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ON THE BOUNDEDNESS OF SOLUTIONS OF NONLINEAR SECOND – ORDER DIFFERENTIAL EQUATIONS WITH PARAMETR

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ABSTRACT. This paper establishes sufficient conditions for the boundedness of solutions of a one-parameter differential equation $y'' - q(t)\,y = f(t, y, y'', \mu)$ either on a halfline $\langle t_1, \infty)$ or on R satisfying conditions either $y(t_1) = y(t_2) = 0\,(t_2 > t_1)$ or $y(t_1) = 0$, respectively.

1. INTRODUCTION

We consider the second-order differential equations

$$y'' - q(t)\,y = f_1(t, y, \mu) \tag{1}$$

and

$$y'' - q(t)\,y = f_2(t, y, y', \mu) \tag{2}$$

with $q \in C^0(J)$, $f_1 \in C^0(J \times R \times I)$, $f_2 \in C^0(J \times R^2 \times I)$, $q(t) > 0$ for $t \in J$, where $J \subset R$ is either a halfline $\langle t_1, \infty)$ or R, $I = \langle \alpha, \beta\rangle$ $(-\infty < \alpha < \beta < \infty)$, containing a parameter μ.

For $y \in C^0(J)$ define $\|y\| := \sup\{|y(t)|\,;\, t \in J\}$. If $J = \langle t_1, \infty)$ is a halfline and $t_2 > t_1$ is a number, the problem is considered to determine sufficient conditions on g, f_1, f_2 such that it is possible to choose the parameter μ so that there exists a solution y_1 (y_2) of (1) ((2)) satisfying either the boundary conditions

$$y_1(t_1) = y_1(t_2) = 0 \qquad (y_2(t_1) = y_2(t_2) = 0) \tag{3}$$

or the initial conditions

$$y_1(t_1) = y_1'(t_1) = 0 \qquad (y_2(t_1) = y_2'(t_1) = 0)$$

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and

$$(4) \qquad \|y_1\| < \infty \qquad (\|y_2\| + \|y_2'\| < \infty).$$

If $J = R$ and $t_1 \in R$ is a number, the problem is considered to determine sufficient conditions on q, f_1, f_2 for the existence of a $\mu_0 \in I$ such that equation (1) ((2)) with $\mu = \mu_0$ has a solution y_1 (y_2) satisfying

$$(5) \qquad y_1(t_1) = 0 \qquad (y_2(t_1) = 0)$$

and (4).

It is discussed also the uniqueness of solutions y_1 and y_2 satisfying either (3) (4) for a halfline J or (4), (5) for $J = R$.

By using the technique of the two-point boundary value problem Bebernes and Jackson [1], Belova [2] and Corduneanu [3], [4] have been studied the existence (and uniqueness) of bounded solutions of the equation $y'' = f(x, y)$ and Kiguradze [6] of a system of differential equations either on the halfline $\langle 0, \infty)$ or on R and in the case of the halfline $\langle 0, \infty)$ with the further condition $y(0) = y_0$. In contradiction to them in this paper there are studied second-order differential equations (1) and (2) depending on the parameter μ and using the technique of the three-point boundary value problem it is investigated boundary solutions satisfying the above conditions. The three-point boundary value problem $y(a) = y(b) = y(c) = 0$ only for homogeneous second-order linear differential equations with two parameters has been investigated in [5].

2. Lemmas

Lemma 1. *Let r be a positive constant. If the assumptions*

$$(6)\ |f_1(t, y, \mu)| \le rq(t) \quad \text{for} \quad (t, y, \mu) \in D_1 \times I, \quad \text{where} \quad D_1 := J \times \langle -r, r\rangle,$$

$$(7) \quad f_1(t, y, \cdot) \text{ is an increasing function on } I \text{ for every fixed } (t, y) \in D_1,$$

$$(8) \quad f_1(t, y, \alpha) f_1(t, y, \beta) \leqq 0 \quad \text{for} \quad (t, y) \in D_1,$$

hold, then for any three numbers a, b, $c \in J$, $a < b < c$ there exist μ_0, $\mu_1 \in I$ such that equation (1) with $\mu = \mu_0$ and $\mu = \mu_1$ has a solution y_0 and a solution y_1, respectively, satisfying

$$(9) \qquad \begin{aligned} y_0(a) &= y_0(b) = y_0(c) = 0, \\ y_1(a) &= y_1'(a) = y_1(c) = 0, \end{aligned}$$

and

$$|y_i(t)| \leqq r \quad \text{for} \quad t \in \langle a, c\rangle \quad \text{and} \quad i = 0, 1.$$

For the proof see [7].

Lemma 2. *Let r_1, r_2 be positive constants. If the assumptions*

$$|f_2(t, y_1, y_2, \mu)| \le r_1 q(t) \quad \text{for} \quad (t, y_1, y_2, \mu) \in D_2 \times I, \text{ where}$$

(10) $$D_2 := J \times \langle -r_1, r_1 \rangle \times \langle -r_2, r_2 \rangle,$$

$f_2(t, y_1, y_2, .)$ *is an increasing function on* I *for every fixed*

(11) $$(t, y_1, y_2) \in D_2,$$

(12) $$f_2(t, y_1, y_2, \alpha)\, f_2(t, y_1, y_2, \beta) \leqq 0 \quad \text{for} \quad (t, y_1, y_2) \in D_2,$$

$$2\sqrt{r_1}\sqrt{A_2 + r_1 \|q\|} \leqq r_2, \quad \text{where} \quad A_2 := \sup\{|f_2(t, y_1, y_2, \mu)|;$$

(13) $$(t, y_1, y_2, \mu) \in D_2 \times I\},$$

hold, then for any a, b, $c \in J$, $a < b < c$ there exist μ_0, $\mu_1 \in I$ such that equation (2) with $\mu = \mu_0$ and $\mu = \mu_1$ has a solution y_0 and a solution y_1, respectively, satisfying (9) and

$$|y_j^{(i)}(t)| \leqq r_{i+1} \quad \text{for} \quad t \in \langle a, c \rangle \quad \text{and} \quad i, j = 0, 1.$$

For the proof see [7].

Remark 1. It follows from Lemma 2: Assume $A_2 := \sup\ \{|f_2(t, y_1, y_2, \mu)|;\ (t, y_1, y_2, \mu) \in J \times \langle -r_1, r_1 \rangle \times R \times I\} < \infty$ for a positive constant r_1. If $\|q\| \leqq \infty$ and assumptions (10) – (12) are fulfilled for $D_2 = J \times \langle -r_1, r_1 \rangle \times R$, then for any three numbers a, b, $c \in J$, $a < b < c$ there are μ_0, $\mu_1 \in I$ such that equation (2) with $\mu = \mu_0$ and $\mu = \mu_1$ has a solution y_0 and a solution y_1, respectively, satisfying (9),

$$|y_i(t)| \leqq r_1 \quad \text{for} \quad t \in \langle a, c \rangle, \quad i = 0, 1$$

and, of course, $|y_i'(t)| \leqq 2\sqrt{r_1}\sqrt{A_2 + r_1 \sup\{q(t);\ t \in \langle a, c \rangle\}}$ for $t \in \langle a, c \rangle$, $i = 0, 1$.

3. Boundedness and uniqueness of solutions on halfline

In this part we shall assume that $J = \langle t_1, \infty)$ is a halfline on R and $t_2 \in (t_1, \infty)$ is an arbitrary but fixed number.

Theorem 1. *Assume that assumptions (6) – (8) are fulfilled for a positive constant r. Then there are μ_0, $\mu_1 \in I$ such that equation (1) with $\mu = \mu_0$ and $\mu = \mu_1$ has a solution y_0 and a solution y_1, respectively, satisfying*

(14) $$y_0(t_1) = y_0(t_2) = 0,$$

(15) $$y_1(t_1) = y_1'(t_1) = 0$$

and

(16) $$\|y_i\| \leqq r \quad \text{for} \quad i = 0, 1.$$

If, in additional,

$$\|q\| < \infty, \tag{17}$$

then

$$\|y_i'\| \leqq 2\sqrt{2r(r\|q\| + A_1)} \quad \text{for} \quad i = 0, 1, \tag{18}$$

where $A_1 := \sup \{|f_1(t, y, \mu)| \, ; (t, y, \mu) \in D_1 \times I\} \quad (\leqq r\|q\|)$.

Proof. Let $\{a_n\}$ be an increasing sequence, $a_1 > t_2 \lim_{n\to\infty} a_n = \infty$. Then, by Lemma 1, there is $\{\mu_n\}$, $\mu_n \in I$ such that equation (1) with $\mu = \mu_n$ has a solution y_n satisfying

$$y_n(t_1) = y_n(t_2) = y_n(a_n) = 0 \tag{19}$$

and

$$|y_n(t)| \leqq r \quad \text{for} \quad t \in \langle t_1, a_n \rangle. \tag{20}$$

Setting $Q_n := \max\{q(t); t \in \langle t_1, a_n\rangle\}$, then $|y_n''(t)| \leqq 2rQ_m$ for $t \in \langle t_1, a_m\rangle$, $m \leqq n$. Let $\xi_n \in (t_1, t_2)$ be a such number that $y_n'(\xi_n) = 0$. From the equalities

$$y_n'(t) = \int_{\xi_n}^t y_n''(s)\, ds \quad \text{for} \quad t \in \langle t_1, a_n\rangle, \quad n \in N,$$

we get

$$|y_n'(t)| \leqq 2rQ_m(a_m - t_1) \quad \text{for} \quad t \in \langle t_1, a_m\rangle, \quad m \leqq n.$$

Consequently, $\{y_n^{(i)}(t)\}_{n=k}^{\infty}$ is equicontinuous and uniformly bounded on $\langle t_1, a_k\rangle$ for $k \in N$ and $i = 0, 1$. Thus by the Ascoli's theorem we may choose a "diagonal" subsequence of $\{y_n(t)\}$ which for short we denote again $\{y_n(t)\}$ such that $\{y_n^{(i)}(t)\}$ locally uniformly convergent on J for $i = 0, 1$. Since I is a compact interval without any loss of generality we may assume $\{\mu_n\}$ is a convergent sequence, $\lim_{n\to\infty} \mu_n = \mu_0$. From the equalities

$$y_n''(t) = q(t)y_n(t) + f_1(t, y_n(t), \mu_n) \quad \text{for} \quad t \in \langle t_1, a_n\rangle, \quad n \in N, \tag{21}$$

we see $\{y_n''(t)\}$ is locally uniformly convergent on J and for $y_0(t) := \lim_{n\to\infty} y_n(t)$, $t \in J$, we have $\lim_{n\to\infty} y_n^{(i)}(t) = y_0^{(i)}(t)$ locally uniformly on J for $i = 0, 1, 2$. If we pass to the limit for $n \to \infty$ in (21), we get

$$y_0''(t) = q(t)y_0(t) + f_1(t, y_0(t), \mu_0), \quad t \in J,$$

and therefore y_0 is a solution of (1) with $\mu = \mu_0$ satisfying (14) and (16) for $i = 0$.

Let assumption (17) be satisfied. Then $\|y_0''\| \leqq r\|q\| + A_1$ and from the Landau's inequality $\|y_0'\|^2 \leqq 8\|y_0\| \, \|y_0''\|$ we obtain (18).

The proof of the existence of a solution y_1 having the properties demanded in Theorem 1 is very similar to that above and therefore it is omitted.

Example 1. Let ν, m be positive constants and let $k \in R$. Consider the differential equation

$$(22) \qquad y'' = q(t)\,y + \frac{k}{1+t^2}|y|^\nu + \varphi(t) + \mu$$

with q, $\varphi \in C^0(J)$, $|q(t)| \geqq 2(m+|k|)$, $|\varphi(t)| \leqq m$ for $t \in J$ and $\mu \in \langle -|k|-m, |k|+m\rangle =: I_1$. Equation (22) satisfies the assumptions of Theorem 1 with $r = 1$ and thus there are μ_0, $\mu_1 \in I_1$ such that equation (22) with $\mu = \mu_0$ ($\mu = \mu_1$) has a solution y_0 (y_1) satisfying (14) ((15)) and $\|y_i\| \leqq 1$ for $i = 0, 1$. If, in additional, $\|q\| < \infty$ then with respect to the inequality

$$\sup\left\{ |\frac{k}{1+t^2}|y|^\nu + \varphi(t) + \mu|;\ (t, y, \mu) \in J \times \langle -1, 1\rangle \times I_1 \right\} \leqq 2(m+|k|)$$

we obtain $\|y_i'\| \leqq 2\sqrt{2(\|q\| + 2(m+|k|))}$ for $i = 0, 1$.

Theorem 2. *Let assumptions (10) – (13) be fulfilled for positive constants r_1, r_2. Then there are μ_0, $\mu_1 \in I$ such that equation (2) with $\mu = \mu_0$ and $\mu = \mu_1$ has a solution y_0 and a solution y_1 satisfying (14) and (15), respectively, and*

$$(23) \qquad \|y_j^{(i)}\| \leqq r_{i+1} \quad \text{for} \quad i, j = 0, 1.$$

Proof. Since the proofs of the existence of solutions y_0, y_1 are very similar we shall prove only the existence of y_0. Let $\{a_n\}$ be defined as in the proof of Theorem 1. By Lemma 2 there is a sequence $\{\mu_n\}$, $\mu_n \in I$ such that equation (2) with $\mu = \mu_n$ admits a solution y_n satisfying (19), $|y_n^{(i)}(t)| \leqq r_{i+1}$ for $t \in \langle t_1, a_n\rangle$, $n \in N$, $i = 0, 1$ and $|y_n''(t)| \leqq r_1\|q\| + A_2$ for $t \in \langle t_1, a_n\rangle$, $n \in N$. Using the Ascoli's theorem and the Cauchy's diagonal method we may assume $\{y_n^{(i)}(t)\}$ locally uniformly convergent on J for $i = 0, 1$ and (since I is a compact interval) $\{\mu_n\}$ is a convergent sequence, $\lim_{n\to\infty} \mu_n = \mu_0$. From the equalities

$$y_n''(t) = q(t)\,y_n(t) + f_2(t, y_n(t), y_n'(t), \mu_n), \quad t \in \langle t_1, a_n\rangle, \quad n \in N,$$

we obtain that $\{y_n''(t)\}$ locally uniformly convergent on J. Thus the function y_0, $y_0(t) := \lim_{n\to\infty} y_n(t)$ for $t \in J$, is a solution of (2) with $\mu = \mu_0$ satisfying (14) and (23).

Remark 2. Let $\|q\| \leqq \infty$ and let $\sup\{|f_2(t, y_1, y_2, \mu)|;\ (t, y_1, y_2, \mu) \in J \times \langle -r_1, r_1\rangle \times R \times I\} < \infty$, where r_1 is a positive constant. If assumptions (10) – (12) are fulfilled for r_1 and an arbitrary positive constant r_2 then there exist $\mu_1, \mu_2 \in I$ such that equation (2) with $\mu = \mu_1$ ($\mu = \mu_2$) has a solution y_1 (y_2) such that $y_1(t_1) = y_1(t_2) = 0$, $\|y_1\| \leqq r_1$ ($y_2(t_1) = y_2'(t_1) = 0$, $\|y_2\| \leqq r_1$). This follows immediately from Remark 1 and the proofs of Theorems 1 and 2.

Example 2. Let $\nu > 0$ be a positive constant and let $m > 0$ be a positive integer. The differential equation

$$y'' = q(t)\,y + |y|^{\nu}\sin(y') + \frac{\arctan(t)}{1+(y')^{2m}} + \mu \tag{24}$$

with $q \in C^0(J)$, $q(t) \geqq 2+\pi$ for $t \in J$, $\|q\| \leqq \infty$, where $\mu \in \langle -1-\frac{\pi}{2}, 1+\frac{\pi}{2}\rangle =: I_1$, satisfies assumptions (10) – (12) with $r_1 = 1$ and an arbitrary $r_2 > 0$. Thus by Remark 2 there are μ_1, $\mu_2 \in I_1$ such that equation (24) with $\mu = \mu_1$ ($\mu = \mu_2$) has a solution y_1 (y_2), $y_1(t_1) = y_1(t_2) = 0$, $\|y_1\| \leqq 1$ ($y_2(t_1) = y_2'(t_1) = 0$, $\|y_2\| \leqq 1$). Assume $\|q\| < \infty$. Since $\||y_1|^{\nu}\sin(y_2) + \dfrac{\arctan(t)}{1+y_2^{2m}} + \mu| \leqq 2+\pi$ for $(t, y_1, y_2, \mu) \in J \times \langle -1, 1\rangle \times R \times I_1$, assumption (13) holds for $r_2 = 2\sqrt{2+\pi+\|q\|}$ and thus by Theorem 2 we have $\|y_i'\| \leqq 2\sqrt{2+\pi+\|q\|}$ for $i = 1, 2$.

Theorem 3. *Let r_1, r_2 be positive constants and let*

$$|f_2(t, y_1, y_2, \mu) - f_2(t, z_1, z_2, \mu)| \leqq h_1(t)|y_1 - z_1| + h_2(t)|y_2 - z_2|$$

for (t, y_1, y_2, μ), $(t, z_1, z_2, \mu) \in \langle t_1, t_2\rangle \times \langle -r_1, r_1\rangle \times \langle -r_2, r_2\rangle \times I$, where $h_i \in C^0(\langle t_1, t_2\rangle)$, $i = 1, 2$. Let the initial problem (2), $y^{(i)}(t_0) = \lambda_i$ has the (locally) unique solution for all $t_0 \in \langle t_2, \infty)$ and $|\lambda_i| \leqq r_{i+1}$ ($i = 0, 1$). Moreover, assume that at least one from the following conditions

$$\int_{t_1}^{t_2} \exp\int_{t_1}^{s} h_2(\nu)\,d\nu)\int_{t_1}^{s}(q(\tau)+h_1(\tau))\,d\tau\,ds \leqq 1,$$

$$\int_{t_1}^{t_2}[(q(s)+h_1(s))\,(s-t_1)+h_2(s)]\,ds \leqq 1$$

$$\int_{t_1}^{t_2}(\exp\int_{s}^{t_2} h_2(\nu)\,d\nu)\int_{s}^{t_2}(q(\tau)+h_1(\tau))\,d\tau\,ds \leqq 1$$

$$\int_{t_1}^{t_2}[(q(s)+h_1(s))\,(t_2-s)+h_2(s)]\,ds \leqq 1,$$

holds.

If there exists a $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution y_0 satisfying (14) and (23), then this solution is unique in the set $\{y : y \in C^2(J),\ \|y^{(i)}\| \leqq r_{i+1}$ for $i = 0, 1\}$.

Proof. If y_1 is a further solution of (2) with $\mu = \mu_0$, $y_1(t_1) = y_1(t_2) = 0$, $\|y_1^{(i)}\| \leqq r_{i+1}$ ($i = 0, 1$) then analogous to [7] we may prove $y_0(t) = y_1(t)$ for $t \in \langle t_1, t_2\rangle$. The locally uniqueness of solutions implies $y_0(t) = y_1(t)$ for $t \in J$.

Corollary 1. *Let*

$$|f_1(t,y,\mu) - f_1(t,z,\mu)| \leqq h(t)|y-z|$$

for $(t,y,\mu), (t,z,\mu) \in \langle t_1, t_2\rangle \times \langle -r, r\rangle \times I$, *where* $h \in C^0(\langle t_1, t_2\rangle)$, *be satisfied for a positive constant* r. *Let the initial problem (1),* $y^{(i)}(t_0) = \lambda_i$ *has the (locally) unique solution for all* $t_0 \in \langle t_2, \infty)$, $|\lambda_0| \leqq r$ *and* $\lambda_1 \in R$. *Finally, let at least one from the following conditions*

$$\int_{t_1}^{t_2}\int_{t_1}^{s} (q(\tau) + h(\tau))\,d\tau\,ds \leqq 1,$$

$$\int_{t_1}^{t_2} (q(s) + h(s))\,(s - t_1)\,ds \leqq 1,$$

$$\int_{t_1}^{t_2}\int_{s}^{t_2} (q(\tau) + h(\tau))\,d\tau\,ds \leqq 1,$$

$$\int_{t_1}^{t_2} (q(s) + h(s))\,(t_2 - s)\,ds \leqq 1,$$

be satisfied.

If for a $\mu_0 \in I$ *equation (1) with* $\mu = \mu_0$ *has a solution* y_0 *satisfying (14) and (16) then this solution is unique in the set* $\{y;\, y \in C^2(J),\, \|y\| \leqq r\}$.

Lemma 3. *Let assumption (11) be fulfilled for positive constants* r_1, r_2 *and let* $\dfrac{\partial f_2}{\partial y_1}, \dfrac{\partial f_2}{\partial y_2} \in C^0(D_2 \times I)$. *Assume*

$$q(t) + \frac{\partial f_2}{\partial y_1}(t, y_1, y_2, \mu) \geqq m \quad \text{for} \quad (t, y_1, y_2, \mu) \in D_2 \times I, \tag{25}$$

where $m \geqq 0$ *is a non-negative constant and*

$$(L :=) \inf\{\frac{\partial f_2}{\partial y_2}(t, y_1, y_2, \mu);\quad (t, y_1, y_2, \mu) \in D_2 \times I\} > -\infty. \tag{26}$$

If at least one from the conditions

$$m > 0, \tag{27}$$

$$(K :=) \inf\left\{\int_z^t p(s)\,ds;\quad t_2 \leqq z \leqq t\right\} > -\infty, \quad \text{where } p(t) =$$

$$= \min\{\frac{\partial f_2}{\partial y_2}(t, y_1, y_2, \mu);\quad (y_1, y_2, \mu) \in \langle -r_1, r_1\rangle \times \langle -r_2, r_2\rangle \times I\} \tag{28}$$
$$\text{for} \quad t \in \langle t_2, \infty),$$

$$\inf\ \{|f_2(t, y_1, y_2, \mu_1) - f_2(t, y_1, y_2, \mu_2)|;\quad (t, y_1, y_2) \in D_2\} > 0 \tag{29}$$
$$\text{for} \quad \mu_1, \mu_2 \in I, \mu_1 \neq \mu_2,$$

holds, then there is at most one $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution y_0 satisfying (14) and (23). In the positive case the solution y_0 is unique in the set $\{y;\ y \in C^2(J),\ \|y^{(i)}\| \leqq r_{i+1},\quad i = 0,1\}$.

Proof. Assume y_1 and y_2 are solutions of (2) with $\mu = \mu_1$ and $\mu = \mu_2$, respectively, $\mu_1, \mu_2 \in I$, $\mu_1 \leqq \mu_2$, $y_j(t_1) = y_j(t_2) = 0$, $\|y_j^{(i)}\| \leqq r_{i+1}$ for $i = 0,1$ and $j = 1,2$. Putting $w = y_1, y_2$ then

$$\begin{aligned} w''(t) = q(t)w(t) &+ (f_2(t, y_1(t), y_1'(t), \mu_1) - f_2(t, y_2(t), y_1'(t), \mu_1)) + \\ &+ (f_2(t, y_2(t), y_1'(t), \mu_1) - f_2(t, y_2(t), y_2'(t), \mu_1)) + \\ &+ (f_2(t, y_2(t), y_2'(t), \mu_1) - f_2(t, y_2(t), y_2'(t), \mu_2)), \end{aligned}$$

consequently,

$$w''(t) = (q(t) + g(t))w(t) + h(t)w'(t) + a(t) \quad \text{for} \quad t \in J, \tag{30}$$

where $g, h, a \in C^0(J)$, $q(t) + g(t) \geqq m$ ($\geqq 0$) (by (25)), $h(t) \geqq L$ (by (26)) and $a(t) \leqq 0$ (by(11)) for $t \in J$. If $\mu_1 < \mu_2$ ($\mu_1 = \mu_2$) then $a(t) < 0$ ($a(t) = 0$) for $t \in J$.

Let $\mu_1 = \mu_2$. Since $q(t) + g(t) \geqq 0$ for $t \in J$, the equatin $y'' = (q(t) + g(t))y + h(t)y'$ is disconjugate on J and thus $w = 0$.

Let $\mu_1 < \mu_2$ and let $w(\tau) = 0$, $w'(\tau) \leqq 0$ for some $\tau \in \langle t_1, t_2)$. If $w'(\tau) = 0$ then using (30) we get $w''(\tau) < 0$ and thus $w(t) < 0$, $w'(t) < 0$ in a right neighbourhood of the point τ, likewise as in the case, when $w'(\tau) < 0$. Since $w''(\xi) < 0$ in any point $\xi \in (\tau, \infty)$ where $w(\xi) \leqq 0$, $w'(\xi) = 0$, we obtain $w(t) < 0$, $w'(t) < 0$ on (τ, ∞) which contradicting $w(t_2) = 0$. Consequently, $w(t) < 0$, $w'(t) < 0$ for $t > t_2$. Next, from (30) we get equality

$$\begin{aligned} w(t) = \int_{t_2}^{t} (\exp \int_{t_2}^{s} h(\tau)\, d\tau) \big[w'(t_2) + \\ + \int_{t_2}^{s} (\exp(-\int_{t_2}^{\tau} h(\nu)\, d\nu)) ((q(\tau) + g(\tau))w(\tau) + a(\tau))\, d\tau \big] ds,\ t \in J, \end{aligned}$$

and thus

$$w(t) \leqq \int_{t_2}^{t} \int_{t_2}^{s} (\exp \int_{\tau}^{s} h(\nu)\, d\nu) ((q(\tau) + g(t))\, w(\tau) + a(\tau))\, d\tau\, ds, \quad t \geqq t_2. \tag{31}$$

If $m > 0$ then for some $t_3, t_3 > t_2$ we obtain

$$w(t) < m \int_{t_3}^{t} \int_{t_3}^{s} (\exp \int_{\tau}^{s} h(\nu)\, d\nu)\, w(\tau) d\tau\, ds \leqq m w(t_3) \int_{t_3}^{t} \int_{t_3}^{s} \exp(L(s - \tau))\, d\tau\, ds$$

for $t > t_3$ and since $\int_{t_3}^{\infty} \int_{t_3}^{s} \exp(L(s-\tau))\, d\tau\, ds = \infty$ we have $\lim_{t\to\infty} w(t) = -\infty$.

If $K > -\infty$, then using (31) we have

$$w(t) \leqq \int_{t_2}^{t}\int_{t_2}^{s}(\exp\int_{\tau}^{s} h(\nu)\,d\nu)\,a(\tau)\,d\tau\,ds \leqq e^K \int_{t_2}^{t}\int_{t_2}^{s} a(\tau)\,d\tau\,ds$$

and since $\int_{t_2}^{\infty}\int_{t_2}^{s} a(\tau)\,d\tau\,ds = -\infty$, we get $\lim_{t\to\infty} w(t) = -\infty$.

If $a(t) \leqq A < 0$ for $t \geqq t_2$, where A is a negative constant, then

$$w(t) \leqq A\int_{t_2}^{t}\int_{t_2}^{s}(\exp\int_{\tau}^{s} h(\nu)\,d\nu\,ds \leqq A\int_{t_2}^{t}\int_{t_2}^{s}\exp(L(s-\tau))\,d\tau\,ds$$

and $\lim_{t\to\infty} w(t) = -\infty$.

Thus we see if at least one from conditions (27)– (29) is fulfilled then $\lim_{t\to\infty} w(t) = -\infty$ contradicting $\|w\| \leqq 2r_1$. This completes the proof.

Corollary 2. *Assume assumption (7) is fulfilled for a positive constant r, $\frac{\partial f_1}{\partial y} \in C^0(D_1 \times I)$ and*

$$q(t) + \frac{\partial f_1}{\partial y}(t, y, \mu) \geqq 0 \qquad \text{for} \qquad (t, y, \mu) \in D_1 \times I. \tag{32}$$

Then there is at most one $\mu_0 \in I$ such that equation (1) with $\mu = \mu_0$ has a solution y satisfying (14) and (16). In the positive case y is unique in the set $\{y;\ y \in C^2(J),\ \|y\| \leqq r\}$.

Theorem 4. *Let assumptions (10) – (13) be satisfied for positive constants r_1, r_2 and let $\frac{\partial f_2}{\partial y_1}, \frac{\partial f_2}{\partial y_2} \in C(D_2 \times I)$. If assumptions (25), (26) and at least one from conditions (27) – (29) hold, then there are unique μ_0, $\mu_1 \in I$ such that equation (2) with $\mu = \mu_0$ and $\mu = \mu_1$ has a solution y_0 and a solution y_1 satisfying (14) and (15), respectively, and (23). This solutions are unique in the set $\{y;\ y \in C^2(J),\ \|y^{(i)}\| \leqq r_{i+1},\quad i = 0, 1\}$.*

Proof. The proof folows from Theorem 2 and Lemma 3 (for y_1 with an evident modification of the proof of Lemma 3).

Theorem 5. *Let assumptions (6) – (8) be satisfied for a positive constant r and let $\frac{\partial f}{\partial y} \in C^0(D_1 \times I)$. If assumption (32) is satisfied, then there are unique μ_0, $\mu_1 \in I$ such that equation (1) with $\mu = \mu_0$ and $\mu = \mu_1$ has a solution y_0 and a solution y_1 satisfying (14) and (15), respectively, and (16). This solutions are unique in the set $\{y;\ y \in C^2(J),\ \|y\| \leqq r\}$.*

Proof. The proof follows from Theorem 1 and Corollary 2 (for y_1 with an evident modification of the proof of Lemma 3).

Example 3. Consider the differential equation

$$y'' - (\exp(|\sin(t)| - 1))y = t^{-5}\cos(e^{-1}y) + t^{-1}\arctan(y') + \mu \tag{33}$$

on the interval $J := \langle 1, \infty)$, where $\mu \in I := \langle -1 - \frac{\pi}{2}, 1 + \frac{\pi}{2}\rangle$. Assume $t_2 \in (1, \infty)$ and r_1, r_2 are positive constants, $r_1 \geqq (2+\pi)e$, $r_2 \geqq 3r_1$. It is easy to verify that assumptions (10) – (13), (29), (25) with $m = 0$ and (26) with $L = 1$ are fulfilled. Therefore by Theorem 4 there are unique μ_0, $\mu_1 \in I$ such that equation (33) with $\mu = \mu_0$ ($\mu = \mu_1$) has a solution y_0 (y_1) satisfying $y_0(1) = y_0(t_2) = 0$, $y_1(1) = y_1'(1) = 0$ and $\|y_j\| \leqq (2+\pi)e, \|y_j'\| \leqq 3e(2+\pi)$ for $j = 0, 1$. This solutions y_0, y_1 are unique even in the set $\{y; y \in C^2(J), \|y\| + \|y'\| < \infty\}$.

4. Boudedness and uniqueness of solutions on R

In this part we shall assume $J = R$ and $t_1 \in R$ is arbitrary but fixed number.

Theorem 6. *Let assumptions (6) – (8) be fulfilled for a positive constant r. Then there is a $\mu_0 \in I$ such that equation (1) with $\mu = \mu_0$ has a solution y satisfying*

$$y(t_1) = 0 \tag{34}$$

and

$$\|y\| \leqq r. \tag{35}$$

If, in additional,

$$\|q\| < \infty, \tag{36}$$

then

$$\|y'\| \leqq 2\sqrt{(r\|q\| + A_1)r}, \tag{37}$$

where $A_1 = \sup\{|f_1(t, y, \mu)|;\ (t, y, \mu) \in D_1 \times I\}$.

Proof. Let $\{a_n\}$ be a decreasing sequence and let $\{b_n\}$ be an increasing sequence, $\lim_{n\to\infty} a_n = -\infty$, $\lim_{n\to\infty} b_n = \infty$, $a_1 < t_1 < b_1$. By Lemma 1 there is a sequence $\{\mu_n\}, \mu_n \in I$ such that equation (1) with $\mu = \mu_n$ has a solution $y_n, y_n(a_n) = y_n(t_1) = y_n(b_n) = 0$ and $|y_n(t)| \leqq r$ for $t \in \langle a_n, b_n\rangle$, $n \in N$. Next we have $|y_n''(t)| \leqq 2rQ_n$ for $t \in \langle a_n, b_n\rangle$, $n \in N$, where $Q_n = \max\{q(t);\ t \in \langle a_n, b_n\rangle\}$. From the mean value theorem follows the existence of a $\xi_n \in (a_1, b_1)$ such that $y_n(b_1) - y_n(a_1) = y_n'(\xi_n)(b_1 - a_1)$, consequently $|y_n'(\xi_n)| \leqq \dfrac{2r}{b_1 - a_1}$ and the equality $y_n'(t) = y_n'(\xi_n) + \int_{\xi_n}^t y_n''(s)\,ds$ implies

$$|y_n'(t)| \leqq \frac{2r}{b_1 - a_1} + 2Q_m r(b_m - a_m) \quad \text{for} \quad t \in \langle a_m, b_m\rangle, \quad m \leqq n.$$

Using the Ascoli's theorem and the Cauchy diagonal method we may choose a subsequence of $\{y_n(t)\}$, for short we denote this subsequence again $\{y_n(t)\}$, such that $y(t) := \lim_{n\to\infty} y_n(t)$ locally uniformly on R. Since I is a compact interval we may assume that $\{\mu_n\}$ is a convergent sequence and $\lim_{n\to\infty} \mu_n = \mu_0$. Analogous to the proof of Theorem 1 it is posible to prove that y is a solution of (1) with $\mu = \mu_0$ having properties (34) and (35).

If (36) holds then from the Landau's inequelity $\|y'\|^2 \leqq 4\|y\|\|y''\|$ and using the inequality $\|y''\| \leqq r\|q\| + A_1$ we obtain (37).

Theorem 7. *Let assumptions (10) – (13) be satisfied for positive constants r_1, r_2. Then there exist a $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution y satisfying (34) and*

$$\|y^{(i)}\| \leqq r_{i+1} \quad \text{for} \quad i = 0, 1. \tag{38}$$

Proof. Let $\{a_n\}$, $\{b_n\}$ be defined as in the proof of Theorem 6. Then by Lemma 2 there is a sequence $\{\mu_n\}$, $\mu_n \in I$ such that equation (2) with $\mu = \mu_n$ has a solution y_n, $y_n(a_n) = y_n(t_1) = y_n(b_n) = 0$ and $|y_n^{(i)}(t)| \leqq r_{i+1}$ for $t \in \langle a_n, b_n\rangle, i = 0, 1$ and $n \in N$. Since $|y_n''(t)| \leqq r_1\|q\| + A_2$ for $t \in \langle a_m, b_m\rangle$ and $m \leqq n$, the next part of the proof is analogous to that of Theorem 2 and therefore it is omitted.

Theorem 8. *Let assumptions (10) – (13) be satisfied for positive constants r_1, r_2. Assume that $\frac{\partial f_2}{\partial y_1}, \frac{\partial f_2}{\partial y_2} \in C^0(D_2 \times I)$,*

$$q(t) + \frac{\partial f_2}{\partial y_1}(t, y_1, y_2, \mu) \geqq 0 \quad \text{for} \quad (t, y_1, y_2, \mu) \in D_2 \times I \tag{39}$$

and

$$(K_1 :=) \ \inf\left\{-\int_s^{t_1} p_1(\tau)\,d\tau; s \leqq t_1\right\} > -\infty,$$

$$(K_2 :=) \ \inf\left\{\int_s^{t} p_2(\tau)\,d\tau; t_1 \leqq s \leqq t\right\} > -\infty, \tag{40}$$

where $p_1(t) = \max\left\{\frac{\partial f_2}{\partial y_2}(t, y_1, y_2, \mu); (y_1, y_2, \mu) \in \langle -r_1, r_1\rangle \times \langle -r_2, r_2\rangle \times I\right\}$ for $t \in (-\infty, t_1)$ and $p_2(t) = \min\left\{\frac{\partial f_2}{\partial y_2}(t, y_1, y_2, \mu);\ (y_1, y_2, \mu) \in \langle -r_1, r_1\rangle \times \langle -r_2, r_2\rangle \times I\right\}$ for $t \in \langle t_1, \infty)$.

Then there is the unique $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution y satisfying (34) and (38). This solution is unique in the set $\{y; y \in C^2(R), \|y^{(i)}\| \leq r_{i+1}$ for $i = 0, 1\}$.

Proof. By Theorem 7 there is some $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution y satisfying (34) and (38). Suppose that there is some $\mu_1 \in I$, $\mu_0 \leqq \mu_1$, such that equation (2) with $\mu = \mu_1$ has a solution y_1, $y_1(t_1) = 0$, $\|y_1^{(i)}\| \leqq r_{i+1}$ for $i = 0, 1$. Setting $w = y - y_1$ then

$$w''(t) = (q(t) + g(t))w(t) + h(t)w'(t) + a(t) \quad \text{for} \quad t \in R, \tag{41}$$

where $a, g, h \in C^0(R)$, $q(t) + g(t) \geqq 0$ (by(39)), $a(t) \leqq 0$ (by (11)) for $t \in R$, $\inf\left\{-\int_s^{t_1} h(\tau)\,d\tau;\ s \leqq t_1\right\} \geqq K_1$, $\inf\left\{\int_s^t h(\tau)\,d\tau;\ t_1 \leqq s \leqq t\right\} \geqq K_2$ (by(40)) and if $\mu_0 < \mu_1$ ($\mu_0 = \mu_1$) then $a(t) < 0$ ($a(t) = 0$) for $t \in R$. Using (41) we have

$$\begin{aligned} w(t) = &\int_{t_1}^t (\exp \int_{t_1}^s h(\nu)\,d\nu)\big[w'(t_1) + \\ &+ \int_{t_1}^s (\exp(-\int_{t_1}^\tau h(\nu)\,d\nu)\,((q(\tau) + g(\tau))w(\tau) + a(\tau))\,d\tau\big]\,ds, \quad t \in R \end{aligned} \tag{42}$$

and

$$\begin{aligned} w'(t) = &(\exp \int_{t_1}^t h(\nu)\,d\nu)\big[w'(t_1) + \\ &\int_{t_1}^t (\exp(-\int_{t_1}^s h(\nu)\,d\nu))\,((q(s) + g(s))w(s) + a(s))\,ds\big], \quad t \in R. \end{aligned} \tag{43}$$

Let $w'(t_1) < 0$. Then from (42) and (43) we get $w(t) < 0$, $w'(t) < 0$ for $t \in (t_1, \infty)$, consequently,

$$w(t) \leqq w'(t_1) \int_{t_1}^t (\exp \int_{t_1}^s h(\nu)\,d\nu)\,ds \leqq w'(t_1) \exp(K_2)(t - t_1) \quad \text{for} \quad t \geqq t_1$$

and thus $\lim_{t\to\infty} w(t) = -\infty$ contradicting

$$\|w\| \leqq 2r_1. \tag{44}$$

Let $w'(t_1) > 0$. Then from (42) and (43) it follows $w(t) < 0$, $w'(t) > 0$ for $t \in (-\infty, t_1)$, consequently,

$$w(t) \leqq -w'(t_1) \int_t^{t_1} (\exp(-\int_s^{t_1} h(\nu)\,d\nu))\,ds \leqq -w'(t_1) \exp(K_1)\,(t_1 - t), \quad t \leqq t_1$$

and thus $\lim_{t\to-\infty} w(t) = -\infty$ contradicting (44).

Let $w'(t_1) = 0$. If $\mu_0 = \mu_1$ then $a(t) = 0$ for $t \in R$ and $w = 0$ by virtue of the uniqueness of the initial value problem for the equation $y'' = (q(t)+g(t))y+h(t)y'$. If $\mu_0 < \mu_1$ then $a(t) < 0$ on R and from (41) it follows $w(t) < 0$, $w'(t) < 0$ for $t \in (t_1, \infty)$. Consequently,

$$\begin{aligned} w(t) = &\int_{t_1}^t \int_{t_1}^s (\exp \int_\tau^s h(\nu)\,d\nu((q(\tau) + g(\tau))w(\tau) + a(\tau))\,d\tau\,ds \leqq \\ &\leqq \exp(K_2) \int_{t_1}^t \int_{t_1}^s a(\tau)\,d\tau\,ds \end{aligned}$$

and since $\int_{t_1}^\infty \int_{t_1}^s a(\tau)\,d\tau\,ds = -\infty$ we obtain $\lim_{t\to\infty} w(t) = -\infty$ contradicting (44). This completes the proof of the theorem.

Corollary 4. *Let assumptions (6) – (8) be fulfilled for a positive constant* r. *Assume that* $\frac{\partial f_1}{\partial y} \in C^0(D_1 \times I)$ *and*

$$q(t) + \frac{\partial f_1}{\partial y}(t, y, \mu) \geqq 0 \quad \text{for} \quad (t, y, \mu) \in D_1 \times I.$$

Then there is the unique $\mu_0 \in I$ *such that equation (1) with* $\mu = \mu_0$ *has a solution* y *satisfying (34) and (35). This solution* y *is unique in the set* $\{y;\ y \in C^2(R),\ \|y\| \leqq r\}$.

Example 4. Consider the differential equation

$$(45) \qquad y'' - q(t)\,y = \exp(-y^2)\sin(t) + k \cdot \exp(-|t|)\ln(1 + (y')^2) + \mu p(t),$$

where $p,\ q \in C^0(R), 1 \leqq p(t) \leqq 2,\ 8 \leqq q(t) \leqq 13$ for $t \in R, \mu \in \langle -8, 8\rangle =: I$ and $k \in R, |k| \leqq 1$. Let $t_1 \in R$. Assumptions (10) – (13) hold with $r_1 = 3$ and $r_2 = 31$. Putting $f_2(t, y_1, y_2, \mu) := \exp(-y_1^2)\sin(t) + k \cdot \exp(-|t|)\ln(1 + y_2^2) + \mu p(t)$ for $(t, y_1, y_2, \mu) \in R^3 \times I$, we have $\frac{\partial f_2}{\partial y_1}(t, y_1, y_2, \mu) \geqq -6$, $q(t) + \frac{\partial f_2}{\partial y_1}(t, y_1, y_2, \mu) \geqq 2$ for $(t, y_1, y_2, \mu) \in R \times \langle -3, 3\rangle \times \langle -31, 31\rangle \times I$, $|\frac{\partial f_2}{\partial y_2}(t, y_1, y_2, \mu)| \leqq \exp(-|t|)$ for $(y_1, y_2, \mu) \in \langle -3, 3\rangle \times \langle -31, 31\rangle \times I$, $t \in R$ and since $\int_s^t \exp(-|\tau|)\,d\tau \leqq 2$ for $s \leqq t$, assumption (40) holds. By Theorem 8 there is the unique $\mu_0 \in I$ such that equation (45) with $\mu = \mu_0$ has a solution y satisfying $y(t_1) = 0, \|y\| \leqq 3, \|y'\| \leqq 31$. This solution y is unique in the set $\{y;\ y \in C^2(R),\ \|y\| \leqq 3,\ \|y'\| \leqq 31\}$.

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