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EVERY CONNECTED ACYCLIC DIGRAPH OF HEIGHT 1 IS NEIGHBOURHOOD-REALIZABLE

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Abstract. A digraph H is called neighbourhood-realizable, if there exists a digraph G with the property that for each vertex v of G the set of terminal vertices of edges outgoing from v induces a subgraph isomorphic to H. The assertion in the title of the paper is proved.

Key words. Acyclic digraph, neighbourhood realization.

MS classification. 05 C 20.

The study of the neighbourhood realizability of graphs was initiated by a problem proposed by A. A. Zykov at the Symposium on Graph Theory in Smolenice in 1963 [1]. This problem was formulated for undirected graphs. Here we shall deal with its digraph variant.

Let H be a directed graph (shortly digraph). A digraph G is called a neighbourhood realization of H, if for each vertex v of G the set of terminal vertices of edges outgoing from v induces a subgraph isomorphic to H. If there exists a neighbourhood realization of H, the digraph H is called neighbourhood-realizable.

The digraph variant of the problem of A. A. Zykov is the problem to characterize neighbourhood-realizable digraphs. Some results were obtained in [2] and [3]. Here we shall consider acyclic digraphs of height 1.

A digraph is called acyclic, if it does not contain a cycle (directed circuit). Its height is the maximum length of a directed path in it. If the height of an acyclic diagraph H is 1, then its vertex set can be partitioned into the set A of sources and the set B of sinks.

We shall prove a theorem.

Theorem 1. *Every connected acyclic digraph of height* 1 *has an infinite neighbourhood realization.*

P r o o f. Let H be an acyclic digraph of height 1, let A be its source set, let B be its sink set. Choose two disjoint sets $\tilde{A}$, $\tilde{B}$ such that $A \subset \tilde{A}$, $B \subset \tilde{B}$. If $|A|$

(or $|B|$) is infinite, we choose $\tilde{A}$ (or $\tilde{B}$) in such a way that $|\tilde{A} - A| = |A|$ (or $|\tilde{B} - B| = |B|$ respectively). If $|A|$ (or $|B|$) is finite, then $|\tilde{A}| = \chi_0$ (or $|\tilde{B}| = \chi_0$). We shall construct a graph G_0 with the vertex set $V(G_0) \subseteq \tilde{A} \cup \tilde{B}$. The graph H with the vertex set $A \cup B$ is a subgraph of G_0. Put $A' = A$, $B' = B$. Choose an element $a \in A$, a set $A(a) \subset \tilde{A} - A'$ such that $|A(a)| = |A|$ and a set $B(a) \subset \tilde{B}$ such that $B(a) \cup B'$ is the set of terminal vertices of edges outgoing from a and $|B(a)| = |B|$. We add edges from $A(a)$ to $B(a)$ in such a way that the set $A(a) \cup B(a)$ might induce a graph $H(a)$ isomorphic to H. We add edges from a to all vertices of $A(a) \cup (B(a) - B')$. Then we change the notation in such a way that A' will denote the former set $A' \cup A(a)$ and B' will denote the former set $B' \cup B(a)$. Now we choose another vertex of A and proceed in the same way. After exhausting all vertices of A we continue by taking other vertices of A'. Thus we may proceed into infinity, until we obtain a graph G_0 with the vertex set $\hat{A} \cup \hat{B}$ with $\hat{A} \subseteq \tilde{A}$, $\hat{B} \subseteq \tilde{B}$ in which the neighbourhood (set of terminal vertices of outgoing edges) of every vertex of A induces a subgraph isomorphic to H. Now let r be an integer, $r \geqq 3$. We take r copies $G_1, \ldots, G_r$ of the graph G_0. For each $i = 1, \ldots, r$ we lead edges from all vertices of G_i corresponding to vertices of B into all vertices of G_{i+1} corresponding to vertices of $A \cup B$; the sum $i + 1$ is taken modulo r. The graph G thus obtained is a neighbourhood realization of H. □

A natural question is the following,

Problem. *Has every finite connected acyclic diagraph of height 1 a finite neighbourhood realization?*

We shall show a class of such digraphs for which this assertion is true.

Let p, q be positive integers. By the symbol $K(p \to q)$ we denote the digraph whose vertex set is the union of two disjoint sets P, Q such that $|P| = p$, $|Q| = q$ and whose edge set is the set of all edges going from a vertex of P into a vertex of Q.

Theorem 2. *For any two positive integers p, q the graph $K(p \to q)$ has infinitely many finite neighbourhood realizations.*

Proof. Let r be an integer, $r \geqq 3$. Let $P_1, \ldots, P_r, P'_1, \ldots, P'_r, Q_1, \ldots, Q_r$ be pairwise disjoint sets, let $|P_i| = P'_i| = p$, $|Q_i| = q$ for $i = 1, \ldots, r$. Let $V = \bigcup_{i=1}^{r} P_i \cup P'_i \cup Q_i$. We shall construct a graph G with the vertex set V. In G for each $i = 1, \ldots, r$ edges go from all vertices of $P_i \cup P'_i$ to all vertices of Q_i and each vertex of P_i is joined with each vertex of P_i by a pair of oppositely directed edges (by a double edge). Further from each vertex of Q_i edges go to all vertices of $P_{i+1} \cup Q_{i+1}$, the sum $i + 1$ being taken modulo r. No other edges than those described are in G. The graph G is a neighbourhood realization of $K(p \to q)$. As there are infinitely many possibilities to choose r, there are infinitely many neighbourhood realizations of $K(p \to q)$. □

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