Antoni Chronowski

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ON EXTENSIONS OF HOMOMORPHISMS AND HOMOTOPIES OF COMMUTATIVE n–ADIC GROUPS

Antoni Chronowski

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Abstract. This paper is mainly concerned with two topics: some properties of retracts of n-adic groups and extensions of homomorphisms and homotopies commutative divisible n-adic groups.

In the paper we deal with some theorems on extensions of homomorphisms and homotopies of commutative n-adic groups under additional assumptions of divisibility of n-adic groups. The definitions, theorems, and notations related to the n-adic group theory are based on papers [1], [2], [3], [5], [7], [8]. The symbol a denotes a skew element in an n-adic group.

According to the Hosszú theorem (cf. [5]) for an arbitrary n-adic group $A(\ )$ there exist a binary group $(A, \cdot)$, an automorphism $\alpha \in \mathrm{Aut}(A, \cdot)$, and an element $a \in A$ such that $\alpha(a) = a$, $\alpha^{n-1}(x) = a\, x\, a^{-1}$ for every $x \in A$, and $(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \ldots \alpha^{n-2}(x_{n-1}) \cdot a \cdot x_n$ for all $x_1, x_2, x_3, \dots, x_{n-1}, x_n \in A$.

The system $(A, \cdot, \alpha, a)$ is said to be a binary retract of the n-adic group $A(\ )$ (cf. [3]). For the sake of simplicity a binary retract we shall call a retract and often treat it as a group. Insted of $(A, \cdot, \alpha, a)$ we shall also write (A, α, a).
The retract can be used to the construction of n-adic groups (cf. [5]).

Notice that if $(A, \cdot, \alpha, a)$ and $(A, \cdot, \alpha_1, a_1)$ are retracts (with the same operation $\cdot$) of an n-adic group $A(\ )$, then $\alpha = \alpha_1$ and $a = a_1$. Ideed, $(x_1, x_2, \dots, x_{n-1}, x_n) = x_1 \cdot \alpha(x_2) \cdot \ldots \cdot \alpha^{n-2}(x_{n-1}) \cdot a \cdot x_n$ and $(x_1, x_2, \dots, x_{n-1}, x_n) = x_1 \cdot \alpha_1(x_2) \cdot \ldots \cdot \alpha_1^{n-2}(x_{n-1}) \cdot a_1 \cdot x_n$ for all $x_1, x_2, \dots, x_{n-1}, x_n \in A$. Putting $x_1 = 1, x_3 = 1, \dots, x_{n-1} = 1, x_n = 1$ we get $\alpha(x_2) \cdot a = \alpha_1(x_2) \cdot a_1$ for an arbitrary $x_2 \in A$. For $x_2 = 1$ we have $a = a_1$, hence $\alpha = \alpha_1$.

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Sokolov (cf. [8]) gave a very useful method of constructing a retract $(A, \circ, \alpha, a)$ for an n-adic group $A(\ )$. Namely,

$$x \circ y = (x, p^{n-2}, y),$$
$$\alpha(x) = (\bar{p}, x, p^{n-2}),$$
$$a = (\bar{p}^n)$$

for an arbitrary fixed element $p \in A$ and for all $x, y \in A$. The set A with the operation $\circ$ forms a group for which $\bar{p}$ is an identity.

We shall present a few remarks on the Sokolov method of constructing retracts because it will play an important role in our considerations.

Proposition 1. *A retract (A, α, a) of an n-adic group $A(\ )$ can be constructed by means of the Sokolov method if and only if there exists an element $p \in A$ such that*

(a) $p^{n-2} = a^{-1}$,
(b) $\alpha(p) = p$.

Proof. Assume that the retract (A, α, a) can be constructed by means of the Sokolov method. Then there exists an element $p \in A$ such that $xy = (x, p^{n-2}, y) = x\,\alpha(p)\,\alpha^2(p)\ldots\alpha^{n-2}(p)\,ay$ and $\alpha(x) = (\bar{p}, x, p^{n-2})$ for every $x, y \in A$. Notice that $\alpha(p) = (\bar{p}, p^{n-1}) = p$. Setting $x = y = 1$ in the above equality we have $p^{n-2} = a^{-1}$.

Assume that conditions (a) and (b) are fulfilled. Then $x \circ y = (x, p^{n-2}, y) = x\alpha(p)\,\alpha^2(p)\ldots\alpha^{n-2}(p)\,ay = xp^{n-2}ay = xy$ for every $x, y \in A$. Thus the retract $(A, \circ, \alpha, a)$ constructed by means of the Sokolov method with respect to the element p is identical with the retract (A, α, a). □

Proposition 1 yields immediately the following

Corollary 1. *Every retract (A, α, a) of a 3-adic group $A(\ )$ can be constructed by means of the Sokolov method.*

From the proof of Proposition 1 we obtain the folowing

Corollary 2. *A retract (A, α, a) of an n-adic group $A(\ )$ can be constructed by means of the Sokolov method with respect to an element $p \in A$ if and only if the following conditions are satisfied*

(a) $p^{n-2} = a^{-1}$,
(b) $\alpha(p) = p$.

Proposition 2. *For every natural number $n > 3$ there exist n-adic groups and their retracts which cannot be constructed by means the Sokolov method.*

Proof. Consider the group $(Z_{n-2}, +)$ of integers modulo $n - 2$ for $n > 3$. In the set Z_{n-2} we define the n-ary operation as follows:

$$(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n + 1$$

for all $x_1, x_2, \ldots, x_n \in Z_{n-2}$.
$Z_{n-2}(\ )$ is an n-adic group for which $(Z_{n-2}, +, id_{Z_{n-2}}, 1)$ is a retract. From Proposition 1 we immediately deduce that the retract $(Z_{n-2}, +, id_{Z_{n-2}}, 1)$ cannot be constructed by means of the Sokolov method.

If an n-adic group $A(\ )$ is commutative, then every retract of $A(\ )$ is of the form (A, id_A, a) for a certain $a \in A$.
Indeed, let (A, α, a) be a retract of a commutative n-adic group $A(\ )$.
Then $(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = x_1\alpha(x_2)\,\alpha^2(x_3)\ldots\alpha^{n-2}(x_{n-1})\,ax_n$ for all $x_1, x_2, x_3\ldots, x_{n-1}, x_n \in A$. Taking $x_3 = x_4 = \cdots = x_{n-1} = 1$ and $x_n = a^{-1}$ we obtain $(x_1, x_2, 1, \ldots, 1, a^{-1}) = (x_2, x_1, 1, \ldots, 1, a^{-1})$ for every $x_1, x_2 \in A$. Hence $x_1\alpha(x_2) = x_2\alpha(x_1)$ for all $x_1, x_2 \in A$. Putting $x_2 = 1$ we get $\alpha(x_1) = x_1$ for every $x_1 \in A$.

Now we pass on to the definition and some properties of devisible n-adic groups. For the sake of the uniform notation we shall use the multiplicative notation, also for the divisible groups (cf. [4]) and so instead of the symbol nx we shlal write x^n for $n \in N$.

Let $A(\ )$ be an n-adic group. We begin with the following inductive definition:
(i) $(x, p^{n-2}, x)^{(0)} = x$
(ii) $(x, p^{n-2}, x)^{(k+1)} = ((x, p^{n-2}, x)^{(k)}, p^{n-2}, x)$
for arbitrary $p, x \in A$ and $k \in N_0$.

We say that an n-adic group $A(\ )$ is divisible by a natural number $k \in N$ if

$$\bigvee_{p\in A} \bigvee_{y\in A} \bigvee_{x\in A} (x, p^{n-2}, x)^{(k-1)} = y\,. \tag{1}$$

Theorem 1. *An n-adic group $A(\ )$ is a divisible by a natural number $k \in N$ if and only if there exists a retract of the n-adic group divisible by the number k.*

Proof. (i) Assume that an n-adic group $A(\ )$ is divisible by $k \in N$ i.e. condition (1) holds. We construct the retract $(A, \circ, \alpha, a)$ of $A(\ )$ by means of the Sokolov method with respect to the element $p \in A$ fulfilling condition (1):

$$\begin{aligned} x \circ y &= (x, p^{n-2}, y), \\ \alpha(x) &= (\bar{p}, x, p^{n-2}), \\ a &= (\bar{p}^n) \end{aligned}$$

for arbitrary $x, y \in A$.
By condition (1) we get

$$\bigwedge_{y\in A} \bigvee_{x\in A} x^k = y\,,$$

where $x^k = x \circ x \circ \cdots \circ x$ (k times).
Thus, the retract $(A, \circ, \alpha, a)$ is a group divisible by $k \in N$.

(ii) Assume that there exists a retract (A, α, a) of the n-adic group $A(\ )$ which is a group divisible by $k \in N$. Let $p \in A$ be an arbitrary fixed element. We construct the retract $(A, \circ, \beta, b)$ of $A(\ )$ by means of the Sokolov method putting

$$x \circ y = (x, p^{n-2}, y),$$
$$\beta(x) = (\bar{p}, x, p^{n-2}),$$
$$b = (\bar{p}^n)$$

for all $x, y \in A$.
Since the retracts (A, α, a) and $(A, \circ, \beta, b)$ are isomorphic, the retract $(A, \circ, \beta, b)$ is a group divisible k, i.e.

$$\bigwedge_{y \in A} \bigvee_{x \in A} x^k = y\ ,$$

where $x^k = x \circ x \circ \cdots \circ x$ (k times).
In virtue of this condition we get condition (1) and so $A(\ )$ is an n-adic group divisible by k. □

Since all retracts of an n-adic group $A(\ )$ are isomorphic, Theorem 1 implies the following corollaries.

Corollary 3. *An n-adic group $A(\ )$ is divisible by a natural number $k \in N$ if and only if all the retracts of $A(\ )$ are groups divisible by the number k.*

Corollary 4. *Let $A(\ )$ be an n-adic group. Condition (1) is equivalent to the following condition:*

$$\bigwedge_{p \in A} \bigwedge_{y \in A} \bigvee_{x \in A} (x, p^{n-2}, x)^{(k-1)} = y\ .$$

Taking into account Proposition 1 we get tha following

Corollary 5. *If a commutative n-adic group $A(\ )$ $(n > 3)$ is divisible by $n-2$, then every retract of $A(\ )$ can be constructed by means the Sokolov method.*

If an n-adic group $A(\ )$ is divisible by every natural number $k \in N$, then $A(\ )$ is called divisible.

In virtue of Theorem 1 and Corollary 3 we obtain

Theorem 2. *An n-adic group $A(\ )$ is divisible if and only if there exists a retract of the n-adic group $A(\ )$ which is a divisible group.*

It follows from the foregoing that the following statement is valid.

Corollary 6. *An n-adic group $A(\ )$ is divisible if and only if the tretracts of $A(\ )$ are divisible groups.*

Now we pass on to the extensions of the homomorphisms and the homotopies of the commutative n-adic groups.

We begin with the following theorem.

Theorem 3. *Let $f : A \to B$ be an epimorphism of a divisible n-adic group $A(\ )$ onto n-adic group $B[\ ]$. Then $B[\ ]$ is a divisible n-adic group.*

To prove this theorem it is enough to notice that the condition

$$f((x, p^{n-2}, x)^{(k)}) = [f(x), f(p)^{n-2}, f(x)]^{(k)}$$

for all $p, x \in A$ and $k \in N$ is fulfilled.

We can formulate Theorem 1 of Corovei [2] in the following equivalent form.

Theorem 4. *Let $A(\ )$ and $B[\ ]$ be n-adic groups with retracts (A, α, a) and (B, β, b), respectively.*
A function $f : A \to B$ is a homomorphism of the n-adic groups $A(\ )$ and $B[\ ]$ if and only if there exists a homomorphism $\varphi : A \to B$ of the groups (A, α, a) and (B, β, b) and an element $a_1 \in B$ such that
(a) $f(x) = a_1\varphi(x)$,
(b) $(\varphi\alpha)(x)\beta(a_1) = \beta(a_1)(\beta\varphi)(x)$,
(c) $\varphi(a) = \beta(a_1)\beta^2(a_1)\ldots\beta^{n-2}(a_1)ba_1$
for every $x \in A$.

Proposition 3. *Let (A, α, a) and (B, β, b) be arbitrary fixed retracts of n-adic groups $(\ )$ and $B[\ ]$, respectively.*
For an arbitrary homomorphism $f : A \to B$ of the n-adic groups $A(\ )$ and $B[\ ]$ there exist a unique homomorphism $\varphi : A \to B$ of the groups (A, α, a) and (B, β, b), and a unique element $a_1 \in B$ such that

(2) $$f(x) = a_1\varphi(x)$$

for every $x \in A$.

Proof. The Corovei theorem guarantees the existence of a homomorphism φ and an element a_1. Suppose that $f(x) = a_1\varphi(x)$ and $f(x) = a_2\varphi'(x)$ for $x \in A$. Putting $x = 1$ we get $a_1\varphi(1) = a_2\varphi'(1)$, i.e. $a_1 = a_2$. Hence $\varphi = \varphi'$. □

Remark 1. On the whole, the homomorphism φ and the element a_1 will be changed in formula (2) if the retracts of the n-adic groups $A(\ )$ and $B[\ ]$ are changed.

We shall give a suitable example.

Example. Let $A = \{e, a, b, c\}$ be the Klein group i.e.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

In the set A we define the 3-ary operations ( ) and [ ] as follows:

$$(x_1, x_2, x_3) = x_1 x_2 x_3 a$$
$$[x_1, x_2, x_3] = x_1 x_2 x_3 b$$

for all $x_1, x_2, x_2 \in A$.
$A(\ )$ and $B[\ ]$ are 3-adic groups with the retracts (A, id_A, a) and (A, id_A, b). The function $\varphi : A \to A$ is defined by setting: $\varphi(e) = e$, $\varphi(a) = b$, $\varphi(b) = c$, $\varphi(c) = a$. The function φ is a homomorphism (an automorphism) of the groups (A, id_A, a) and (A, id_A, b). Notice that the homomorphism φ and the element $c \in A$ satisfy conditions (b) and (c) of Theorem 4, and so the function $f(x) = c\varphi(x)$ for $x \in A$ is a homomorphism of the 3-adic groups $A(\ )$ and $A[\ ]$. For the 3-adic group $A(\ )$ we construct the retract by means of the Sokolov method with respect to the element $c \in A$:

$$x \circ y = (x, c, y) = xyb$$

for all $x, y \in A$.
Since $\bar{c} = b$ we have $(\bar{c}, \bar{c}, \bar{c}) = (b, b, b) = c$. We have obtained the retract $(A, \circ, id_A, c)$ of the 3-adic group $A(\ )$ with the identity b. It follows from Theorem 1 of Corovei [1] that there exists a homomorphism $\psi : A \to A$ of the groups $(A, \circ, id_A, c)$ and (A, id_A, b) such that $f(x) = f(b)\,\psi(x)$ for $x \in A$. Since $f(b) = e$ we get $f(x) = e\,\psi(x)$ for $x \in A$, hence $\psi = f$. Thus $f(x) = c\,\varphi(x)$ and $f(x) = e\,\psi(x)$ for $x \in A$, where $c \neq e$ and $\varphi \neq \psi$.

Theorem 5. *Let $A_1(\ )$ be an n-adic subgroup of a commutative n-adic group $A(\ )$. Let $f : A_1 \to B$ be an arbitrary homomorphism of the n-adic group $A_1(\ )$ into a commutative divisible n-adic group $B[\ ]$. Then there exists a homomorphism $\bar{f} : A \to B$ of the n-adic groups $A(\ )$ and $B[\ ]$ which is an extension of the homomorphism f.*

Proof. We construct the retract (A, id_A, a) of the n-adic group $A(\ )$ by means of the Sokolov method with respect to an arbitrary fixed element $p \in A_1$:

$$x\,y = (x, p^{n-2}, y),$$
$$a = (\bar{p}^n)$$

for all $x, y \in A$.
Then (A_1, id_{A_1}, a) is a retract of the n-adic group $A_1(\ )$. Notice that (A_1, id_{A_1}, a) is a subgroup of the group (A, id_A, a). Let (B, id_B, b) be an arbitrary fixed retract of the n-adic group $B[\ ]$. It follows from Theorem 4 that there exists a homomorphism $\varphi : A_1 \to B$ of the groups (A_1, id_{A_1}, a) and (B, id_B, b), and an element $a_1 \in B$ such that $f(x) = a_1\,\varphi(x)$ and $\varphi(a) = (a_1)^{n-1}b$. Since the group (B, id_B, b) is a commutative divisible group, it follows from Theorem 16.1 (cf. [4], p. 59) that there exists a homomorphism $\bar{\varphi} : A \to B$ of the groups (A, id_A, a) and (B, id_B, b) which is an extension of the homomorphism φ. Let us put $\bar{f}(x) = a_1\,\bar{\varphi}(x)$ for $x \in A$.

Since conditions (a), (b) and (c) of Theorem 4 are satisfied for every $x \in A$, then $\bar{f} : A \to B$ is a desired homomorphism of the n-adic groups $A(\ )$ and $B[\ ]$.

In general, the homomorphism $\bar{\varphi}$ is not uniquely determined (cf. [4], p. 69), thus the homomorphism $\bar{f}$ is also not uniquely determined.

We shall consider a homotopy of n-adic groups.

A homotopy of n-adic groups $A(\ )$ and $B[\ ]$ we call a sequence of functions $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} : A \to B$ such that $\alpha_{n+1}((x_1, x_2, \ldots, x_n)) = [\alpha_1(x_1), \alpha_2(x_2), \ldots$ $\ldots, \alpha_n(x_n)]$ for all $x_1, x_2, \ldots, x_n \in A$.

A homotopy will be denoted by $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$.

For commutative n-adic groups Theorem 6 of Chronowski [1] has the following form.

Theorem 6. *Let $A(\ )$ and $B[\ ]$ be commutative n-adic groups with retracts (A, id_A, a) and (B, id_B, b), respectively. If a sequence of functions $(\alpha_1, \alpha_2, \ldots$ $\ldots, \alpha_{n+1})$ is a homotopy of the n-adic group $A(\ )$ into the n-adic group $B[\ ]$, then there exist a homomorphism $\varphi : A \to B$ of the groups (A, id_A, a) and (B, id_B, b), and elements $a_1, a_2, \ldots, a_n \in B$ such that*

$$\begin{aligned} \alpha_k(x) &= a_k\varphi(x) \text{ for } k = 1, 2, \ldots, n-1, \\ \alpha_n(x) &= a_n\varphi(ax), \\ \alpha_{n+1}(x) &= a_1^n b\varphi(x) \end{aligned} \tag{3}$$

for every $x \in A$.

If $\varphi : A \to B$ is a homomorphism of the groups (A, id_A, a) and (B, id_B, b), and $a_1, a_2, \ldots, a_n \in B$ are arbitrary elements, then the sequence of functions $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ of the form (3) is a homotopy of the n-adic group $A(\ )$ into the n-adic group $B[\ ]$.

Remark 2. Let (A, id_A, a) and (B, id_B, b) be arbitrary fixed retracts of commutative n-adic groups $A(\ )$ and $B[\ ]$, respectively.

For an arbitrary homotopy $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ of the n-adic group $A(\ )$ into the n-adic group $B[\ ]$ there exist uniquely a homomorphism $\varphi : A \to B$ of the groups (A, id_A, a) and (B, id_B, b), and elements $a_1, a_2, \ldots, a_n \in B$ such that (3) holds.

Theorem 7. *Let $A_1(\ )$ be n-adic subgroup of a commutative n-adic group $A(\ )$. Let $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be a homotopy of the n-adic group $A_1(\ )$ into a commutative divisible n-adic group $B[\ ]$. Then there exists a homotopy $(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n+1})$ of the n-adic group $A(\ )$ into the n-adic group $B[\ ]$ which is an extension of the homotopy $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$.*

Proof. We construct the retract (A, id_A, a) of the n-adic group $A(\ )$ by means of the Sokolov method with respect to an arbitrary fixed element $p \in A_1$:

$$\begin{aligned} xy &= (x, p^{n-2}, y), \\ a &= (\bar{p}^n) \end{aligned}$$

for all $x, y \in A$.
(A_1, id_{A_1}, a) is a retract of the n-adic group $A_1(\ )$. Moreover, (A_1, id_{A_1}, a) is a subgroup of the group (A, id_A, a). Let (B, id_B, b) be an arbitrary fixed retract of the n-adic group $B[\ ]$.
It follows from Theorem 6 that there exist a homomorphism $\varphi : A_1 \to B$ of the groups (A_1, id_{A_1}, a) and (B, id_B, b), and elements $a_1, a_2, \ldots, a_n \in B$ such that (3) is valid for $x \in A_1$. Since the group (B, id_B, b) is commutative and divisible, it follows from Theorem 16.1 (cf. [4], p. 59) that there exists a homomorphism $\bar{\varphi} : A \to B$ of the groups (A, id_A, a) and (B, id_B, b) which is an extension of the homomorphism φ. To construct a homotopy $(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n+1})$ which is an extension of the homotopy $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ we put:

$$\begin{aligned} \bar{\alpha}_k(x) &= a_k\bar{\varphi}(x) \text{ for } k = 1, 2, \ldots, n-1, \\ \bar{\alpha}_n(x) &= a_n\bar{\varphi}(ax), \\ \bar{\alpha}_{n+1}(x) &= a_1^n b\bar{\varphi}(x), \end{aligned}$$

for an arbitrary $x \in A$. □

We say that n-adic group $A(\ )$ is uniquely divisible by a natural number $k \in N$ if

$$(4) \qquad \bigvee_{p\in A} \bigwedge_{y\in A} \bigvee_{x\in A}{}^{1} (x, p^{n-2}, x)^{(k-1)} = y$$

Proposition 4. *An n-adic group $A(\ )$ is uniquely divisible by a natural number $k \in N$ if and only if there exists a retract of the n-adic group $A(\ )$ uniquely divisible by the number k.*

The proof of this proposition is similar to the proof of Theorem 1.

Since all retracts of an n-adic group $A(\ )$ are isomorphic, the following corollaries are valid.

Corollary 7. *An n-adic group $A(\ )$ is uniquely divisible by a natural number $k \in N$ if and only if every retract of $A(\ )$ is uniquely divisible by the number k.*

Corollary 8. *Let $A(\ )$ be an n-adic group. Condition (4) is equivalent to the following condition:*

$$\bigwedge_{p\in A} \bigwedge_{y\in A} \bigvee_{x\in A}{}^{1} (x, p^{n-2}, x)^{(k-1)} = y\,.$$

Let $A_1(\ )$ be an n-adic subgroup of an n-adic group $A(\ )$. Consider the following condition:

$$(5) \qquad \bigvee_{p\in A_1} \bigvee_{k\in N} \bigwedge_{x\in A} (x, p^{n-2}, x)^{(k-1)} \in A_1\,.$$

Theorem 8. *Let $A_1(\ )$ be an n-adic subgroup of a commutative n-adic group $A(\ )$ for which condition (5) is fulfilled. Let $B[\ ]$ be a commutative n-adic group uniquely divisible by the natural number $k \in N$ (from condition (5)). Let $f : A_1 \to B$ be a homomorphism of the n-adic groups $A_1(\ )$ and $B[\ ]$. The there exists a unique homomorphism $\bar{f} : A \to B$ of the n-adic groups $A(\ )$ and $B[\ ]$ which is an extension of the homomorphism f.*

Proof. Let (A, id_A, a) be a retract of the n-adic group $A(\ )$ constructed by means of the Sokolov method with respect to the element $p \in A_1$ which satisfies condition (5). Let (B, id_B, b) be an arbitrary fixed retract of the n-adic group $B[\ ]$. Notice that (A_1, id_{A_1}, a) is a retract of the n-adic group $A_1(\ )$. Moreover, (A_1, id_{A_1}, a) is a subgroup of the group (A, id_A, a). It follows from condition (5) that there exists a number $k \in N$ such that $x^k \in A_1$ for every $x \in A$. By Theorem 4 $f(x) = a_1\varphi(x)$ for $x \in A$, where $\varphi : A_1 \to B$ is a certain homomorphism of the groups (A_1, id_{A_1}, a) and (B, id_B, b), and $a_1 \in B$. According to Theorem 4 (cf. [6], p. 481) there exists a unique homomorphism $\bar{\varphi} : A \to B$ of the groups (A, id_A, a) and (B, id_B, b) which is an extension of the homomorphism φ. Thus, it is enough to put $\bar{f}(x) = a_1\bar{\varphi}(x)$ for all $x \in A$ and we get the desired homomorphism $\bar{f}$. Let a homomorphism $g : A \to B$ of the n-adic groups $A(\ )$ and $B[\ ]$ be an extension of the homomorphism f. It follows from Proposition 3 that there exist a unique homomorphism $\psi : A \to B$ of the groups (A, id_A, a) and (B, id_B, b), and a unique element $b_1 \in B$ such that $g(x) = b_1\psi(x)$ for $x \in A$. Since $g|A_1 = f$, applying Proposition 3 we obtain $a_1 = b_1$ and $\psi|A_1 = \varphi$. Thus, if $g(c) \neq \bar{f}(c)$ for a certain $c \in A \setminus A_1$, then $\psi(c) \neq \bar{\varphi}(c)$ which contradicts Theorem 4 (cf. [6], p. 481). We have obtained the uniqueness of the extension of the homomorphism f. □

Theorem 9. *Let $A_1(\ )$ be an n-adic subgroup of a commutative n-adic group $A(\ )$ for which condition (5) is fulfilled. Let $B[\ ]$ be a commutative n-adic group uniquely divisible by the number $k \in N$ (from condition (5)). Let $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be a homotopy of the n-adic group $A_1(\ )$ into the n-adic group $B[\ ]$. The there exists a unique homotopy $(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n+1})$ of the n-adic group $A(\ )$ into the n-adic group $B[\ ]$ which is an extension of the homotopy $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$.*

The proof of this theorem is similar to the proof of Theorem 8. It is enough to use Theorem 6, Theorem 4 (cf. [6], p. 481) and Remark 2.

Remark 3. Using formula (16) (cf. [6], p. 481) we can give the formulas for the extensions $\bar{f}$ of the homomorphism f and $(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n+1})$ of the homotopy $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ occuring in Theorems 8 and 9.

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ANTONI CHRONOWSKI
WYZSZA SZKOŁA PEDAGOGICZNA
INSTYTUT MATEMATYKI
UL. PODCHORĄZYCH 2
30 - 084 KRAKÓW
POLAND