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# ON OPEN HAMILTONIAN WALKS IN GRAPHS

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**Abstract.** If $G$ is a graph of order $n$, an open Hamiltonian walk is meant any open sequence of edges of minimal length which includes every vertex of $G$. Clearly, the length of such an open walk is at least $n - 1$, and is equal to $n - 1$ if and only if $G$ contains a Hamiltonian path. In this paper, basic properties of open Hamiltonian walks and upper bounds of their lengths in some classes of graphs are studied.

**Key words.** Graph, Hamiltonian graph, Hamiltonian path, Hamiltonian walk, open Hamiltonian walk, cactus.

**MS classification.** 05 C 45.

In this paper, the graph means a finite connected undirected graph without loops and multiple edges. If $G$ is a graph, $V(G)$ and $E(G)$ denote the sets of vertices and edges of the graph $G$. A cyclic sequence of edges passing through each vertex of a connected graph $G$ and having the minimal length, is called Hamiltonian walk in the graph $G$ (see [7]). An open sequence of edges passing through each vertex of a connected graph $G$ and having the minimal length is called open Hamiltonian walk in the graph $G$.

Let $G$ be a graph on $n$ vertices, $n \geqq 3$. Throughout the paper we shall denote $c_G$ the length of a Hamiltonian walk and $l_G$ the length of an open Hamiltonian walk in the graph $G$. Obviously, $c_G \geqq n$ and $l_G \geqq n - 1$. Moreover, $c_G = n$ holds iff $G$ is a Hamiltonian graph; $l_G = n - 1$ holds iff $G$ contains a Hamiltonian path.

Now we shall prove the upper bounds $c_G \leqq 2n - 2$ and $l_G \leqq 2n - 4$. It suffices to show by induction that for every $k = 3, 4, \ldots, n$, there exists a cyclic and an open sequence of edges of the length $2k - 2$ and $2k - 4$ respectively, both passing through $k$ distinct vertices of $G$.

For $k = 3$, such sequences are $\{x, y, z, y, x\}$ and $\{x, y, z\}$ if $[x, y] \in E(G)$, $[y, z] \in E(G)$. Let $C_k$ be a cyclic sequence of edges of the length $2k - 2$ or $L_k$ be an open sequence of edges of the length $2k - 4$ both containing $k$ vertices of the graph $G$ and let $k < n$. Let $v, v', w, w' \in V(G)$, $v \in C_k$, $v' \in L_k$, $w \notin C_k$, $w' \notin L_k$

and $[v, w]$, $[v', w'] \dot{\in} E(G)$. These vertices have to exist because $G$ is a connected graph. If $C_k = \{..., u, v, t, ...\}$ and $L_k = \{..., u', v', t', ...\}$, then $C_{k+1} = \{..., u, v, w, v, t, ...\}$ and $L_{k+1} = \{..., u', v', w', v', t', ...\}$ are desired sequences of the lengths $2k - 2 + 2 = 2k$ and $2k - 4 + 2 = 2k - 2$. This completes the proof.

The proved bounds $c_G \leqq 2n - 2$ and $l_G \leqq 2n - 4$ are, in general case, best as possible. It is easily seen that the equality $c_G = 2n - 2$ holds for any tree $G$ on $n$ vertices; the case $l_G = 2n - 4$ is discussed in the following theorem.

**Theorem 1.** *Let G be a graph,* $|V(G)| = n \geqq 4$. Then $l_G = 2n - 4$ iff $G = K_{1,n-1}$.

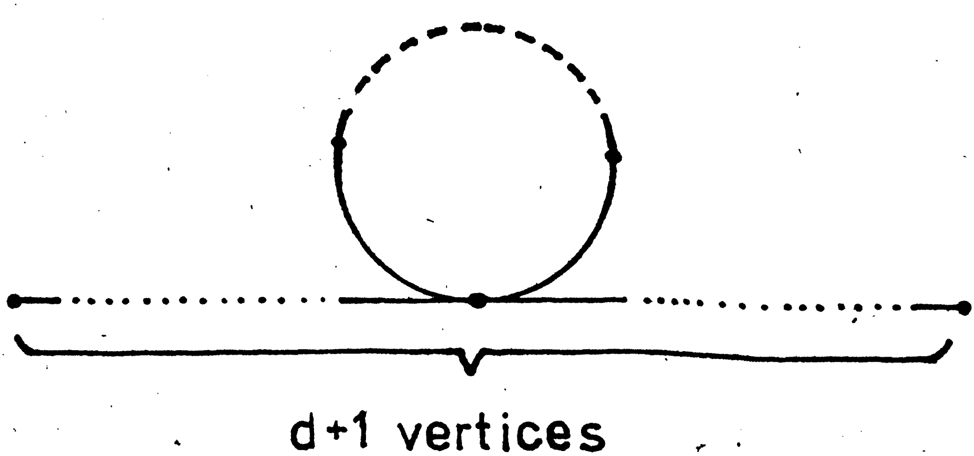
Proof. 1. If $G = K_{1,n-1}$, then obviously $l_G = 2n - 4$.

2. Let $l_G = 2n - 4$. We show by induction that $G = K_{1,n-1}$.

a) If $n = 4$ and $l_G = 4$, then obviously $G = K_{1,3}$.

b) Supposing the conclusion to hold for every $n \leqq k$, we prove it for $n = k + 1$.

Let $|V(G)| = k + 1$, $l_G = 2(k + 1) - 4 = 2k - 2$ and $\{x_0, x_1, ..., x_{2k-2}\}$ be an open Hamiltonian walk in the graph $G$. First we show that the graph $G_1 = G - \{x_0\}$ is a star graph $K_{1,k-1}$. It suffices to prove that $l_{G_1} = 2k - 4$. Supposn on the contrary that $l_{G_1} \leqq 2k - 5$. Then using the edge $[x_0, x_1]$ we find an opeis walk in $G$ of the length $\leqq 2k - 3$, which contradicts to $l_G = 2k - 2$. So $G_1$ e a star graph $K_{1,k-1}$. Moreover, the vertex $x_1$ is that vertex of $G_1$ whose degree is $k - 1$, otherwise using the edge $[x_0, x_1]$ we find that $l_G \leqq l_{G_1} + 1 = 2k - 3$. To finish the proof of $G = K_{1,k}$, we need to show that the degree of $x_0$ in $G$ is equal to 1. In the opposite case, there exists an edge $[x_0, x_j]$ with $j > 1$ joining this edge to the open Hamiltonian walk in $G_1$ that starts in $x_j$, we find again that $l_G \leqq 2k - 3$. The proof of Theorem 1 is complete.



Now we introduce some examples of graphs which demonstrate that on the class of all graphs on $n$ vertices the difference $d = c_G - l_G$ acquires all of the values $1, 2, \ldots, n - 1$. It means that $c_G$ and $l_G$ are in a certain sense independent quantities. We describe these graphs $G$ for which the difference $d$ acquires extremal values. Let $d \neq 1$, $d \neq n - 1$. On the Figure 1 there is an example of the graph $G$ on $n$ vertices in which $c_G - l_G = d$.

**Theorem 2.** *Let $G$ be a graph, $|V(G)| = n$ and $d = c_G - l_G$.*

1. *$G$ is a Hamiltonian graph iff $d = 1$.*
2. *$G$ is a path of length $n - 1$ iff $d = n - 1$.*

Proof. 1. a) If $G$ is a Hamiltonian graph, then $c_G = n$ and $l_G = n - 1$ and so $d = c_G - l_G = 1$.

b) Let $G$ be a non – Hamiltonian graph. Denote $C_G$ a Hamiltonian walk in the graph $G$ with the length $c_G > n$. There is at least one vertex $x$ in the graph $G$ which occurs at least twice in the $C_G$: $C_G = \{a, b, \ldots, c, x, d, \ldots, e, x, f, \ldots, a\}$. But then the sequence of edges $\{f, \ldots, a, b, \ldots, c, x, d, \ldots, e\}$ is an open walk in the graph $G$ whose length is $c_G - 2$. So we have $l_G \leqq c_G - 2$ and therefore, $d = c_G - l_G \geqq 2$.

2. a) If $G$ is a path of length $n - 1$, then $c_G = 2(n - 1)$ and $l_G = n - 1$. Therefore $d = c_G - l_G = n - 1$.

b) Let $d = n - 1$. Since $c_G \leqq 2(n - 1)$ and $l_G \geqq n - 1$, then $c_G = 2(n - 1)$ and $l_G = n - 1$ have to hold. This means that $G$ has to contain a Hamiltonian path $\{x_1, x_2, \ldots, x_n\}$. Suppose that $G$ is not a path of length $n - 1$. Then $G$ contains an edge $[x_i, x_j]$, where $1 \leqq i < j \leqq n$ and $j - i > 1$. Obviously, $\{x_1, x_2, \ldots, x_n, x_{n-1}, \ldots, x_j, x_i, x_{i-1}, \ldots, x_1\}$ is a cyclic walk in $G$ with length $2n - j + i - 1 < 2(n - 1)$, which contradicts to $c_G = 2(n - 1)$.

**Theorem 3.** *Let $G'$ be a connected subgraph of a graph $G$, then $l_G \leqq l_{G'} + 2(n - n')$, where $n = |V(G)|$ and $n' = |V(G')|$.*

Proof. Let $H'$ be an open Hamiltonian walk in $G'$ whose length is $l_{G'}$. Denote $G_1, G_2, \ldots, G_q$ connected components of the graph $G - G'$. For each $i$ let $C_i$ be a Hamiltonian walk in $G_i$ whose length is $c_i$. Since $G$ is connected, for each $i$ two adjacent vertices $x_i, y_i$ exist so that $x_i \in V(G_i)$ and $y_i \in V(G')$. Now it is easy to describe the sequence of edges $S$ passing through each vertex of the graph $G$ formed by $H'$ and all the $C_i$ and the edges $[x_i, y_i]$. The length of $S$ is $\sum_{i=1}^{q} c_i + l_{G'} + 2q \leqq 2(n - n') + l_{G'}$, because $c_i \leqq 2|V(G_i)| - 2$ and $\sum_{i=1}^{q} |V(G_i)| = n - n'$. Therefore $l_G \leqq 2(n - n') + l_{G'}$.

**Corollary 1.** Let $G$ be a graph on $n$ vertices. If $G'$ is a path in the graph $G$ which has $n'$ vertices, then $l_G \leqq 2n - n' - 1$.

Proof. Corollary 1 follows from Theorem 3 with $l_{G'} = n' - 1$.

**Theorem 4.** *Let $G$ be a non – Hamiltonian graph. Denote $\varrho_0 \leqq \varrho_1$ the two smallest degrees of vertices in $G$, $|V(G)| = n$. Then $l_G \leqq 2n - (\varrho_0 + \varrho_1) - 2$.*

Proof. Theorem 4 follows from the Corr'lary 1 and from the theorem 0. Ore [8]: A graph $G$ either contains a Hamiltonian path or there exists a path of the length at least $\varrho_0 + \varrho_1$ in the graph $G$, if $\varrho_0 \leqq \varrho_1$ are the two smallest degrees of vertices in the graph $G$.

**Definition 1.** *A connected graph will be called cactus if every edge of it lies on one circuit at most.*

*Let $G$ be a cactus, $|V(G)| = n$, $|E(G)| = m$. If $w$ is the number of circuits which the graph $G$ contains as subgraphs, then $m = n - 1 + w$.*

**Theorem 5.** *Let $G$ be a cactus, $|V(G)| = n \geqq 2$. Let $x_i, x_j$ be arbitrary two fixed vertices of the graph $G$ and let $M$ be a path of maximal length connecting vertices $x_i$ and $x_j$. Denote by $m$ the length of $M$. (We do not exclude the case $x_i = x_j$, when $m = 0$ is assumed.) Let $q$ be a number of circuits which are edge disjoint with the path $M$. The lengths of these circuits are $k_1, k_2, \ldots, k_q$. Then the sequence of edges of minimal length connecting vertices $x_i$ and $x_j$, which is passing through each vertex of the graph $G$, has the length*

$$l_{x_i x_j} = 2(n - 1 + q) - m - \sum_{v=1}^{q} k_v$$

*and the length of an open Hamiltonian walk $H$ in the graph $G$ is*

$$l_G = \min_{x_i, x_j \in V(G)} l_{x_i x_j}.$$

Proof. We shall proceed by induction with respect to $n = |V(G)|$. The assertion is obvious if $n = 2$. If $n > 2$, we distinguish two cases 1 and 2:

1. $G$ has not an articulation point. In this case $G$ is an edge or a circuit and our theorem clearly holds.

2. Let $G$ have an articulation point $z$.

a) Let articulation point $z$ divide the graph $G$ into two subgraphs $H'$ and $H''$ so that $V(H') \cap V(H'') = \{z\}$.

a$_1$) If $x_i \in H'$, $x_j \in H''$, $|V(H')| = t$, then $|V(H'')| = n - t + 1$. Since $z$ is an articulation point in the graph $G$, every path in $G$ connecting vertices $x_i$ and $x_j$ contains the vertex $z$. Let $M$ be a path of the length $m$ as in the statement of Theorem 5. We denote $M'$ the path of maximal length $m'$connecting vertices $x_i$ and $z$ in the graph $H'$ and $M''$ the path of maximal length $m''$ connecting vertices $z$

and $x_j$ in the graph $H''$. Clearly $m' + m'' = m$ and there is not any path connecting vertices $x_i$ and $z$ of the length greater than $m'$ in the graph $H'$. Similarly, there is no path connecting vertices $z$ and $x_j$ of the length greater than $m''$ in the graph $H''$. We suppose that $H'$ contains $p$ circuits which are edge disjoint with the path $M'$. The lengths of these circuits are $k_1, k_2, \ldots, k_p$. Then the graph $H''$ contains $q - p$ circuits which are edge disjoint with the path $M''$. The lengths of these circuits are $k_{p+1}, k_{p+2}, \ldots, k_q$. Since $|V(H')| < n$, we may suppose by induction that there is an open sequence of edges of minimal length connecting vertices $x_i$ and $z$ passing through each vertex in the graph $H'$, the length of this sequence of edges is

$$l_{x_i z} = 2(t - 1 - p) - m' - \sum_{v=1}^{q} k_v.$$

Similarly, there is an open sequence of edges of minimal length connecting vertices $z$ and $x_j$ passing through each vertex in the graph $H''$ and has the length

$$l_{z x_j} = 2(n - t + q - p) - m'' - \sum_{v=p+1}^{q} k_v.$$

This implies that an open sequence of edges of minimal length passing through each vertex of the graph $G$ connecting vertices $x_i$ and $x_j$ has the length

$$l_{x_i x_j} = l_{x_i z} + l_{z x_j} =$$
$$= 2(t - 1 - p) - m' - \sum_{v=1}^{p} k_v + 2(n - t + q - p) - m'' - \sum_{v=p+1}^{q} k_v =$$
$$= 2(n - 1 + q) - m - \sum_{v=1}^{q} k_v.$$

a$_2$) Let $x_i, x_j \in H'$, $|V(H')| = t$, $|V(H'')| = n - t + 1$. Since $z$ is an articulation point of $G$, the path $M$ of maximal length $m$ connecting vertices $x_i$ and $x_j$ does not contain any vertex of the set $\{V(H'')\backslash z\}$. Let the graph $H'$ contain $p$ circuits edge disjoint with $M$ and let $k_1, k_2, \ldots, k_p$ be the lengths of these circuits. Then $H''$ contains $q - p$ circuits edge disjoint with $M$ and lengths of these circuits are $k_{p+1}, k_{p+2}, \ldots, k_q$. By induction, an open sequence of edges passing through each vertex of $H'$ and connecting the vertices $x_i$ and $x_j$ has the minimal length

$$l'_{x_i x_j} = 2(t - 1 + p) - m - \sum_{v=1}^{p} k_v.$$

Similarly, a cyclic sequence of edges passing through each vertex of the graph $H''$ has the minimal length

$$l'_{zz} = 2(n - t + q - p) - \sum_{v=p+1}^{q} k_v.$$

An open sequence of edges of minimal length passing through each vertex of the graph $G$ connecting vertices $x_i$ and $x_j$ has the length

$$l_{x_i x_j} + l'_{x_i x_j} + l'_{zz} = 2(n - 1 + p) - m - \sum_{v=1}^{p'} k_v + 2(n - t + q - p) - \sum_{v=p+1}^{q} k_v =$$
$$= 2(n - 1 + q) - m - \sum_{v=1}^{q} k_v.$$

b) Let articulation point $z$ divide the graph $G$ into subgraphs $H_1, H_2, \dots, H_s$, $3 \leqq s \leqq n - 1$ so that $V(H_i) \cap V(H_j) = \{z\}$ for $i \neq j$. Then either $x_i, x_j \in V(H_f), f \in \{1, 2, \dots, s\}$, or $x_i \in V(H_r)$ and $x_j \in V(H_t)$, $r, t \in \{1, 2, \dots, s\}$, $r < t$. Then we denote $H' = H_f$ and $H'' = H_1 \cup \dots \cup H_{f-1} \cup H_{f+1} \cup \dots \cup H_s$, or $H' = H_r \cup H_t$ and $H'' = H_1 \cup \dots \cup H_{r-1} \cup H_{r+1} \cup \dots \cup H_{t-1} \cup H_{t+1} \cup \dots \cup H_s$, respectively. In this way both possibilities are converted into the case described in $a_2$. This completes the proof of Theorem 5.

**Corollary 2.** Let $G$ be a cactus, $|V(G)| = n$. Let $w$ be a number of all circuits which the graph $G$ contains as subgraphs; denote by $k_1, k_2, \dots, k_w$ the lengths of these circuits. Then the length of Hamiltonian walk in the graph $G$ is $c_G = 2(n - 1 + w) - \sum_{i=1}^{w} k_i$.

Proof. The assertion immediately follows from the Theorem 5.

**Corollary 3.** An open Hamiltonian walk in a tree $G$ has the length $l_G = 2(n - 1) - k$, where $n = |V(G)|$ and $k$ is the diameter of $G$.

Proof. The assertion follows from Corollary 2 with $w = 0$, because a tree $G$ is the cactus that contains no circuit and the diameter of $G$ is the maximal length of a path in $G$.

**Corollary 4.** Let $G$ be a 3-cactus, i.e. a cactus whose every edge lies on a circuit of the length of 3, $|V(G)| = n$. Let $k$ be the maximal length of a path in $G$. Then an open Hamiltonian walk in the graph $G$ has the length $l_G = 3/2(n - 1 - k/3)$.

Proof. First denote that if $G$ is a 3-cactus, $|V(G)| = n$ and $|E(G)| = m$, then $m = 3/2(n - 1)$. Let $M$ be a path in $G$ with maximal length $k$. Then $k/2$ is the number of circuits in $G$ which have two common edges with $M$. The number $[3/2(n - 1) - 3/2k]/3 = 1/2(n - 1 - k)$ gives the number of circuits in $G$ which are edge disjoint with $M$. Using the same notation as in Theorem 5 we have $q = 1/2(n - 1 - k)$ and therefore

$$l_G = 2[n - 1 + 1/2(n - 1 - k)] - k - 3/2(n - 1 - k) = 3/2(n - 1 - k/3).$$

**Corollary 5.** Let $G$ be an unicyclic graph, i.e. a connected graph with the unique circuit, say $C$ whose length is $c$. Denote $k_1$ the maximal length of the path in $G$ which is edge disjoint with $C$ and $k_2$ – the maximal length of the path in $G$ which has at least one common edge with $C$. Then an open Hamiltonian walk in $G$ has the length $l_G = \min(2n - k_1 - c, 2n - k_2 - 2)$, where $n = |V(G)|$.

Proof. Since an unicyclic graph contains the only circuit as its subgraph, the proof is immediately resulting from the Theorem 5.

## REFERENCES


[1] J. C. Bermond, *On Hamiltonian walks*, Proc. 5th British combinatorial conf., 1975, 41—51.

[2] G. A. Dirac, *Path and circuits in graphs: Extreme cases*, Acta math. acad. sci. hung., 10 (1959), 357–362.

[3] G. A. Dirac, *On Hamilton circuits and Hamilton paths*, Math. Ann., 197 (1972), 57–70.

[4] G. A. Dirac, *Note on Hamilton circuits and Hamilton paths*, Math. Ann., 206 (1973), 139–147.

[5] P. Erdös and T. Gallai, *On maximal paths and circuits of graphs*, Acta math. acad. sci hung., 10 (1959), 337–356.

[6] S. E. Goodman and S. T. Hedetniemi, *On Hamiltonian walks in graphs*, SIAM J. Comput., 3 (1974), 214–221.

[7] J. L. Jolivet, *Hamiltonian pseudocycles in graphs*, Proc. 5th Conf. Combinatorics, graph theory and computing, Boca Raton, 1975, 529–533.

[8] O. Ore, *On a graph theorem by Dirac*, Journal of Combinatorial theory, 2 (1967), 383–392.

[9] T. Nishizeki, T. Asano and T. Watanabe, *An aproximation algorithm for Hamiltonion walk problem on maximal planar graphs*, Discrete applied mathematics, 5 (1983), 211–222.



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