrak{g}^*) &= \{\{\alpha_1 \wedge \alpha_2 \wedge \alpha_3\}, \{\alpha_1 \wedge \alpha_2 \wedge \alpha_4\}, \{\alpha_1 \wedge \alpha_3 \wedge \eta\}, \\
&\quad \{\alpha_1 \wedge \alpha_3 \wedge \alpha_4\}, \{\alpha_1 \wedge \alpha_4 \wedge \eta\}, \{\alpha_2 \wedge \alpha_3 \wedge \eta\}, \{\alpha_2 \wedge \alpha_4 \wedge \eta\}\}, \\
H_4^{can}(\mathfrak{g}^*) &= \{\{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4\}, \{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \eta\}, \{\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \eta\}, \\
&\quad \{\alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta\}\}, \\
H_5^{can}(\mathfrak{g}^*) &= \{\{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta\}\}.
\end{aligned}$$

Theorem 16.2. *For the first spectral sequence $\{E^1(M^5)\}$ we have*

$$E^1_{0,1}(M^5) \ncong E^2_{0,1}(M^5)$$

Proof. In fact, α_3 defines a nontrivial homology class $\{\alpha_3\}$ in $H_1^{can}(\mathfrak{g}^*)$. Then, α_3 defines a nontrivial homology class in $H_1^{can}(M^5)$. Moreover, $E^1_{0,1}(M^5) \cong H_1^{can}(M^5)$, so that α_3 represents a nontrivial class in $E^1_{0,1}(M^5)$.

However, $d\alpha_3 = \alpha_2 \wedge \eta$; and we know that $\alpha_2 \wedge \eta$ defines a nontrivial class in $H_2^{can}(\mathfrak{g}^*)$. Therefore, $\alpha_2 \wedge \eta$ represents a nontrivial homology class in $H_2^{can}(M^5)$. This implies that

$$\alpha_2 \wedge \eta \notin \delta(\Lambda^3(M^5)).$$

Thus, α_3 does not live in $E^2_{0,1}(M^5)$. □

17. About the Problems A and B

In this section, we shall show that Problems A and B of Brylinski have independent answers. Firstly, we prove that Problem A does not imply Problem B. For this, we consider the Kodaira-Thurston manifold KT defined by the equations (2).

Define a Poisson structure G on KT by

$$G = X_3 \wedge X_4 . \tag{4}$$

Let δ be the differential operator determined by G. From a straightforward computation we obtain that $\delta(\rho) = 0$ for any left invariant form ρ. Using again Nomizu's theorem we see that any de Rham cohomology class has a left invariant representative which is δ-coclosed. This proves the following.

Theorem 17.1. *Let KT be the Kodaira-Thurston manifold with the Poisson tensor G given by (4). Then, any de Rham cohomology class of KT has a representative which is harmonic for the Poisson structure.*

In order to show that the first spectral sequence $\{E^r(KT)\}$ does not degenerate at the term $E^1(KT)$, we need to prove that the canonical homology groups $H_*^{can}(KT)$ have finite dimension.

Consider the subspaces $\Lambda^q_{34}(KT)$, $\Lambda^q_1(KT)$, $\Lambda^q_2(KT)$ and $\Lambda^q_{12}(KT)$ of $\Lambda^q(KT)$, $(1 \leq q \leq 4)$, defined by

$$\begin{aligned}
\Lambda^q_{34}(KT) &= \{\lambda \in \Lambda^q(KT) \,|\, i_{X_1}\lambda = i_{X_2}\lambda = 0\} = \Lambda^q(\alpha_3, \alpha_4) , \\
\Lambda^q_1(KT) &= \{\lambda \in \Lambda^q(KT) \,|\, \alpha_1 \wedge \lambda = 0, i_{X_2}\lambda = 0\} = \alpha_1 \wedge \Lambda^{q-1}(\alpha_3, \alpha_4) ,
\end{aligned}$$

$$\begin{aligned}\Lambda^q_2(KT) &= \{\lambda \in \Lambda^q(KT) \,|\, i_{X_1}\lambda = 0, \alpha_2 \wedge \lambda = 0\} = \alpha_2 \wedge \Lambda^{q-1}(\alpha_3, \alpha_4)\ ,\\ \Lambda^q_{12}(KT) &= \{\lambda \in \Lambda^q(KT) \,|\, \alpha_1 \wedge \lambda = \alpha_2 \wedge \lambda = 0\} = \alpha_1 \wedge \alpha_2 \wedge \Lambda^{q-2}(\alpha_3, \alpha_4)\ ,\end{aligned}$$

where $\Lambda^*(\alpha_3, \alpha_4)$ denotes the exterior algebra generated by α_3 and α_4.

Now, for $\lambda \in \Lambda^q(KT)$ it is easy to see that

$$\begin{aligned}&\lambda = (\lambda - \alpha_1 \wedge i_{X_1}\lambda - \alpha_2 \wedge i_{X_2}\lambda + \alpha_1 \wedge \alpha_2 \wedge i_{X_2}i_{X_1}\lambda) + (\alpha_1 \wedge i_{X_1}\lambda - \alpha_1 \wedge \alpha_2\\ &\wedge i_{X_2}i_{X_1}\lambda) + (\alpha_2 \wedge i_{X_2}\lambda - \alpha_1 \wedge \alpha_2 \wedge i_{X_2}i_{X_1}\lambda) + \alpha_1 \wedge \alpha_2 \wedge i_{X_2}i_{X_1}\lambda\ ,\end{aligned}$$

with $(\lambda - \alpha_1 \wedge i_{X_1}\lambda - \alpha_2 \wedge i_{X_2}\lambda + \alpha_1 \wedge \alpha_2 \wedge i_{X_2}i_{X_1}\lambda) \in \Lambda^q_{34}(KT)$, $(\alpha_1 \wedge i_{X_1}\lambda - \alpha_1 \wedge \alpha_2 \wedge i_{X_2}i_{X_1}\lambda) \in \Lambda^q_1(KT)$, $(\alpha_2 \wedge i_{X_2}\lambda - \alpha_1 \wedge \alpha_2 \wedge i_{X_2}i_{X_1}\lambda) \in \Lambda^q_2(KT)$ and $\alpha_1 \wedge \alpha_2 \wedge i_{X_2}i_{X_1}\lambda \in \Lambda^q_{12}(KT)$.

Therefore, the space $\Lambda^q(KT)$ becomes:

$$\Lambda^q(KT) = \Lambda^q_{34}(KT) \oplus \Lambda^q_1(KT) \oplus \Lambda^q_2(KT) \oplus \Lambda^q_{12}(KT)\ . \tag{5}$$

It follows that

$$\left\{\begin{aligned}&\delta(\alpha_1 \wedge \lambda) = \alpha_1 \wedge \delta(\lambda)\ ,\\ &\delta(\alpha_2 \wedge \lambda) = \alpha_2 \wedge \delta(\lambda)\ ,\\ &\delta(\alpha_1 \wedge \alpha_2 \wedge \lambda) = \alpha_1 \wedge \alpha_2 \wedge \delta(\lambda)\ .\end{aligned}\right. \tag{6}$$

Now, from (6) we obtain that δ preserves the decomposition (5), that is, $\delta(\Lambda^q_{34}(KT)) \subset \Lambda^{q-1}_{34}(KT)$, $\delta(\Lambda^q_s(KT)) \subset \Lambda^{q-1}_s(KT)$, $(s = 1, 2)$ and $\delta(\Lambda^q_{12}(KT)) \subset \Lambda^{q-1}_{12}(KT)$.

Therefore, we have the differential complexes $(\Lambda^*_{34}(KT), \delta)$, $(\Lambda^*_s(KT), \delta)$ $(s = 1, 2)$ and $(\Lambda^*_{12}(KT), \delta)$, each one of which is a subcomplex of the canonical complex of KT. Denote by $H^{can}_{34,*}(KT)$, $H^{can}_{s,*}(KT)$ $(s = 1, 2)$ and $H^{can}_{12,*}(KT)$ the homology of the complexes $(\Lambda^*_{34}(KT), \delta)$, $(\Lambda^*_s(KT), \delta)$ $(s = 1, 2)$ and $(\Lambda^*_{12}(KT), \delta)$, respectively.

Let us now consider the homomorphisms $\widehat{i_{X_s}} : \Lambda^q_s(KT) \longrightarrow \Lambda^{q-1}_{34}(KT)$, $(s = 1, 2)$, and $\widehat{i_{X_2}}\widehat{i_{X_1}} : \Lambda^q_{12}(KT) \longrightarrow \Lambda^{q-2}_{34}(KT)$ given by

$$\begin{aligned}&\widehat{i_{X_s}}(\lambda) = i_{X_s}\lambda\ ,\\ &\widehat{i_{X_2}}\widehat{i_{X_1}}(\mu) = i_{X_2}i_{X_1}\mu\ ,\end{aligned}$$

for $\lambda \in \Lambda^q_s(KT)$, $(s = 1, 2)$, and $\mu \in \Lambda^q_{12}(KT)$.

Using (6) one can check that each one of the homomorphisms $\widehat{i_{X_s}}$, $(s = 1, 2)$, and $\widehat{i_{X_2}}\widehat{i_{X_1}}$ commutes with the differential δ, and moreover the homomorphisms induced in homology are isomorphisms. Now, from (5), we obtain the isomorphism:

$$\begin{aligned}H^{can}_q(KT) &\cong H^{can}_{34,q}(KT) \oplus H^{can}_{34,q-1}(KT)\\ &\oplus H^{can}_{34,q-1}(KT) \oplus H^{can}_{34,q-2}(KT)\ .\end{aligned} \tag{7}$$

Next, we study the homology $H^{can}_{34,*}(KT)$. First, we need to introduce the map $\hat{d} : \Lambda^q_{34}(KT) \longrightarrow \Lambda^{q+1}_{34}(KT)$, $(q \geq 0)$, defined by

$$\hat{d}(\lambda) = d\lambda - \alpha_1 \wedge i_{X_1} d\lambda - \alpha_2 \wedge i_{X_2} d\lambda + \alpha_1 \wedge \alpha_2 \wedge i_{X_2} i_{X_1} d\lambda \ , \tag{8}$$

for $\lambda \in \Lambda^q_{34}(KT)$.

A direct computation, by using (8), shows that

$$\begin{cases} \hat{d}^2 = 0 \ , \\ \hat{d}(\lambda \wedge \mu) = \hat{d}(\lambda) \wedge \mu + (-1)^q \lambda \wedge \hat{d}\mu \ , \end{cases}$$

for $\lambda \in \Lambda^q_{34}(KT)$.

Thus, we have the differential complex $(\Lambda^*_{34}(KT), \hat{d})$. Denote by $\hat{H}^*(KT)$ the cohomology of this complex.

Proposition 17.1. *The differential complex $(\Lambda^*_{34}(KT), \hat{d})$ is elliptic. Therefore, the cohomology groups $\hat{H}^q(KT)$ have finite dimension.*

Proof. The complex is elliptic in degree q if for all points x in KT and for all 1-form non-zero $\mu \in \Lambda^1_{34}(KT)$ at x the complex

$$\cdots 0 \stackrel{\mu\wedge}{\longrightarrow} (\Lambda^0_{34})_x(KT) \stackrel{\mu\wedge}{\longrightarrow} (\Lambda^1_{34})_x(KT) \stackrel{\mu\wedge}{\longrightarrow} (\Lambda^2_{34})_x(KT) \stackrel{\mu\wedge}{\longrightarrow} 0 \cdots$$

is exact, where $(\Lambda^q_{34})_x(KT)$ is the space of the q-forms λ at x such that $i_{X_1(x)}\lambda = i_{X_2(x)}\lambda = 0$.

If we consider Darboux coordinates (q^1, q^2, p_1, p_2) defined on some contractible neighborhood of x and such that $G = \dfrac{\partial}{\partial q^2} \wedge \dfrac{\partial}{\partial p_2}$, then $(\Lambda^k_{34})_x(KT)$ is spanned by $\{dq^2, dp_2\}$. This implies the exactness of the above complex. □

Now, imitating the definition of the symplectic star operator given in [7], we define the operator $\star_{34} : \Lambda^q_{34}(KT) \longrightarrow \Lambda^{2-q}_{34}(KT)$, $(0 \leq q \leq 2)$, by the condition $\lambda \wedge (\star_{34}\mu) = \Lambda^q(G)(\lambda, \mu)\sigma$, for $\lambda, \mu \in \Lambda^q_{34}(KT)$, and where σ is the 2-form $\sigma = \alpha_3 \wedge \alpha_4$.

Moreover, we consider the operator $\hat{\delta} : \Lambda^q_{34}(KT) \longrightarrow \Lambda^{q-1}_{34}(KT)$ given by

$$\hat{\delta} = [i(G), \hat{d}] = i(G) \circ \hat{d} - \hat{d} \circ i(G) \ .$$

Then, the same proofs given by Brylinski for the symplectic star operator (see [7], Lemma 2.1.2, Theorem 2.2.1, pp. 100-101) show that for $\lambda \in \Lambda^q_{34}(KT)$ hold:

$$\begin{cases} \star_{34}(*_{34}\lambda) = \lambda \ , \\ \hat{\delta}(\lambda) = (-1)^{q+1} \star_{34} \hat{d}(\star_{34}\lambda) \ . \end{cases} \tag{9}$$

Moreover, from (6) and (9), we have

Proposition 17.2. *The differential δ of KT satisfies*

$$\delta\lambda = [i(G), \hat{d}](\lambda) ,$$

for $\lambda \in \Lambda^q_{34}(KT)$ and $q \geq 0$.

From (9) and Proposition 17.2, it follows that $\star_{34}$ defines an isomorphism of groups $\hat{H}^q(KT) \cong H^{can}_{34,2-q}(KT)$ $(0 \leq q \leq 2)$. This proves, by using (8), the following

Proposition 17.3. *For the Kodaira-Thurston manifold KT, the canonical homology groups $H^{can}_q(KT)$ have finite dimension. Therefore, the term $E^1(KT)$ of the first spectral sequence has also finite dimension.*

Theorem 17.2. *Let KT be the Kodaira-Thurston manifold with Poisson tensor given by (4). Then, for the first spectral sequence we have $E^1(KT) \ncong E^2(KT)$.*

Proof. Consider the differential 2-form ω defined by

$$\omega = \alpha_3 \wedge \alpha_4 .$$

Then $\delta\omega = 0$ so ω represents a class in $E^1_{0,2}(KT) \cong H^{can}_2(KT)$. Moreover $\omega \notin \delta(\Lambda^3(KT))$. In fact, suppose that

$$\omega = \delta\theta ,$$

for some differential 3-form $\theta \in \Lambda^3(KT)$. It follows that

$$\omega = \delta\theta = i(G)d\theta - d(i(G)\theta) = f\alpha_1 \wedge \alpha_2 - d(i(G)\theta) , \tag{10}$$

for some function $f \in F(M)$. Taking in (10) the wedge product by $\alpha_1 \wedge \alpha_2$, we get

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 = \omega \wedge \alpha_1 \wedge \alpha_2 = -d(i(G)\theta) \wedge \alpha_1 \wedge \alpha_2 = d(-i(G)\theta \wedge \alpha_1 \wedge \alpha_2) ,$$

which is a contradiction with (3). Therefore, ω represents a non-trivial class in $E^1_{0,2}(KT)$.

Next, we shall prove that ω does not define a class in $E^2_{0,2}(KT)$ which is equivalent to show that $d\omega = -\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \notin \delta(\Lambda^4(KT))$. Suppose that

$$d\omega = -\alpha_1 \wedge \alpha_2 \wedge \alpha_4 = \delta(\nu) ,$$

for some differential 4-form $\nu \in \Lambda^4(KT)$. We have

$$d\omega = -\alpha_1 \wedge \alpha_2 \wedge \alpha_4 = -i(G)d\nu + d(i(G)\nu) = d(i(G)\nu) . \tag{11}$$

Let us now consider the differential 2-form $\gamma \in \Lambda^2(KT)$ defined by

$$\gamma = \omega - i(G)\nu .$$

From (11) it follows that γ is closed, so it represents a de Rham cohomology class $[\gamma] \in H^2_{DR}(KT)$. Now, there are two possibilities:

- CASE I. γ defines the zero class in $H^2_{DR}(KT)$. In this case we have $\gamma = d\gamma_1$ for some $\gamma_1 \in \Lambda^1(KT)$, and hence

$$\omega = i(G)\nu + d\gamma_1 .$$

In this equality taking the wedge product by $\alpha_1 \wedge \alpha_2$ we conclude that

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 = \omega \wedge \alpha_1 \wedge \alpha_2 = d(\gamma_1 \wedge \alpha_1 \wedge \alpha_2) \ ,$$

which is a contradiction with (3).

- CASE II. γ defines a non-trivial class in $H^2_{DR}(KT)$. In this case must be:

$$(12)\gamma = \omega - i(G)\nu = \lambda_1\alpha_1 \wedge \alpha_3 + \lambda_2\alpha_1 \wedge \alpha_4 + \lambda_3\alpha_2 \wedge \alpha_3 + \lambda_4\alpha_2 \wedge \alpha_4 + d\gamma_2 \ ,$$

for some $\lambda_i \in \mathbb{R}$, $(1 \leq 1 \leq 4)$, and some 1-form $\gamma_2 \in \Lambda^1(KT)$. Again, taking in (12) the wedge product by $\alpha_1 \wedge \alpha_2$, we deduce that $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4$ defines the zero class in $H^4_{DR}(KT)$, which is a contradiction. This completes the proof. □

Now, from Theorem 17.1 and Theorem 17.2, we have

Corollary 17.1. *For the Kodaira-Thurston manifold with the Poisson tensor G given by (4), Problem A does not imply Problem B.*

That Problem B does not imply Problem A follows directly from Theorem 9.2 and Theorem 13.1.

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