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CONVEX LINES IN MEDIAN GROUPS

Milan Kolibiar

Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. There is proved that a convex maximal line in a median group G, containing 0, is a direct factor of G.

1. Introduction

The present paper is related to the paper [5]. The aim of it is to extend the main result in [5] to a class of all median groups.

A basic notion in both papers is that of median algebra. By a median algebra is meant an algebra with a single ternary operation satisfying the identities

(1) $(a, a, b) = b$,

(2) $((a, b, c), d, c) = ((d, c, b), a, c)$.

Such algebras were investigated under various names by several authors. A survey of results is e.g. in [1]. Let $L = (L; \wedge, \vee)$ be a distributive lattice. Consider the operation

(3) $(a, b, c) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

$M(L) = (L; (\wedge, \vee))$ is a median algebra. According to [7] each median algebra is isomorphic to a subalgebra of an algebra $M(L)$.

In an l-group $G = (G; +, -, 0, (, ,))$ the operations (3) and + are related by the identity

(4) $u + (a, b, c) + v = (u + a + v,\ u + b + v,\ u + c + v)$.

Definition. By a median group (m. group) there is meant an algebra $(G; +, -, 0, (, ,))$ where $(G; +, -, 0)$ is a group, $(G; (, ,))$ is a median algebra and the identity (4) in G holds.

If G is an l-group then the m. group $(G; +, -, 0, (, ,))$ where the ternary operation is given by (3), is said to be associated with G. There are median groups which are not associated with any l-group. Examples of such m. groups are in [5].

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2. Some properties of median algebras and median groups

Let $A = (A; (, ,))$ be a median algebra. If $a, b, c \in A$ and $(a, b, c) = b$ we say that b is between a and c (in symbols abc). If $a_1, a_2, \dots, a_n \in A$ and $a_i a_j a_k$ holds for $1 \leqq i \leqq j \leqq k \leqq n$ we denote it by $a_1 a_2 \dots a_n$. A subset K of A is said to be convex if $a, b \in K$, $u \in A$ and aub imply $u \in K$.

Given an element $u \in A$, then the rule $x \wedge y = (x, u, y)$ gives an idempotent, commutative and associative operation in A and $(A; \wedge)$ is a semilattice with the least element u. In what follows we shall use such operation in median groups setting $u = 0$. Then $x \leqq y$ in the semilattice $(G; \wedge)$ iff $0xy$. (a, b) will denote the set $\{x \in A : axb\}$. The algebra $((a, b); \wedge, \vee)$, where $x \wedge y = (x, a, y)$, $x \vee y = (x, b, y)$ is a distributive lattice with the least and the greatest elements a, b, respectively [7]. Call a mapping $f : A \to B$ between two median algebras A, B betweenness preserving if abc implies $(fa)(fb)(fc)$. A subset L of a median algebra A is called a line if there is a betweenness preserving injective mapping f from a chain C to A such that $L = \{fa : a \in c\}$.

2.1[3, Proposition 2]. *A subset L of a median algebra with* $\operatorname{card} L \neq 4$ *is a line iff for any $a, b, c \in L$ one of the relations abc, bca, acb holds. Obviously a subset of a line is a line. If a is an element of a line L such that for each $b, c \in L$ either abc or acb holds, we say that a is an end element of L.*

2.2. Let A be a line in a median algebra and $0, a \in A$, $a \neq 0$. Denote $A' = \{x \in A : x0a\}$, $A_a = A - A'$. Then $A = A' \cup A_a$ and $x \in A'$ together with $y \in A_a$ imply $x0y$. Routine proof omitted.

2.3. Definition [4]. A subset C of a median algebra is called a Čebyšev subset if for each $a \in A$ an element $a_C \in C$ exists such that aa_Cx for any $x \in C$.

Obviously a Čebyšev set is a convex subset of A.

2.4 [5; 2.7]. Any convex maximal line in a median algebra is a Čebyšev subset.

Some elementary properties of median algebras and median groups are in [5]. Let us recall some of them.

$(a, b, c) = (b, a, c) = (b, c, a)$,
$((a, b, c), d, e) = ((a, d, e), b, (c, d, e))$,
abc implies cba,
abc and buc imply $abuc$,
abc and acb imply $b = c$,
aub, buc and cua hold iff $u = (a, b, c)$.
G will denote an m. group.

These properties as well as the lemmas 2.5, 2.6 and 2.7 below will be used freely in what follows.

The following lemma is obvious.

2.5. Lemma. *Let a, b, c, u be elements of an m. group then abc implies $(a+u)(b+u)(c+u)$, $(u+a)(u+b)(u+c)$.*

2.6. Lemma. *Let $a, b, u \in G$. If (a, b) is a line then $(a+u, b+u)$ and $(u+a, u+b)$ are lines too.*

Proof. The lemma is an immediate corollary of 2.1 and 2.5. □

The following assertion is easy to proove.

2.7. Let a, b, c, d be elements of a line and let abc, bcd and $b \neq c$ hold. Then $abcd$ holds.

3. Direct factors

In this paragraph G denotes a median group.

3.1. We say that a subset A of G forms a direct factor of G whenever a direct product decomposition $f : G \to B \times C$ exists such that $A = f^{-1}(\{(b, 0) : b \in B\})$.

3.2. Lemma [6; 3.9]. *A subset A in G forms a direct factor of G if and only if it is a Čebyšev subset in $M(G)$ and forms a subgroup of the group $(G; +, -, 0)$.*

3.3. Theorem. *Any convex maximal line in G, containing 0, is a direct factor of G.*

In view of 3.2 and 2.4 it suffices to prove the following lemma.

3.4. Lemma. *Any convex maximal line L in $M(G)$ forms a subgroup of the group $(G; +, -, 0)$ whenever $0 \in L$.*

The proof of lemma 3.4 is divided into a series of lemmas and ends in 3.15.

3.5. Remark. A short proof of lemma 3.4 has been given (not yet published) by T. Marcisová.

3.6. Let $a \in G$ and let $(0, a)$ be a line. Then one of the cases

$$(-a)0a, \quad 0(-a)a, \quad 0a(-a)$$

occurs.

Proof. Denote $u := (-a, 0, a)$. From $0u(-a)$ it follows that $a(a+u)0$ and, since $0ua$ and $(0, a)$ is a line, one of the cases

a) $0u(a+u)a$, $b)0$(a+u)ua

occurs. In the case a) we get $(-a)(-a+u)u$, which together with $a(a+u)u$ and $au(-a)$ yields $a(a+u)u(-a+u)(-a)$. From $(a+u)u(-a+u)$ it follows $a0(-a)$. In the case b) we get $(-u)a0(a-u)$ and, according to $au0$, $(-u)au0(a-u)$. Since $(0, a)$ is a line, $(-u, a-u)$ is a line, too, (see 2.6) and, according to $(-u, u) \subset (-u, a-u)$, $(u, -u)$ is a line.

We shall show that

(i) $$a, -a \in (u, -u).$$

First from the above relation we get $(-u)au$. From $0(a+u)a$ we get $u(a+2u)(a+u)$ which together with $au(a+u)$ yields $au(a+2u)$, hence (add $-a$ on the left and $-u$ on the right side) $(-u)(-a)u$. Hence (i) holds. Since $(u,-u)$ is a line, using 2.1 we get that one of the following cases occurs.

b1) $ua(-a)(-u)$, b2) $u(-a)a(-u)$.

In the case b1) we get $ua(-a)$ and, since $au(-a)$, $u = a$ hence $(0,a,-a) = a$, i.e. $0a(-a)$. In the case b2) $u(-a)a$ and $au(-a)$ yield $u = -a$ hence $0(-a)a$. This proves the assertion 3.6. □

3.7. Let $(0,a)$, $(0,b)$ be lines and neither $0ab$ nor $0ba$ hold. Then $a \wedge b = 0$ (i.e. $a0b$).

Proof. Let $a \wedge b = (a,0,b) = u$. According to 3.6 there occurs one of the cases

1. $a0(-a)$, 2. $0a(-a)$, 3. $0(-a)a$

and one of the cases

1′. $b0(-b)$, 2'. $0b(-b)$, 3′. $0(-b)b$.

Case (1.1′). From the assumptions we get $au0(-a)$ hence $(2a)(a+u)a0$. From this we get $(a+u)au0$ and $a(a-u)0$. Similarly $b(b-u)0$.

Denote $a' = a-u$, $b' = b-u$. Then $a' \wedge b' = (a-u, 0, b-u) = (a,u,b) - u = u - u = 0$, $0a'a$ and $0b'b$. Since $0ua$ and $0ub$, there hold

$$\text{either a)} \quad 0a'u \text{ or b)} \quad ua'a$$

and

$$\text{either a}') \quad 0b'u \text{ or b}') \quad ub'b.$$

a) and a′) yield (since $(0,u)$ is a line) $0a'b'$ or $0b'a'$ hence $a' = a' \wedge b' = 0$ i.e. $a = u$ or $b' = 0$ i.e. $b = u$ and we get that $0ab$ or $0ba$ - a contradiction. The case a) and b′) yields $0a'b'$ - a contradiction as above. The case b) and a′) is symmetric. In the case b) and b′) we get $0 = a' \wedge b'$ (since $u \leqq a' \leqq a$, $u \leqq b' \leqq b$ and $a \wedge b = u$).

Case (1.3′). Again denote $u = (0,a,b)$. There are two possibilities:

a) $0(-b)u$, b) $u(-b)b$.

The case a) yields $0(-b)ua$ and

(1) $(-a)(-b-a)(-a)0$.

From $(-a)0a$ and $0(-b)a$ we get $(-a)0(-b)a$. This together with (1) yields $(-a)(-b-a)(u-a)0(-b)a$. From this we get successively

$$0(-b)ua(-b+a)(2a), \quad b0(b+u)(b+a)a(b+2a).$$

From this we get $b0a$ hence $u = (a,0,b) = 0$. Since $0(-b)u$, we get $0(-b)0$ hence $b = 0$ - a contradiction.

In the case b) we get $0(-b)b$, $u(-b)b$ and $0ub$. This yields $0u(-b)$ hence $b(u+b)0$ so that

$$(*) \qquad u, \quad u+b \in (0,b).$$

$a0(-a)$ and $au0$ yield $au0(-a)$. From $au0$ we get $0(u-a)(-a)$. From this we get successively $au0(u-a)(-a)$, $(-u+a)0(-u)(-a)(-u-a)$, $(-u+2u)a(-u+a)0(-u)$. From $au0$ and $a0(-u)$ we get

$$(+) \qquad au0(-u).$$

According to $(*)$ there are two cases possible
b1) $0(u+b)ub$, b2) $0u(u+b)b$
Case b1) yields

$$(1) \qquad (-u)b0(-a+b)$$

From $bu0$ and $u(-b)b$ we get

$$(2) \qquad b(-b)u0.$$

But from (1) $(-u)b0$. This together with (1) yields $(-u)b(-b)u0$. From this we get $(-u)u0$. But according to (+) $u0(-u)$ hence $u = 0$.

In the case b2) from $0u(u+b)b$ it follows $(-u)0b(-u+b)$. From this we get successively $(-u)0ub(-u+b)$, $(-2u)0(-u+b)$ and $0u(2u)b$. Combining the last two relations we get $(-2u)0ub(-u+b)$. From aub and $u(2u)b$ we get $au(2u)$. Hence the elements $0, u, a\ 2u$ fulfil the conditions in the case $(1.1')$. $0 = a\ (2u) = a\ b = u$. This completes the case $(1.3')$.

In the case $(1.2')$ $0 \leqq b \leqq -b$ hence $u = a \wedge b \leqq a \wedge (-b) = v$ so that $0 \leqq u \leqq v \leqq a$, $0uv(-b)$ and $0ub(-b)$.

$(0, b)$ is a line hence $(-b, 0) = -b + (0, b)$ is a line. Since $b, v \in (0, -b)$, uvb or $bv(-b)$ hold. The second case yields $0ubv(-b)$ hence $0 \leqq b \leqq v$. Since $u \leqq v \leqq a$, we get $0 \leqq b \leqq a$ i.e. $0ba$ - a contradiction. Hence uvb holds. Then $v \leqq a$ and $v \leqq b$ yield $v \leqq a \wedge b = u$. Since $u \leqq v$, we get $u = v$. The elements $0, u, a, -b$ fulfil the conditions of the case $(1.3')$, hence $u = 0$.

Case $(3.3')$. Recall that $u = (0, a, b)$, $0ua$, $0ub$ hence $0(u-a)(-a)$, $0(u-b)(-b)$. Since $(0, a)$, $(0, b)$ are lines and u belongs to both $(0, a)$ and $(0, b)$ the following cases are possible.

1. $0u(u-a)(-a)a$, $0u(u-b)(-b)b$,
2. $0u(u-a)(-a)a$, $0(u-b)(-b)b$,
3. $0u(u-a)(-a)a$, $0(u-b)(-b)ub$,
4. $0(u-a)u(-a)a$, $0u(u-b)(-b)b$,
5. $0(u-a)u(-a)a$, $0(u-b)u(-b)b$,
6. $0(u-a)u(-a)a$, $0(u-b)(-b)ub$,
7. $0(u-a)(-a)ua$, $0u(u-b)(-b)b$,
8. $0(u-a)(-a)ua$, $0(u-b)u(-b)b$,
9. $0(u-a)(-a)ua$, $0(u-b)(-b)ub$.

Because of the symmetry it suffices to settle the cases 1,2,3,5,6,9.

Case 1. From the suppositions we get $(u-a, 0, u-b) = u$, $(-a, -u, -b) = -u + (u-a, 0, u-b) = 0$, $u = (0, -a, -b) = ((-u, -a, -b), -a, -b) = (-u, -a, -b) = 0$.

In the case 2 $(0, -a, -b) = u$. But $0(u-b)u(u-a)$ hence $(-u)(-b)0(-a)$ so that $u = (0, -a, -b) = 0$.

In the case 3 we have $0u(u-a)$, $0(u-b)u$ hence $(u-a)u(u-b)$ and $(-a)0(-b)$. From $0(-b)u$ and $0u(-a)$ we get $0(-b)(-a)$. Combining this with the above relation we get $b = 0 \in (0, a)$ - a contradiction.

Case 5. Let e.g. $0(u-b)(u-a)a$ (the second possibility is symmetric to this). Then from $u(u-a)(u-b)$ we get $0(-a)(-b)$ hence $u = (0, -a, -b) = -a$. Then from $0(u-b)u$ we get $0(u-b)(-a)$ so that $a(-b)0$ and $-a = u = (a, 0, -b) = -b$ hence $a = b$ - a contradiction.

In the case 6 we have $(u-b)(-b)u$ hence (add $-u$ on the left and b on the right side)

(1) $0(-u)b$.

Next $(u-a)u(-a)$ gives $(-a)0(-u-a)$ and $0a(-u)$. This together with (1) gives $0ab$ - a contradiction.

Case 9. Let e.g. $0(-a)(-b)$ (the case $0(-b)(-a)$ is symmetric). Then $u(u-a)(u-b)$ which together with $u(u-b)0$ gives $u(u-a)(u-b)0$. Combining these relations with $au(-a)(u-a)0$ we get $au(-b)(-a)(u-a)(u-b)0$. From the relation $au(-a)(u-a)$ we get $(2a)(u+a)0u$, $(2a)(u+a)0(u-b)u$. From the last relation we get $au(u-b-a)$. But from $0(u-b)(u-a)$ we get a $(u-b+a)u$, which together with the above relation gives $a(u-b+a)u(u-b-a)$. From this we get $(-b+a)0(-b-a)$ hence $ab(-a)$ so that $b \in (a, -a) \subset (0, a)$ - a contradiction.

This settles the case $(3.3')$.

Case $(2.2')$. We have $0 \leqq u \leqq a \leqq -a$, $0 \leqq u \leqq b \leqq -b$. We claim that $-a \notin (0, -b)$. Suppose $-a \in (0, -b)$. Then $0a(-a)(-b)$. Since $b \in (0, -b)$ and $0ba$ do not hold the possibility $0ab(-b)$ remains which is a contradiction. Symmetrically, $-b \notin (0, -a)$. Using the consideration in the case $(3, 3')$ for the intervals $(0, -a)$ and $(0, -b)$ we get $(-a) \wedge (-b) = 0$ hence $a \wedge b = 0$.

In the remaining case $(2, 3')$ we have $0a(-a)$ and $0(-b)b$. $-a \in (0, b)$ would give $0ab$ - a contradiction. Hence $-a \notin (0, b)$. Suppose $b \in (0, -a)$ i.e. $0b(-a)$. Since $a \in (0, -a)$ one of the relations $0ab$ and $0ba(-a)$ would hold which is a contradiction. Hence $b \notin (0, -a)$. The elements $b_1 = b$ and $a_1 = -a$ fulfil the conditions of the case $(3, 3')$ so that $a_1 \wedge b_1 = 0$ hence also $a \wedge b = 0$.

Summarizing the results, we proved the assertion 3.7 in the cases $(1, 1')$, $(1, 3')$, $(1, 2')$, $(3, 3')$, $(2, 2')$ and $(2, 3')$. Because of the symmetry this settles also the cases $(3, 1')$, $(2, 1')$ and $(3, 2')$. This completes the proof. □

3.8. Let A and B be lines with the end element 0. If neither $A \subset B$ nor $B \subset A$ holds then $a \wedge b = 0$ for any $a \in A$, $b \in B$.

Proof. The assertion is a corollary of 3.7. □

3.9. Let A be a convex maximal line in G and $0, a \in A$. Then $-a \notin A$ or $a0(-a)$.

Proof. According to 3.6 one of the following three cases occurs.

1) $0(-a)a$, 2) $0a(-a)$, 3) $a0(-a)$.

Case 1) yields $-a \in A$.

Case 2). We use the notations used in 2.2. There are two possibilities:

2a) $A_a \subset (0,-a)$, 2b) $A_a - (0,a) \notin \emptyset$.

Case 2a). Let $b \in A'$. Set $t := (b,a,-a)$. There are two possibilities:

2a1) $bt0$, 2a2) $0ta$.

In the case 2a1) $b0a$ and $bt0$ imply $t0a$. But $at(-a)$ and $0a(-a)$ yield $0at$. Hence $a = 0$ and $-a \in A$.

In the case 2a2) $0ta$ and $0a(-a)$ yield $ta(-a)$. Since $at(-a)$, we get $t = a$, hence $(b,a,-a) = a$ so that $ba(-a)$ and $b0a(-a)$. From this it follows that $A' \cup (0,-a)$ is a line. Combining $A = A' \cup A_a$ and the supposition 2a) we get $A \subseteq A' \cup (0,-a)$. This and the maximality of A yields $A = A' \cup (0,-a)$, hence $-a \in A$.

Case 2b). Let $c \in A_a - (0,-a)$. If $(0,-a) \subset A_a$ then $-a \in A$. If $(0,-a) \not\subset A_a$ then, according to 3.8, $c0(-a)$ holds. Since $c \in A_a$, $0ca$ or $0ac$ holds. The first relation together with $0a(-a)$ yields $0c(-a)$ i.e. $c \in (0,-a)$ - a contradiction. Summarizing the above procedure we get that either $-a \in A$ or $a0(-a)$ hold. This completes the proof of 3.9. □

3.10. Let A be a convex maximal line in G and $0, a \in A$, $-a \in A$. Then $(-a)_A = 0$.

Proof. Denote $(-a)_A = t$. There are three cases possible:

1) $0at$, 2) $0ta$, 3) $t0a$.

In the case 1) the relations $0at$ and $(-a)t0$ yield $(-a)a0$. But by 3.9 $a0(-a)$, hence $a = 0$ and $-a \in A$ - a contradiction.

In the case 2) $0ta$ and $a0(-a)$ (see 3.9) yield $t0(-a)$. But $(-a)t0$ according to the definition of t. Hence $t = 0$.

Case 3). According to 3.8 there are three possibilities (we use the notation from 2.2): 3a) $(0,-a) \subset A'$, 3b) $A' \subset (0,-a)$, 3c) $x0y$ for each $x \in A'$ and $y \in (0,-a)$.

In the case 3a) $-a \in A$ - a contradiction.

Case 3b). Let $b \in A_a$. Then either $0ba$ or $0ab$ holds. In the first case $(-a)ta$, $t0a$ and $0ba$ yield $(-a)t0ba$, hence $(-a)0b$. In the second case $t0a$ and $0ab$ yield $t0b$ (see 2.7). This together with $(-a)tb$ yield $(-a)0b$. Hence for any $b \in A_a$ $(-a)0b$ holds. This follows that $(-a,0) \cup A_a$ is a line. Using the supposition $A' \subset (0,a)$ we get $A \subset (-a,0) \cup A_a$ so that $A = (-a,0) \cup A_a$, hence $-a \in A$ - a contradiction.

In the case 3c) we get $t0(-a)$ ($t \in A'$!). This and $(-a)t0$ yield $t = 0$. This completes the proof of 3.10. □

3.11. Let A be a convex maximal line in G and $0, a \in A$, $a \neq 0$. Then $b \in A$ exists such that $b \neq 0$ and $b0a$.

Proof. If such an element did not exist, then 0 would be an end element of A. According to 3.10 $(-a)0t$ for any $t \in A$, hence $(-a,0) \cup A$ would be a line, so that $(-a,0) \cup A = A$ and $-a \in A$ - a contradiction. □

3.12. If A is a convex maximal line in G and $0 \in A$ then $a \in A$ implies $-a \in A$.

Proof. Assume, on the contrary, that there is $a \in A$ such that $-a \in A$. According to 3.11 $b \in A$ exists such that $b0a$ and $b \neq 0$. Then $0(-b)(a-b)$ and $(0, a-b)$ is a line (see 2.5). According to 3.7 one of the following three cases occurs.

1) $0a(a-b)$, 2) $0(a-b)a$, 3) $a0(a-b)$.

In the case 1) $a \in (0, a-b)$ and, since $b0a$ is a line, there are two possibilities:

1a) $0a(-b)(a-b)$, 1b) $0(-b)a(a-b)$.

The case 1a) yields (we add $-a$ on the left and b on the right side) $b(-a)0$, hence $-a \in A$ - a contradiction.

In the case 1b) we get $b0(a+b)a$ and (adding $-a$ on the left) $(-a)b0$. But $(-a)_A = 0$ (see 3.10) hence $(-a)0b$ and $b = 0$ - a contradiction.

In the case 2) we get $(-a)(-b)0$. According to 3.9 $(-a)0a$. The two last relations yield $(-b)0a$. On the other hand from $a0b$ we get $(a-b)(-b)0$. This together with $0(a-b)a$ yields $a(-b)0$. Combining this with $(-b)0a$ we get $b = 0$ - a contradiction.

In the case 3), using the relation $0(-b)(a-b)$ which follows from $b0a$, we get $a0(-b)(a-b)$. From this we get $b(-a+b)(-a)0$, hence $b(-a)0$ and $-a \in A$ - a contradiction. This completes the proof of 3.12. □

3.13. Let A be a convex maximal line in G and $0 \in A$. Then $a \in A$ implies $2a \in A$.

Proof. There are three possibilities:

1) $0a(-a)$, 2) $0(-a)a$, 3) $a0(-a)$.

The possibility 1) yields $a(2a)0$, hence $2a \in A$. In the case 2) we get $(-a)(-2a)0$, hence $-2a \in A$ and $2a \in A$ according to 3.12. Case 3). The interval $(-a, a)$ is a line, hence $(0, 2a)$ is a line, too (see 2.5). According to 3.8 there are three possibilities: 3a) $A_a \subset (0, 2a)$, 3b) $(0, 2a) \subset A_a$, 3c) $x0y$ for each $x \in A_a$ and $y \in (0, 2a)$. In the case 3a) we get that $A' \cup (0, 2a)$ is a convex line containing A, hence $A = A' \cup (0, 2a)$ so that $2a \in A$. 3b) yields $2a \in A$ immediately. Case 3c). From $a0(-a)$ we get $0a(2a)$ hence $a \in (0, 2a)$ and $a \in A_a$ so that $a0a$ i.e. $a = 0$ and trivially $2a \in A$. □

3.14. Let A be a convex maximal line in G and $0 \in A$. Then $a, b \in A$ imply $a + b \in A$.

Proof. There are three possibilities:

1) $0ab$, 2) $0ba$, 3) $a0b$.

In the case 1) we get $b(a+b)(2b)$, $b(b+a)(2b)$. Since $2b \in A$, $a+b$ and $b+a$ belong to A. The case 2) is similar. In the case 3) we get $(-b)(-a-b)(-b)$. Since $-a, -b$ belong to A (see 3.12), $-(a+b) = -a-b \in A$, hence $a+b \in A$ according to 3.12. □

3.15. From 3.12 and 3.14 we get that a convex maximal line in G, containing 0, forms a subgroup of the group $(G; +, 0, -)$ which completes the proof of lemma 3.4 and the proof of theorem 3.3.

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