Archivum Mathematicum

Tadek Nadzieja

Shadowing lemma for family of ε-trajectories

Archivum Mathematicum, Vol. 27 (1991), No. 1-2, 65--77

Persistent URL: http://dml.cz/dmlcz/107405

SHADOWING LEMMA FOR FAMILY OF ε-TRAJECTORIES

TADEK NADZIEJA

(Received November 1, 1985)

Abstract. The aim of this paper is to show that most of the theorems about local stability of flow near hyperbolic set follows from one fundamental lemma.

Key words. Hyperbolic sets, Shadowing Lemma.

MS Classification. 58 F 15.

Shadowing Lemma is the most important tool for the detailed study of diffeomorphisms or flows near hyperbolic sets. Roughly speaking it states that pseudo orbit (for diffeomorphism or flow) contained in small neighborhood of hyperbolic set Λ can be approximed by orbit of some point from Λ. Some kinds of this lemma were proved by many authors [1], [2], [4], [7]. The most general version for diffeomorphisms can be found in [7].

The aim of this paper is to prove a similar Lemma for flows on a smooth manifold. Our result implies previous versions of Shadowing Lemma and a lot of facts about local stability of hyperbolic sets. Moreover the proofs of the known facts about hyperbolic sets, when using our Lemma, seem to be rather simpler than original ones.

The essential difference between our result and other versions of Shadowing Lemma for flows lies in considering the problem of approximation not only of a particular pseudo orbit but the whole family of pseudo orbits. Similar situation for diffeomorphisms is described in [7].

The proofs and ideas we use in this paper are combinations of ideas of Alekseyev, Anosov, Katok and Moser. Some technical results which were helpful in preparation of this paper can be found in [5], [6].

M will denote a smooth, Riemannian manifold, $X(M)$ the space of all complete vector field on M with C^1-topology. If $X \in X(M)$ then $X(t, .)$ will be the flow generated by X.

A compact subset Λ of M is *hyperbolic* for $X \in X(M)$ if Λ is invariant under the flow $X(t, .)$ and the tangent flow $TX(t, .)$ leaves invariant a continuous splitting

$$T_\Lambda M = E^s \oplus E^0 \oplus E^u \tag{1}$$

and for some λ, $0 < \lambda < 1$, $c > 0$ we have

(1a) if $v \in E^s$ and $t > 0$ then $\| TX(t, v) \| < c\lambda^t \| v \|$,
(1b) if $v \in E^u$ and $t > 0$ then $\| TX(t, v) \| > c\lambda^{-t} \| v \|$,
(1c) E^0 is spanned by the vector field X, i.e. $E^0 = \text{lin}\, X$,

where $\| . \|$ is a Riemannian structure on M.

The following characterization of hyperbolic set will be useful in the next part of the paper.

Remark 1. A compact Λ invariant under $X(t, .)$ is hyperbolic if and only if there exists $t_0 > 0$, a continuous splitting

$$T_\Lambda M = E^s_{t_0} \oplus E^0 \oplus E^u_{t_0},$$

which is invariant under $TX(t_0, .)$ and there are numbers $\lambda, 0 < \lambda < 1$, $\tilde{c} > 0$ such that

(2a) if $v \in E^s_{t_0}$ then $\| TX(nt_0, v) \| < \tilde{c}\lambda^{nt_0} \| v \|$ for $n \in N$,

(2b) if $v \in E^u_{t_0}$ then $\| TX(nt_0, v) \| > \tilde{c}\lambda^{-nt_0} \| v \|$ for $n \in N$,

(2c) $E^0 = \text{lin}\, X$.

Proof. We have to show the existence of a splitting invariant under $TX(t, .)$ and fulfilling the conditions (1 abc).

It is easy to see that the splitting (2) is uniquely determined, i.e. every continuous splitting $\tilde{E}^s \oplus \text{Lin}\, X \oplus \tilde{E}^u$ invariant under $TX(t_0, .)$ and close to the splitting (2) in C^0-sense is equal to (2).

Let for $t \in R$ denote $E^s_t = TX(t, E^s_{t_0})$ and $E^u_t = TX(t, E^u_{t_0})$.

The subbundles E^s_t, E^u_t are invariant under $TX(t_0, .)$. In fact,

$$\begin{aligned} TX(t_0, E^s_t) &= TX(t_0, TX(t, E^s_{t_0})) = \\ &= TX(t, TX(t_0, E^s_{t_0})) = E^s_t. \end{aligned}$$

For small t the bundles E^s_t, E^u_t are near $E^s_{t_0}$, $E^u_{t_0}$ in C^0-sense, respectively. Using the uniqueness of splitting we obtain $E^s_t = E^s_{t_0}$ and $E^u_t = E^u_{t_0}$ for small t and so for all t.

Put $E^s = E^s_{t_0}$, $E^u = E^u_{t_0}$. Then obviously the conditions (1 abc) hold.

Let F be a subbundle of $T_\Lambda M$ invariant under $TX(t_0, .)$. We denote by $\Gamma^0(F)$ the space of continuous sections of F and by $X^*(t_0, .)$ the linear operator on $\Gamma^0(F)$ defined as follows:

$$X^*(t_0, v)(q) = TX(t_0, v(X(-t_0, q))).$$

Remark 2. Let Λ be a compact, invariant under $X(t, .)$ subset of M and $t_0 > 0$. There is a splitting fulfilling the conditions (2abc) if and only if there exists a continuous subbundle E^{su} invariant under $TX(t_0, .)$ such that the linear operator $X^*(t_0, .) : \Gamma^0(E^{su}) \to \Gamma^0(E^{su})$ is hyperbolic.

Proof. The proof is the same as the proof of Mather's characterization of hyperbolic sets of diffeomorphism [9] hence we omit it.

The Remarks 1 and 2 give the characterization of hyperbolic sets which will be useful in the proof of Remark 5.

The *family of ε-trajectories of X* is a triple of elements $\langle P, \Phi, f(t, .)\rangle$, were P is a topological space, Φ is a continuous mapping from P to M and $f(t, .)$ is a continuous flow on P, satisfying:

$$\sup_{t \in [0,1]} d(\Phi(f(t, p)), X(t, \Phi(p))) < \varepsilon \qquad \text{for all } p \in P,$$

where d denotes the Riemannian metric on M generated by the structure $\|\,.\,\|$. The curve $\Phi(f(t, p))$ we will call the *ε-trajectory of X through the point* $\Phi(p)$.

Example 1. Let $\gamma : R \to M$ be a smooth curve such that $\| \dot\gamma(t) - X(\gamma(t)) \| < \varepsilon$ for all $t \in R$. Let $P = \gamma(R)$, $\Phi =$ Identity and $f(t, .)$ is the flow generated on P by $\dot\gamma(t)$. Then the triple $\langle P, \Phi, f(t, .)\rangle$ is a simple example of a family of ε-trajectories of X.

Example 2. Let $X, Y \in \times(M)$ and Y is in ε-C^0-neighborhood of X. Then $\langle M$, Identity, $Y(t, .)\rangle$ is a family of ε-trajectories of X.

The family $\langle P, \Phi, f(t, .)\rangle$ *can be δ-approximated by a family of trajectories* if there exist continuous mappings $\psi : P \to M$ and $\alpha : P \times R \to M$ such that:

(3a) $$\psi(f(t, p)) = X(\alpha(p, t), \psi(p)),$$

(3b) $$\sup_{p \in P} d(\psi(p), \Phi(p)) < \delta,$$

(3c) $$\alpha(p, 0) \doteq 0, \qquad | t - \alpha(p, t) | < \delta \qquad \text{for } t \in [0, 1]$$

$$\lim_{t \to +\infty} \alpha(p, t) = +\infty, \qquad \lim_{t \to -\infty} \alpha(p, t) = -\infty.$$

The conditions (3a), (3b) imply that $d(X((\alpha(p, t), \psi(p)), \Phi(f(t, p))) < \delta$ for all $t \in R$. It means that the trajectory of $\psi(p)$ approximates the ε-trajectory through $\Phi(p)$ after some "reparametrization" of time.

Generally the "reparametrization" $\alpha(p, t)$ is not a homeomorphism of R. Now we are ready to formulate the main result of this paper.

Lemma (Shadowing Lemma for family of ε-trajectories). *Let Λ be a hyperbolic* *set for $X \in X(M)$. Then there exist a neighborhood $U(\Lambda)$ of Λ and a neighborhood $W(X)$ of X satisfying the following property;*

(*) *For every $\delta > 0$ there exists $\varepsilon > 0$ such that if $\hat{X} \in W(X)$, $\langle P, \Phi, f(t, .)\rangle$ is a family of ε-trajectories of $\hat{X}$ and $\Phi(P) \subset U(\Lambda)$ then $\langle P, \Phi, f(t, .)\rangle$ can be δ-approximated by a family of trajectories of $\hat{X}$.*

Proof. We will prove the existence of a neighborhood $U(\Lambda)$ of Λ with following property: for all $\delta > 0$ there exists $\varepsilon > 0$ such that every family of ε-trajectories of X contained in $U(\Lambda)$, i.e. $\Phi(P) \subset U(\Lambda)$, can be δ-approximated by a family of trajectories of X. It will be easy to see that the neighborhood $U(\Lambda)$ has property (*), i.e. for any $\hat{X}$ from some small C^1-neighborhood of X every family of ε-trajectories of $\hat{X}$ which is contained in $U(\Lambda)$ can be δ-approximated by a family of trajectories of $\hat{X}$.

We can to extend the hyperbolic structure on Λ in a continuous way to some neighborhood $\tilde{U}(\Lambda)$ of Λ. Next this continuous extension we approximate in C^0-topology by the smooth splitting

$$T_{\tilde{U}(\Lambda)}M = \tilde{E}^s \oplus E^0 \oplus \tilde{E}^u. \tag{4}$$

In the proof of the Lemma we will use only the continuity of the splitting (4). Its smoothness will be useful in the next part of the paper (Corollary 2).

The splitting (4) may be not invariant under tangent flow $TX(t, .)$, but for every $\varepsilon_0 > 0$ there are a neighborhood $U(\Lambda) \subset \tilde{U}(\Lambda)$, a positive number T_0 such that

$$\| P^s_{X(t,x)} TX(t, v(x)) \| < \frac{1}{2} \| v(x) \|,$$

$$\| P^u_{X(t,x)} TX(t, v(x)) \| < \varepsilon_0 \| v(x) \| \tag{5}$$

$$\text{for } v(x) \in \tilde{E}^s_x, \quad x \in U(\Lambda) \quad \text{and} \quad t \in \left[\frac{1}{4} T_0, T_0\right],$$

$$\| P^u_{X(t,x)} TX(t, v(x)) \| > 2 \| v(x) \|,$$

$$\| P^s_{X(t,x)} TX(t, v(x)) \| < \varepsilon_0 \| v(x) \| \tag{6}$$

$$\text{for } v(x) \in \tilde{E}^u_x, \ x \in U(\Lambda) \quad \text{and} \quad t \in \left[\frac{1}{4} T_0, T_0\right],$$

$$\| P^0_{X(t,x)} TX(t, v(x)) \| < \varepsilon_0 \| v(x) \| \tag{7}$$

$$\text{for } v(x) \in \tilde{E}^s_x \otimes \tilde{E}^u_x, \ x \in U(\Lambda) \quad \text{and} \quad t \in \left[\frac{1}{4} T_0, T_0\right],$$

where P^s_x, P^u_x and P^0_x denotes the projections on the space $E^s_x, E^u_x, E^0_x = \text{lin}\, X(x)$, respectively.

Fix a number $\delta > 0$. We shall show that for sufficiently small $\varepsilon > 0$, every family of ε-trajectories $\langle P, \Phi, f(t, .)\rangle$ of X contained in $U(\Lambda)$ can be δ-approximated by family of trajectories.

Fix a number $t \in \left[\frac{1}{4} T_0, T_0\right]$. We start with the case of "discrete time". We want to solve the functional equation:

$$\psi(f(t, p)) = X((u(p) + 1)\, t, \psi(p)), \tag{8}$$

where $\psi: P \to M$ and $u : P \to R$ are unknown functions.

Equation (8) is a discrete analog of (3a).

By H_Φ we denote the space of continuous, bounded mappings $w : P \to T_{U(\Lambda)}M$ such that $w(p) \in T_{\Phi(p)}M$ for any $p \in P$.

We equip the set H_Φ with the supremum norm

$$\| w \| = \sup_{p \in P} \| w(p) \| .$$

We will try to find a solution of (8) in the form

$$\psi(p) = \exp_{\Phi(p)} v(p),$$

where $v \in H_\Phi$ and $v(p) \in \tilde{E}^s_{\Phi(p)} \oplus \tilde{E}^u_{\Phi(p)}$.

Thus (8) may be rewritten as follows

$$\exp_{\Phi(f(t, p))} v(f(t, p)) = X((u(p) + 1)\, t, \exp_{\Phi(p)} v(p)). \tag{9}$$

Now we will transform (9) to a form convenient for our purposes.

Consider the expression

$$K(v) = \exp^{-1}_{X(t, \Phi(p))} \exp_{\Phi(f(t, p))} v(f(t, p)). \tag{10}$$

This formula gives, for small ε, a well defined, smooth mapping K from some neighborhood of 0 in H_Φ to $H_{X(t, \Phi(\cdot))}$.

Using the Taylor expansion we obtain:

$$K(v) = K(0) + J_\varepsilon v + O(v), \tag{11}$$

where J_ε is the linear part of K at 0 and $\lim_{v \to 0} \frac{O(v)}{\| v \|} = 0$.

It is easy to see that $\| J_\varepsilon \|$ tends to 1 if ε goes to 0. Consequently the left hand side of (9) is equal to

$$\exp_{X(t, \Phi(p))} [K(0) + J_\varepsilon v(p) + O(v)(p)]. \tag{12}$$

Now we want to find a similar form for the right side of (9).

Consider the mapping F from some neighborhood of 0 in H_Φ to $H_{X(t, \Phi(\cdot))}$ defined as follows: every element w of H_Φ can be in the unique way represented in the form $w = v + uX$, where u is a real continuous function on P and $v(p) \in \tilde{E}^s_{\Phi(p)} \oplus \oplus \tilde{E}^s_{\Phi(p)}$. Put

$$F(w) = F(v + uX) = \exp^{-1}_{X(t,\Phi(p))} X((u(p) + 1)\, t, \exp_{\Phi(p)} v(p)). \tag{13}$$

F is the smooth mapping and the derivative of F at 0 taken for $w = v + uX$ is equal

$$DF(0)\,(v + uX) = TX(t, v(p)) + tu(p)\, X(X(t, \Phi(p))). \tag{14}$$

Hence

$$\begin{aligned} X((u(p) + 1)\, t, \exp_{\Phi(p)} v(p)) &= \exp_{X(t,\Phi(p))}[TX(t, v(p)) + \\ &+ tu(p)\, X(X(t, \Phi(p)) + F(0) + 0(w)\,(p)], \end{aligned} \tag{15}$$

where $\lim\limits_{w \to 0} \frac{0(w)}{\| w \|} = 0$.

Combining (12) and (15) we obtain the following form of the equation (8)

$$\begin{aligned} J_\varepsilon v(p) - (TX(t, v(p)) - tu(p)\, X(X(t, \Phi(p))) &= \\ = F(0) - K(0) + 0(w)\,(p) &= a(w)\,(p). \end{aligned} \tag{16}$$

The left hand side D of (16) is a linear operator from H_Φ to $H_{X(t,\Phi(\cdot))}$.
We want to show that D is invertible.
Using the decomposition (4) we can present D in the form

$$\begin{bmatrix} D^{ss}, & D^{su}, & D^{s0} \\ D^{us}, & D^{uu}, & D^{u0} \\ 0, & 0, & D^{00} \end{bmatrix}.$$

To prove invertibility of D it is enough to prove that operator $\hat{D}$

$$\begin{bmatrix} D^{ss}, & 0, & 0 \\ 0, & D^{uu}, & 0 \\ 0, & 0, & D^{00} \end{bmatrix},$$

is invertible.

In fact, the set of invertible operators is open and $\| D - \hat{D} \| < \varepsilon_0$ (for ε_0 see (5), (6), (7)). Hence for sufficiently small ε_0 the operator D is invertible.

Remark that in this moment we have fixed ε_0 and chosen the neighborhood $U(\Lambda)$.

Now we will prove invertibility of $\hat{D}$.

Define the following subspaces of H_Φ:

$$\begin{aligned} H^s_\Phi &= \{v \in H_\Phi : v(p) \in E^s_{\Phi(p)}\}, \\ H^u_\Phi &= \{v \in H_\Phi : v(p) \in E^u_{\Phi(p)}\}, \\ H^0_\Phi &= \{v \in H_\Phi : v(p) \in \mathrm{lin}\, X(\Phi(p))\}. \end{aligned}$$

Obviously

$$H_\Phi = H_\Phi^s \oplus H_\Phi^0 \oplus H_\Phi^u.$$

The operator $D^{ss} : H_\Phi^s \to H_\Phi^s$ defined by

$$D^{ss}v(p) = P^s_{X(t,\Phi(p))}[J_\varepsilon v(p) - TX(t, v(p))]$$

is invertible for small ε, due to the fact that

$$P^s_{X(t,\Phi(p))}TX\ (t, .) : H_\Phi^s \to H^s_{X(t,\Phi\ (\cdot))}$$

is a contraction and $\| J_\varepsilon \|$ tends to 1 if ε goes to 0.

In a similar way we can show that D^{uu} is invertible.

Note that

$$(D^{00}uX)\,(X(t, \Phi(p)) = tu(p)\,X(X(t, \Phi(p))).$$

Hence D^{00} is invertible too. Thus $\hat{D}$ is invertible.

Consider the mapping $G^t = D^{-1} . a$.

We claim that G^t maps some small neighborhood of 0 in H_Φ into itself.

In fact,

$$\| G^t w \| \leqq \| D^{-1} \| \; \| a(w) \| \leqq \| D^{-1} \| \; \| K(0) - F(0) \| + \| D^{-1} \| \; \| 0\,(w) \|,$$

$w(\varepsilon) = \| K(0) - F(0) \|$ depends on ε and $\lim_{\varepsilon \to 0} w(\varepsilon) = 0$. The mapping $\| 0(w) \|$ may be represented in the form $\| 0(w) \| = \gamma(w) . \| w \|$ where $\lim_{w \to 0} \gamma(w) = 0$. Hence

$$\| G^t w \| \leqq \| D^{-1} \| \; \omega(\varepsilon) + \gamma(w) \; \| w \|.$$

So for sufficiently small ε and r

$$\| G^t w \| < r \qquad \text{for } \| w \| < r.$$

Hence our claim is proved.

Obviously G^t is a contraction. Hence G^t has a fixed point $w^t = v^t + u^t X$ in small neighborhood of 0 in H_Φ.

Hence the functions $\psi^t = \exp_{\Phi\,(p)} v^t(p)$ and $u^t(p)$ are the solution of (8).

From the proof it follows that equation (8) has the unique solution of the form $\psi(p) = \exp_{\Phi(p)} v(p)$.

It is not hard to see that v^t and u^t are continuous functions of t (for detailes see [5]). Roughly speaking this fact follows from "continuous dependence" of the contraction G^t in t.

Define

$$\alpha(p, t) = (u^t(p) + 1) . t \qquad \text{for } t \in \left[\frac{1}{4} T_0, T_0\right].$$

We are going to show that $\psi^t = \psi^{T_0}$ for $t \in \left[\frac{1}{4} T_0, T_0\right]$. This part of the proof is like steps 3 and 4 in the proof of Theorem A in [5]. We will present here this proof for the sake of completness and we will omit some technical details which can be found in [5].

We shall show that the equality $\psi^{t_1} = \psi^{t_2}$ implies that

$$\psi^{t_1+t_2} = \psi^{t_1}$$

and

$$\alpha(f(t_2, p), t_1) + \alpha(p, t_2) = \alpha(p, t_1 + t_2) \tag{17}$$
$$\text{for } t_1, t_2, t_1 + t_2 \in \left[\frac{1}{4} T_0, T_0\right].$$

By easy calculation we obtain

$$\psi^{t_1}(f(t_1 + t_2, p)) = X(\alpha(f(t_2, p), t_1), \psi^{t_1}(f(t_2, p))) =$$
$$= X(\alpha(f(t_2, p), t_1) + \alpha(p, t_2), \psi^{t_1}(p)).$$

Consequently, by uniqueness of solutions of (8), we get (17).

In a similar way we can prove that the equality $\psi^{t_1} = \psi^{t_2}$ implies that $\psi^{t_1} = \psi^{t_1-t_2}$ for $t_1, t_2, t_1 - t_2 \in \left[\frac{1}{4} T_0, T_0\right]$.

Using this fact we obtain $\psi^{T_0} = \psi^{\frac{k}{2^n} \cdot T_0}$ for $k, n \in N$ and $\frac{k}{2^n} \cdot T_0 \in \left[\frac{1}{4} T_0, T_0\right]$.

Hence $\psi^t = \psi^{T_0}$ for $t \in \left[\frac{1}{4} T_0, T_0\right]$ due to continuity of ψ^t under t.

We put $\psi = \psi^{T_0}$.

For $t \in \left[0, \frac{1}{4} T_0\right]$ we define the function α as follows:

$$\alpha(p, t) = \alpha\left(p, \frac{1}{4} T_0 + t\right) - \alpha\left(f(t, p), \frac{1}{4} T_0\right).$$

We will check that (3a) holds for $t \in \left[0, \frac{1}{4} T_0\right]$.

Note that

$$X\left(\alpha\left(f(t, p), \frac{1}{4} T_0\right), X(\alpha(p, t), \psi(p))\right) =$$
$$= X\left(\alpha\left(f(t, p), \frac{1}{4} T_0\right) + \alpha(p, t), \psi(p)\right) =$$
$$= X\left(\alpha\left(p, t + \frac{1}{4} T_0\right), \psi(p)\right) = \psi\left(f\left(t + \frac{1}{4} T_0, p\right)\right) =$$

$$= X\left(\alpha\left(f(t, p), \frac{1}{4} T_0\right), \psi(f(t, p))\right).$$

Hence

$$X(\alpha(p, t), \psi(p)) = \psi(f(t, p)).$$

For $t \in [-T_0, 0]$ we put

$$\alpha(p, t) = -\alpha(f(t, p), -t)$$

and then by induction on $k \in N$ we put

$$\alpha(p, t) = \alpha(p, t - kT_0) - \sum_{i=0}^{k-1} \alpha(f(t - iT_0, p), -T_0)$$

for $t \in [kT_0, (k+1) T_0]$.

Finally for $t \leqq 0$ we put

$$\alpha(p, t) = -\alpha(f(t, p), -t).$$

Then (3ac) will be fullfilled on $P \times R$.

Remark 3. It follows from the proof of Lemma that the mappings ψ and α are uniquely determined in some sense. More precisely, there exists $\delta_0 > 0$ such that if the mappings ψ_1, α_1 and ψ_2, α_2 fulfill (3abc) with $\delta < \delta_0$ and $\psi_i(p) = \exp_{\Phi(p)} v_i(p)$ where $v_i(p) \in \tilde{E}^s_{\Phi(p)} \in \tilde{E}^u_{\Phi(p)}$, then $\psi_1 = \psi_2$ and $\alpha_1 = \alpha_2$.

Remark 4. If we assume that $\psi_1, \alpha_1, \psi_2, \alpha_2$ fulfill (3abc) with $\delta < \delta_0$, then without any assumption on the form of ψ_i there is a continuous function $\beta : P \to R$ such that

(18) $$\psi_1(p) = X(\beta(p), \psi_2(p)) \quad \text{and} \quad |\beta(p)| < \text{Const}. \delta.$$

Proof. It is enough to prove the thesis in the case ψ_1 has the form $\psi_1(p) = \exp_{\Phi(p)} v(p)$ where $v(p) \in \tilde{E}^s_{\Phi)p(} + \tilde{E}^u_{\Phi(p)}$.

Consider the manifold $M_r = \exp_{\Phi(p)} B_r(\Phi(p))$ where $B_r(\Phi(p))$ is an open bal in $\tilde{E}^s_{\Phi(p)} \oplus \tilde{E}^u_{\Phi(p)}$ of the radius r. For sufficiently small r, M_r is transversal to X and there is only one point $\psi_1(p)$ on M_r the trajectory of which δ-approximates the ε-trajectory through $\Phi(p)$. Because the trajectory of $\psi_2(p)$ also δ-approximates the ε-trajectory through $\Phi(p)$, hence there is a number $\beta(p)$ such that $\psi_1(p) = X(\beta(p), \psi_2(p))$.

Continuity of β and inequality (18) are obvious.

In particular, it follows from Remark 4 that for sufficiently small δ there is only one trajectory which δ-approximates the ε-trajectory through $\Phi(p)$.

We may rewrite the Remark 4 in the following way:

Corollary 1. *The flow on a hyperbolic set is expansive, i.e. there is* $\delta_0 > 0$ *such that if* $\sup_{t \in R} d(X(t, p), X(t, q)) < \delta_0$, *then* $p = X(s, q)$ *for some small s.*

Remark 5. The closure of $\psi(P)$ is a hyperbolic set for X.

Proof. Conditions (3ac) imply that $\psi(P)$ is invariant, hence cl $\psi(P)$ is invariant, too.

The operator $X^*(T_0, .)$ is hyperbolic. In fact, in the proof of Lemma we showed that the operator $cJ_\varepsilon - X^*(T_0, .)$ is invertible for every $c \in C$ such that $| c | = 1$ (compare the proof of invertibility of D). Hence $X^*(T_0, .)$ is hyperbolic on $\Gamma^0(\tilde{E}^{su})$. Due to Remark 2 cl $\psi(P)$ is a hyperbolic set.

Corollary 2 (local stability of hyperbolic set). *Let* Λ_X *be a hyperbolic set for X. For any number* $\delta > 0$ *there exists a neighborhood W of X in* C^1*-topology such that for every* $Y \in W$ *there is an invariant set* Λ_Y, *homeomorphism* $\psi_Y : \Lambda_X \to \Lambda_Y$ *and a function* $\alpha : \Lambda_Y \times R \to R$ *such that* $\sup_{p \in \Lambda_X} d(\psi_Y(p), p) < \delta$ *and the following diagram*

$$
\begin{array}{ccc}
\Lambda_X & \xrightarrow{\;X(t,.)\;} & \Lambda_X \\
\psi_Y \big\downarrow & & \big\downarrow \psi_Y \\
\Lambda_Y & \xrightarrow[\;Y(\alpha(.\,,t),.)\;]{} & \Lambda_Y
\end{array}
\tag{18}
$$

commutes.

Proof. If a vector field Y belongs to a sufficiently small neighborhood W of X, then $\langle \Lambda_X, Id, X(t, .)\rangle$ is the family of ε-trajectories for Y. Denote by ψ_Y and α the mappings which were constructed in the proof of Shadowing Lemma. For such ψ_Y and α the diagram commutes. All that remains is to show that ψ_Y is invertible.

It follows from Remark 4 that the equality $\psi_Y(p) = \psi_Y(q)$ implies $p = Y(\beta(q), q)$. Now, using the fact that decomposition (4) is smooth, we can show that there is $b > 0$ such that $\psi_Y(p) \neq \psi_Y(x(t, p))$ for all $p \in \Lambda_X$ and $t \in (-b, b)$. In fact, let $B_r(p)$ be an open ball in T_pM of the radius r. The mapping $E : \cup \{B_r(X(t, p)) : t \in \in (-b, b)\} \to M$ defined by $E(v) = \exp_{X(t,p)} v$ for $v \in B_r(X(t, p))$ is smooth and derivative of E at p is identity. Hence for sufficiently small r and b the mapping E is a diffeomorphism. Thus ψ_Y is invertible.

The structural stability of Anosov flows follows in an easy way from Corollary 2. In fact we have something more, we showed that there is a C^1-neighborhood W of Anosov flow X such that for $Y \in W$, X and Y are conjugate in the sense of diagram (19).

Corollary 3 (see [5]). *Anosov flows are topologically stable.*

Proof. Let X be a Anosov field on a manifold M. If Y is in some small C^0-neighborhood of X, then $\langle M, Id, Y(t, .)\rangle$ is a family of ε-trajectories of X. From Lemma it follows that there are the continuous mapping $\psi_Y : M \to M$ and $\alpha : M \times R \to R$ such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\;Y(t,.)\;} & M \\ {\scriptstyle\psi_Y}\downarrow & & \downarrow{\scriptstyle\psi_Y} \\ M & \xrightarrow[X(\alpha(.\,,t),.)]{} & M \end{array}$$

commutes. Because ψ_Y is a continuous mapping from some small neighborhood of identity, hence ψ_Y is onto (see [10]).

Now we are going to show that flow on hyperbolic set is semiconjugate to suspension flow of subshift of finite type (see [3]).

We start with recalling some definitions.

For $A = [A_{ij}]$, an $n \times n$ matrix of $0's$ and $1's$, we define

$$\Sigma_A = \{x = \{\bar{x}_i\} \in \{1, \ldots, n\}^Z : A_{x_i x_j} = 1 \qquad \text{for all } i, j \in Z\}$$

and

$$\sigma_A : \Sigma_A \to \Sigma_A \qquad \text{by} \qquad \sigma_A(\{x_i\}) = \{x_i'\} \qquad \text{where} \qquad x_i' = x_{i+1}.$$

If we give $\{1, \ldots, n\}$ the discrete topology and $\{1, \ldots, n\}^Z$ the product topology, then Σ_A becomes a compact space and σ_A a homeomorphism.

(Σ_A, σ_A) is called a *subshift of finite type.*

On the set $\Sigma_A \times R$ we define the flow $\sigma(t, (\bar{x}, s)) = (\bar{x}, t + s)$.

We identify the points $(\bar{x}, 0)$ and $(\sigma_A(\bar{x}), 1)$. After this identification we obtain space $\tilde{\Sigma}_A$ and flow $\tilde{\sigma}(t, .)$ on $\tilde{\Sigma}_A$. For simplicity of notation the elements of $\tilde{\Sigma}_A$ we will denote like elements of $\Sigma_A \times R$, i.e. $(\{x_n\}, t)$ denotes the element of $\tilde{\Sigma}_A$ after the identification described above.

Corollary 4. (see [3]). *Let Λ be a hyperbolic set for X. There exist a continuous surjection $\psi : \tilde{\Sigma}_A \to \Lambda$ and a function $\alpha : \tilde{\Sigma}_A \times R \to R$ such that the diagram*

$$\begin{array}{ccc} \tilde{\Sigma}_A & \xrightarrow{\;\tilde{\sigma}(t,.)\;} & \tilde{\Sigma}_A \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Lambda & \xrightarrow[X(\alpha(t,.)\,.)]{} & \Lambda \end{array}$$

commutes.

Proof. Let δ and ε be the same as in Lemma. We suppose that δ is small such that there is only one orbit which δ-approximates ε-trajectory.

Let $E_1, \ldots, E_n$ be an open covering of Λ such that $\operatorname{diam}(X(t, E_i)) < \varepsilon/2$ for $t \in [0, 1]$.

We define a matrix $A = [A_{ij}]$ in the following way: $A_{ij} = 1$ if and only if, $X(1, E_i) \cap E_j \neq \emptyset$.

Let $\{a_i\}$, $i = 1, \ldots, n$, be a sequence of points such that $a_i \in E_i$.

Note that if $d(X(1, a_i), a_j) < \varepsilon$, then there is a flow $\xi_{ij}(t, .)$ such that $\xi_{ij}(1, a_i) = a_j$ and

$$d(\xi_{ij}(t, a_i), X(t, a_i)) < \varepsilon \qquad \text{for } t \in [0, 1].$$

We define $\Phi : \tilde{\Sigma}_A \to \Lambda$ as follows:

$$\Phi(\{x_n\}, t) = \xi_{ij}(t, a_{x_0}) \qquad \text{for } t \in [k, k + 1].$$

Then $\langle \tilde{\Sigma}_A, \Phi, \tilde{\delta}(t, .) \rangle$ is the family of ε-trajectories for X. Hence there are continuous mappings ψ and α such that the diagram commutes.

Now we remain to show that ψ is a surjection. For $p \in \Lambda$ we define the sequence $\{x_n\}$ as follows: $x_j = i$ if and only if $X(j, p) \in E_i$. The ε-trajectory $\Phi(\tilde{\sigma}(t, \{x_n\}))$ is δ-approximated by the trajectory of the point p, hence $\psi(\{x_n\}, 0) = X(\beta\{p\}, p)$. Thus $p = \psi(\tilde{\sigma}(t, (\{x_n\}, 0)))$ for some small t.

Corollary 5. *Let Λ be a hyperbolic set for X. The periodic points of the flow $X(t, .)|_\Lambda$ are dense in the set of nonwandering points $X(t, .)|_\Lambda$.*

Proof. Let $p \in \Omega(X(t, .)|_\Lambda)$ and $\delta > 0$. There exists a point $q \in \Lambda$ such that $d(p, q) < r$ and $d(X(t, q), p) < r$ for some $T > 1$. If the number r is sufficiently small we can construct a smooth closed curve γ such that: $q \in \gamma$, γ is in a small neighborhood of Λ and $\|\dot{\gamma}(t) - X(\gamma(t))\| < \varepsilon$ where ε is a small number which depends on δ as in Lemma.

To obtain periodic orbit $\tilde{\gamma}$ such that $\operatorname{dist}(p, \tilde{\gamma}) < \delta$ we may apply the Lemma to the case: $P = \gamma$, $\Phi = Id$, $f(t, .)$ is the flow generated on γ by $\dot{\gamma}(t)$.

In similar way we may obtain

Corollary 6 (see Th. B in [4]). *If Λ is a hyperbolic set for X and $X(t, .)$ is chain recurrent, then Λ is contained in the closure of the set of periodic orbits.*

REFERENCES

[1] D. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Proceedings of the Steklov Institute of Mathematical 90 (1967) (American Mathematical Society translation).

[2] R. Bowen, *Periodic orbits for hyperbolic flows*, American Journal of Mathematics 94 (1972) 1–30.

[3] R. Bowen, *Symbolic dynamics for hyperbolic flows*, American Journal of Mathematics 95 (1973), 421–460.
[4] J. E. Franke, J. F. Selgrade, *Hyperbolicity and chain recurrence*, Journal of Differential Equations 26 (1977), 27–36.
[5] K. Kato, A. Morimoto, *Topological stability of Anosov flows and theirs centralizers*, Topology 12 (1973), 255–273.
[6] K. Kato, A. Morimoto, *Topological Ω-stability of Axiom* A *flows with no Ω-explosions*, Journal of Differential Equations 34 (1979), 464–481.
[7] A. Katok, *Local properties of hyperbolic sets* (in Russian), appendix to the Russian edition of: book Z. Nitecki Differentiable Dynamics, Mir, 1975, 219–232.
[8] J. Moser, *On a theorem of Anosov*, Journal of Differential Equations 5 (1969), 411–440.
[9] Z. Nitecki, *Differentiable Dynamics*, The MIT Press, 1972.
[10] P. Walters, *Asosov diffeomorphisms are topologically stable*, Topology 9 (1970), 71–78.

Tadek Nadzieja
Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław
Poland