TOLERANCES OF FRAMES

JOSEF NIEDERLE

(Received June 8, 1988)

Abstract. Frame tolerances are lattice tolerances compatible with arbitrary joins. They turn out to be just lattice tolerances the polars (or blocks) of which contain greatest elements. There exists a semigroup-and-order isomorphism between the lattice of all frame tolerances and the lattice of all extensive $\wedge$-endomorphisms of the frame. Lattices of frame tolerances are frames.

Key words. Frames, distributive lattices, tolerances.

MS Classification. 06 D 99, 08 A 30.

By a *frame* is meant a complete lattice satisfying the join-infinite distributive identity $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$. A *lattice tolerance* on a frame is a reflexive symmetric relation compatible with finite meets and joins. A *frame tolerance* on a frame is a lattice tolerance compatible with arbitrary joins (suprema).

Lattice tolerances on lattices and distributive lattices in general were extensively studied by several authors. It may be hardly anything added in the case of frames except for the Noetherian (alias with ACC) ones ([8]). In this paper we shall prove a few properties of frame tolerances.

By $[a, b]$ we denote the ordered pair of elements a and b, by $\langle a)$ the dual principal ideal generated by an element a, and finally by $T(a)$ the *polar* (alias neighbourhood, alias tolerance class) of an element a in a tolerance T. Recall that $T(a) = \{x \mid [x, a] \in \in T\}$ and X is a *block* of T when $X = \bigcap \{T(a) \mid a \in X\}$.

We shall use an alternative form of the Grätzer – Schmidt characterization of principal congruences on distributive lattices:

The following conditions are equivalent for the principal congruence $\Theta(a, b)$ on a distributive lattice:

(i) $[x, y] \in \Theta(a, b)$;

(ii) $(a \vee b) \vee (x \vee y) = (a \vee b) \vee (x \wedge y)$ *and* $(a \wedge b) \wedge (x \vee y) = (a \wedge b) \wedge (x \wedge y)$;

(iii) $a \vee b \vee x = a \vee b \vee y$ *and* $a \wedge b \wedge x = a \wedge b \wedge y$.

Theorem 1. *For a lattice tolerance T on a frame, the following conditions are equivalent:*

(i) *T is a frame tolerance;*

(ii) *each polar of T has a greatest element;*

(iii) *each block of T has a greatest element.*

Proof. Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are transparent. (iii) $\Rightarrow$ (ii): Let $a \in L$ be an arbitrary element of the frame L, denote $A = T(a) \cap \langle a)$. Clearly $a \in A$. For $x, y \in A$ we obtain $x \vee y \in A$, and consequently $[x, y] \in T$. Therefore the set A must be contained in some block of the tolerance T having a greatest element m by assumption. Then $[m, a] \in T$, hence m is the greatest element of $T(a)$. (ii) $\Rightarrow$ (i): Let $[a_i, b_i] \in T$ $(i \in I)$. Then $[\bigvee_{j \in I} a_j, b_i \vee \bigvee_{j \in I} a_j] = [a_i \vee \bigvee_{j \in I} a_j, b_i \vee \bigvee_{j \in I} a_j] \in T$. Since $T(\bigvee_{j \in I} a_j)$ has a greatest element, $[\bigvee_{j \in I} a_j, \bigvee_{j \in I} a_j \vee \bigvee_{j \in I} b_j] \in T$. Mutually, $[\bigvee_{j \in I} b_j, \bigvee_{j \in I} a_j \vee \bigvee_{j \in I} b_j] \in T$, and finally $[\bigvee_{j \in I} a_j, \bigvee_{j \in I} b_j] \in T$. Q.E.D.

Remark. An analogous statement may be a fortiori proved for frame congruences.

Observation. *For any tolerance T on a lattice L and any $a \in L$, there is a unique block containing $T(a) \cap \langle a)$.*

Proposition. *Principal frame tolerances coincide with principal lattice tolerances on the frame, with principal frame congruences, and with principal lattice congruences on the frame.*

Proof. It suffices to establish that a principal lattice tolerance is a frame congruence. Let $[a, b]$ be an arbitrary ordered pair of elements of the frame. Since principal tolerances on distributive lattices coincide with principal congruences ([2]), it holds $[x, y] \in T(a, b)$ if and only if $a \wedge b \wedge x = a \wedge b \wedge y$ and $a \vee b \vee x = a \vee b \vee y$. Denote $Y = T(a, b)(x) = \{y \mid a \wedge b \wedge x = a \wedge b \wedge y \text{ and } a \vee b \vee x = a \vee b \vee y\}$. We have $a \wedge b \wedge \bigvee Y = \bigvee_{y \in Y} (a \wedge b \wedge y) = a \wedge b \wedge x$, and $a \vee b \vee \bigvee Y = \bigvee_{y \in Y} (a \vee b \vee y) = a \vee b \vee x$. Hence $\bigvee Y$ is the greatest element of $T(a, b)(x)$.

Q.E.D.

Since the set of all frame tolerances of a given frame is closed under arbitrary set intersections, and both the diagonal and the universal relation are frame tolerances, *frame tolerances form a complete lattice, in which all infima coincide with set intersections.*

It may be of interest to investigate algebraic properties of lattices of frame tolerances. In the case of finitary algebras, it follows from some general considerations about relations that lattices of tolerances are complete, compactly generated (alias algebraic) with finitely generated tolerances as compact elements. It is not the case for frame tolerances.

Theorem 2. *For a frame, the following conditions are equivalent:*

(i) *it is Noetherian;*

(ii) *lattice tolerances and frame tolerances coincide;*

(iii) *finitely generated frame tolerances are compact elements in the lattice of all frame tolerances;*

(iv) *principal frame tolerances are compact elements in the lattice of all frame tolerances.*

Proof. Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are transparent. (iv) $\Rightarrow$ (i): Let C_0 be an arbitrary chain in the frame, let $c \in C_0$. Put $C = C_0 \cap \langle c)$. Obviously $\bigvee C = \bigvee C_0$. Denote by T the principal (frame) tolerance generated by $[c, \bigvee C]$ and, for any $x \in C$, by T_x the principal (frame) tolerance generated by $[c, x]$. Then $[c, y] \in \bigvee_{x \in C} T_x$ for any $y \in C$, and so $[c, \bigvee C] = [\bigvee c, \bigvee C] \in \bigvee_{x \in C} T_x$. We have obtained $T = \bigvee_{x \in C} T_x$. As T is compact, $T = T_x$ for some $x \in C$. It follows that $[c, \bigvee C] \in T(c, x)$, i.e. $x \vee \bigvee C = x \vee c = x$. Consequently $\bigvee C_0 = \bigvee C = x \in C_0$.

Q.E.D.

Corollary. *In contrast to lattices of tolerances on finitary algebras, lattices of frame tolerances are not compactly generated.*

Thus it is not possible to follow the general scheme of investigations using properties of compact and relatively maximal elements in compactly generated lattices, which was developed and used in [4], [5] and [7] for finding a representation of tolerances on finite distributive lattices. Nevertheless, it happened to prove some interesting features of lattices of frame tolerances.

Statements of Theorem 1 enable us to apply considerations and results of I. G. Rosenberg and D. Schweigert ([9]). Recall that a mapping α of a lattice into itself is *extensive* if $x \leq \alpha(x)$ for any element x, and *idempotent* if $\alpha(\alpha(x)) = \alpha(x)$.

Theorem 3. *There exists a semigroup (isomorphism) and order isomorphism between the lattice of all frame tolerances and the lattice of all extensive $\wedge$-endomorphisms of a (given) frame. Congruences (i.e. transitive frame tolerances) correspond with idempotent extensive $\wedge$-endomorphisms.*

See [9] for details. The particular result for frame congruences has been already known, see [3]. How is this representation connected with that of [7] given for the finite case, and that of [8] given for the Noetherian case, will be reserved for a later paper.

Remark. Since principal frame tolerances are frame congruences (see the Proposition), it follows that *the set of all frame congruences is supremum-dense* (sensu [10]) *in the lattice of all frame tolerances.*

Remark. In contrast to tolerances on finitary algebras, *transitive hulls of frame tolerances are not frame congruences. Nevertheless, frame congruences still form a closure system in the lattice of frame tolerances, thus the mapping that assigns to any frame tolerance T the least frame congruence that includes T is a closure operator, and as such a complete $\vee$-homomorphism.* The last result was proved in [6] for finitary algebras.

Proofs of the preceding statements are transparent and therefore omitted.

Now, we shall show how suprema in lattices of frame tolerances look like, and prove distributivity. Denote by $\mathrm{FTol}(L)$ the lattice of all frame tolerances on a frame L. Put $\mathrm{FT}(A) = \bigcap \{T \in \mathrm{FTol}(L) \mid A \subseteq T\}$, where A is an arbitrary subset in $L \times L$.

Observation. $\mathrm{FT}(A)$ is the least frame tolerance that includes the set A.

Theorem 4. $\mathrm{FT}(A) = \{\bigvee \{\bigwedge X \mid X \in \mathrm{M}\} \mid \mathrm{M} \subseteq \mathrm{Exp}_{\mathrm{finite}}(A \cup A^{\mathrm{op}} \cup \Delta)\}$. *($A^{\mathrm{op}}$ is the converse of A, and Δ is the diagonal relation.)*

Proof. Denote $B \equiv \{\bigvee \{\bigwedge X \mid X \in \mathrm{M}\} \mid \mathrm{M} \subseteq \mathrm{Exp}_{\mathrm{finite}}(A \cup A^{\mathrm{op}} \cup \Delta)\}$. It is obvious that $A \cup A^{\mathrm{op}} \cup \Delta \subseteq \mathrm{FT}(A)$. Hence $\bigwedge X \in \mathrm{FT}(A)$ for any (finite subset) $X \in \mathrm{Exp}_{\mathrm{finite}}(A \cup A^{\mathrm{op}} \cup \Delta)$ and consequently $\bigvee (\bigwedge X \mid X \in \mathrm{M}\} \in \mathrm{FT}(A)$ for any (system of finite subsets) $\mathrm{M} \subseteq \mathrm{Exp}_{\mathrm{finite}}(A \cup A^{\mathrm{op}} \cup \Delta)$. We have proved $B \subseteq \mathrm{FT}(A)$. Obviously $A \subseteq B$. It remains to show that B is a frame tolerance. Reflexivity and symmetry are transparent. It follows the proof of compatibility. Let $[a_i, b_i] \in B$ $(i = 1, 2)$. There exist $\mathrm{M}_i \subseteq \mathrm{Exp}_{\mathrm{finite}}(A \cup A^{\mathrm{op}} \cup \Delta)$ such that $[a_i, b_i] = \bigvee \{\bigwedge X \mid X \in \mathrm{M}_i\}$ $(i = 1, 2)$. Then $[a_1, b_1] \wedge [a_2, b_2] = \bigvee \{\bigwedge X \mid X \in \mathrm{M}_1\} \wedge \bigvee \{\bigwedge Y \mid Y \in \mathrm{M}_2\} = \bigvee \{(\bigvee \{\bigwedge X \mid X \in \mathrm{M}_1\}) \wedge \bigwedge Y \mid Y \in \mathrm{M}_2\} = \bigvee \{\bigvee \{\bigwedge X \wedge \bigwedge Y \mid X \in \mathrm{M}_1\} \mid Y \in \mathrm{M}_2\} = \bigvee \{\bigwedge Z \mid Z \in \{X \cup Y \mid X \in \mathrm{M}_1, Y \in \mathrm{M}_2\}\} \in B$ as $\{X \cup Y \mid X \in \mathrm{M}_1, Y \in \mathrm{M}_2\} \subseteq \mathrm{Exp}_{\mathrm{finite}}(A \cup A^{\mathrm{op}} \cup \Delta)$. Meet compatibility of B has been proved. Further, let $[a_i, b_i] \in B$ $(i \in I)$. There exist $\mathrm{M}_i \subseteq$

$\subseteq \mathrm{Exp}_{\mathrm{finite}}(A \cup A^{\mathrm{op}} \cup \Delta)$ such that $[a_i, b_i] = \bigvee \{\bigwedge X \mid X \in \mathrm{M}_i\}$ $(i \in I)$. Then $\bigvee_{i \in I} [a_i, b_i] = \bigvee_{i \in I} \bigvee \{\bigwedge X \mid X \in \mathrm{M}_i\} = \bigwedge \bigcup_{i \in I} \{\bigwedge X \mid X \in \mathrm{M}_i\} = \bigvee \{\bigwedge X \mid X \in \bigcup_{i \in I} \mathrm{M}_i\} \in B$ as $\bigcup_{i \in I} \mathrm{M}_i \subseteq \mathrm{Exp}_{\mathrm{finite}}(A \cup A^{\mathrm{op}} \cup \Delta)$. Q.E.D.

Corollary. *Let T_i $(i \in I)$ be frame tolerances on a frame. It holds* $\bigvee_{i \in I} T_i = \{\bigvee \{\bigwedge X \mid X \in \mathrm{M}\} \mid \mathrm{M} \subseteq \mathrm{Exp}_{\mathrm{finite}}(\bigcup_{i \in I} T_i)\}$.

Proof. It is obvious that $\bigvee_{i \in I} T_i = \mathrm{FT}(\bigcup_{i \in I} T_i)$. But $\bigcup_{i \in I} T_i$ is reflexive and symmetric, and we are done. Q.E.D.

The preceding statement resembles that of [1], Theorem 2.

Theorem 5. *Lattices of frame tolerances are frames.*

Proof. It suffices to show that $T \wedge \bigvee_{i \in I} T_i \subseteq \bigvee_{i \in I} (T \wedge T_i)$ for any index set I and frame tolerances T and T_i $(i \in I)$. Let $[a, b]$ be an arbitrary element of $T \wedge \bigvee_{i \in I} T_i$. Inasmuch as meets coincide with set intersections, $[a, b] \in T$ and $[a, b] \in \bigvee_{i \in I} T_i$. By the Corollary of Theorem 4, there exists $\mathrm{M} \subseteq \mathrm{Exp}_{\mathrm{finite}}(\bigcup_{i \in I} T_i)$ such that $[a, b] = \bigvee \{\bigwedge X \mid X \in \mathrm{M}\}$. However, $[a, b] = [a \vee b, a \vee b] \wedge ([a \wedge b, a \wedge b] \vee [a, b]) = [a \vee b, a \vee b] \wedge ([a \wedge b, a \wedge b] \vee \bigvee \{\bigwedge X \mid X \in \mathrm{M}\}) = [a \vee b, a \vee b] \wedge \bigvee \{[a \wedge b, a \wedge b] \vee \bigwedge X \mid X \in \mathrm{M}\} = [a \vee b, a \vee b] \wedge \bigvee \{\bigwedge \{[a \wedge b, a \wedge b] \vee [x, y] \mid [x, y] \in X\} \mid X \in \mathrm{M}\} = \bigvee \{[a \vee b, a \vee b] \wedge \bigwedge \{[a \wedge b, a \wedge b] \vee [x, y] \mid [x, y] \in X\} \mid X \in \mathrm{M}\} = \bigvee \{\bigwedge \{[a \vee b, a \vee b] \wedge ([a \wedge b, a \wedge b] \vee [x, y]) \mid [x, y] \in X\} \mid X \in \mathrm{M}\} = \bigvee \{\bigwedge Z \mid Z \in \{\{[a \vee b, a \vee b] \wedge ([a \wedge b, a \wedge b] \vee [x, y]) \mid [x, y] \in X\} \mid X \in \mathrm{M}\}\}$. Since $\{\{[a \vee b, a \vee b] \wedge ([a \wedge b, a \wedge b] \vee [x, y]) \mid [x, y] \in X\} \mid X \in \mathrm{M}\} \subseteq \mathrm{Exp}_{\mathrm{finite}}(\bigcup_{i \in I} (T \cap T_i))$, it follows $[a, b] \in \bigvee_{i \in I} (T \wedge T_i)$. Q.E.D.

We have obtained a result analogous to that of [1], Theorem 16 even though the lattice of all frame tolerances is not a sublattice in the lattice of all lattice tolerances on the frame.

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J. Niederle
Viniční 60
615 00 Brno 15
Czechoslovakia