Dobiesław Bobrowski

Linear boundary value problem in Banach space with random element in two-point boundary condition

Archivum Mathematicum, Vol. 26 (1990), No. 4, 215--221

Persistent URL: http://dml.cz/dmlcz/107391

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 26, No. 4 (1990), 215–222

LINEAR BOUNDARY VALUE PROBLEM IN BANACH SPACE WITH RANDOM ELEMENT IN TWO-POINT BOUNDARY CONDITION

DOBIESŁAW BOBROWSKI

(Received January 15, 1987)

Abstract. The paper contains probabilistic characterization of a mild sample path solution of a linear two-point boundary value problem with a random element in the boundary condition.

Key words. Two-point boundary value problem, random element in Banach space, evolution operator, mild sample path solution.

MS Classification. 34 F 05.

Very recently the mathematical modelling of physical and other systems leads to linear evolution equation in Banach spaces but the uncertainties of enviroment needs to introduce some randomnesses in the initial or boundary conditions. There is rather few papers devoted to study deterministic boundary value problems in Banach spaces. Here we can mention following papers: [3], [4], [6], [7], [9].

The aim of this paper is a characterization of mild sample path solution of a linear two-point boundary value problem in a Banach space when the boundary condition contain a random element.

Let $(\mathbf{B}, |\,.\,|)$ be a real separable Banach space, and let $\mathbf{B}^*$ be its dual and $\mathbf{B}^{**}$ its second dual spaces. Moreover let $([\mathbf{B}], \|\,.\,\|)$ be the Banach algebra of all continuous linear operators from $\mathbf{B}$ into itself and let $\mathscr{B}$ be the σ-algebra of Borel sets in $\mathbf{B}$ (in the sense of strong topology).

Without the lost of generality we take $J = [0, 1]$ in the place or arbitrary nondegenerate interval of the real line $\mathbf{R}$.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. We say that the mapping

$$\xi.: \Omega \times J \to \mathbf{B}$$

is a continuous $\mathbf{B}$-valued stochastic process if for each $t \in J$ it is a random element in $\mathbf{B}$ and its realizations $t \mapsto \xi_t(\omega)$ are $\mathbf{B}$-valued functions continuous with probability 1. The class of all such processes will be denoted by $C(\Omega \times J, \mathbf{B})$.

We shall consider following linear boundary value problem

$$\begin{cases} \dot{\xi}_t = A(t)\,\xi_t \\ B\xi_0 + C\xi_1 = \eta \end{cases} \qquad t \in J, \tag{1}$$

where $\{\xi_t,\ t \in J\} \in C(\Omega \times J,\ \mathbf{B})$ and $\{\dot{\xi}_t,\ t \in J\}$ is a sample-path derivative (in the strong topology) of the process $\{\xi_t,\ t \in J\}$. For every $t \in J$ $A(t) \in [\mathbf{B}]$ and $\mathrm{Dom}\,(A(t)) = D$ is independent of t and dense in $\mathbf{B}$. The operators $B, C \in [\mathbf{B}]$ and η is a random element in $\mathbf{B}$.

A stochastic process $\{\xi_t,\ t \in J\} \in C(\Omega \times J,\ \mathbf{B})$ is said to be a sample path mild solution of the problem (1) if there exists $\Omega_1 \subset \Omega$ such that

(α) $P(\Omega_1) = 1$,

(β) $\xi_\cdot(\omega)\colon J \to \mathbf{B}$ is a mild solution of the deterministic boundary-value problem

$$\begin{cases} \dot{\xi}_t(\omega) = A(t)\,\xi_t(\omega) \\ B\xi_0(\omega) + c\xi_1(\omega) = \eta(\omega) \end{cases} \qquad t \in J, \tag{2}$$

i.e. a solution of the integral equation

$$\xi_t(\omega) = \xi_0(\omega) + \int_0^t A(s)\,\xi_s(\omega)\,\mathrm{d}s \tag{3}$$

for every $\omega \in \Omega_1$.

Now let $\Phi(t, s)$ be the evolution operator corresponding to the operator $A(t)$, i.e. a solution of the operator differential equation

$$\begin{cases} \dot{u} = A(t)\,u, \\ u(s) = I, \end{cases} \tag{4}$$

where I is the identity operator.

Very significant in our consideration will be the operator

$$\Psi(t) = \Phi(t, 0)\,[B + C\Phi(1, 0)]^{-1}, \qquad t \in J. \tag{5}$$

Theorem 1. *Assume that*

(i) *for each $t \in J$ $A(t) \in [\mathbf{B}]$ is an operator with* Dom $A(t) = D$ *independent of t and dense in* $\mathbf{B}$,

(ii) *there exists* $\left[\vartheta \in \left(\frac{\pi}{2}, \pi\right)\right]$ *such that the resolvent set $\varrho(A(t))$ contains the set S of all complex numbers λ satisfying condition $-\ \vartheta \leqslant \arg\lambda \leqslant \vartheta$, for every $t \in J$,*

(iii) *there exists a positive constant k, independent of $\lambda \in S$ and t such that*

$$\|\,[\lambda I - A(t)]^{-1}\,\| \leqslant k/(1 + |\,\lambda\,|),$$

(iv) *the operator $A(t)\,A^{-1}(s) \in [\mathbf{B}]$ is Hölder continuous in t in the uniform operator topology for each fixed $s \in J$,* i.e.

$$\|\,[A(t) - A(\tau)]\,A^{-1}(s)\,\| \leqslant c\,|\,t - \tau\,|^{\mu},$$

where c and μ are positive constant ($0 < \mu \leqslant 1$) *independent of s, t, and τ for* $0 \leqslant t, s, \tau \leqslant 1$,

(v) *the operator $B + C\Phi(1, 0)$ is bounded below,*

(vi) *random element η is such that*

$$\eta \in range\ (\Psi(t)) \tag{6}$$

with probability 1 *and independently of t.*

Then the problem (1) *has a unique mild path-solution*

$$\xi_t = \Psi(t)\, \eta \tag{7}$$

Proof. If the assumptions (i)–(iv) are satisfied then by Theorem 3.2.1 in [5] the evolution equation

$$\dot{v} = A(t)\, v, \qquad 0 < t \leqslant 1$$

has a unique fundamental solution (evolution operator) $\Phi(t, s)$, moreover if $e \in \mathbf{B}$ then by Lemma 3.5.1 [5] there exists a unique (mild) solution of the abstract Cauchy problem

$$\begin{cases} \dot{v} = A(t)\, v, \\ v(0) = e. \end{cases}$$

This solution has a form

$$v(t) = \Phi(t, 0)\, e.$$

Now let $a \in \mathbf{B}$. By assumption (v) there exists inverse operator $[B + C\Phi(1, 0)]^{-1} \in [\mathbf{B}]$. If

$$e = [B + C\Phi(1, 0)]^{-1}\, a,$$

then

$$v(t) = \Phi(t, 0)\, [B + C\Phi(1, 0)]^{-1}\, a = \Psi(t)\, a$$

is a solution of the differential equation

$$\dot{v} = A(t)\, v, \qquad t \in J$$

and it is quite easy to prove that this solution satisfies the boundary value condition

$$B \vee (0) + C \vee (1) = a.$$

Therefore, under the assumptions (i)–(vi) for arbitrary $\eta(\omega)$, $\omega \in \Omega_1 = \{\omega : \eta(\omega) \in \text{Rang}\ \Psi(t)\}$ the boundary-value problem (2) has a unique solution

$$\xi_t(\omega) = \Phi(t, 0)\, [B + c\Phi(1, 0)]^{-1}\, \eta(\omega).$$

Hence

$$\xi_t(\omega) = \Psi(t)\, \eta(\omega);$$

for fixed $t \in J$ it is a linear transformation of random element η in $\mathbf{B}$ into random element ξ_t in **B**. Therefore $\{\xi_t, t \in J\}$ is a **B**-valued stochastic process such that almost every its realization is a solution of the deterministic problem (2).

As $\{\xi_t, t \in J\}$ is patht- differentiable for $t \in J$, it is a mild path solution of the problem (1).

If $[\tilde{\xi}_t, t \in J]$ is another solution of the problem (1) then the **B**-valued stochastic process $[\zeta_t, t \in J]$ such that

$$\zeta_t = \xi_t - \tilde{\xi}_t, \qquad t \in J$$

is a solution of the boundary-value problem

$$\begin{cases} \dot{\xi}_t = A(t)\,\zeta_t, \\ B\zeta_0 + C\zeta_1 = 0. \end{cases}$$

By assumption (v) this problem has only trivial solution. Hence the processes $\{\xi_t, t \in J\}$ and $\{\tilde{\xi}_t, t \in J\}$ are equivalent. □

Corollary 1. *Under the assumptions of Theorem 1, if the random element η is Gaussian, then the problem* (1) *has unique Gaussian mild path solution.*

Proof. It is a consequence of the fact that ξ_t is a linear transformation of η.

Remark 1. Obviously the assumptions (i)–(iv) may be replaced by any other sufficient conditions on the existence of the evolution operator.

Remark 2. Let the operator $A(t) \equiv A \in [\mathbf{B}]$. If there exists a constant μ such that

$$\| I - \mu(B + C\,e^{A}) \| < 1, \tag{8}$$

then (see [4])

$$\Psi(t) = \mu\, e^{At} \sum_{k=0}^{\infty} D^k, \qquad t \in J, \tag{9}$$

where

$$D = I - \mu(B + C\,e^{A}). \tag{10}$$

Example. Let $A(t) \equiv A \in [\mathbf{B}]$ and let $B = \mu^{-1} I$, $C = I$, $\mu > 0$. If $\| e^A \| < \mu^{-1}$ (or $\| e^A \| < \| B \|$) then the operator $\Psi(t)$ exists and has the form

$$\begin{aligned} \Psi(t) = e^{At} \sum_{k=0}^{\infty} \mu^{k+1} e^{kA} = \mu\, e^{At} \exp(\mu\, e^{A}) = \\ = \mu \exp[At + \mu\, e^{A}], \qquad t \in J. \end{aligned}$$

If $\| e^A \| < \mu^{-1}$ and $\lambda \in \sigma(A)$ (spectrum of the operator A) the (see [2]) $\lambda < -\ln \mu$. On the other hand if A is closed and $\operatorname{Re} \lambda \neq -\ln \mu$, then the operator

$\Psi(t)$ exists and has the form

$$\Psi(t) = \mu\, e^{At}[I + \mu\, e^{A}]^{-1}, \qquad t \in J.$$

Remark 3. Let $\dim \mathbf{B} < \infty$, i.e. $\mathbf{B} = \mathbf{R}^n$, then the representation of the operator $\Psi(t)$ is the matrix

$$Q(t)\,[B + CQ(1)]^{-1}, \qquad t \in J,$$

which exists if and only if the matrix $B + CQ(1)$ is nonsingular. It means there are no nontrivial solution of the deterministic boundary value problems under consideration. Here $Q(t)$ denotes the fundamental matrix conected with the matrix $A(t)$.

It is important for random differential equations to known not only that there exists a solution in any given sense, but also to known its probabilistic characterization, in particulary its finite dimensional distributions and moments.

Theorem 2. *Under the assumptions of Theorem 1, the distribution of the solution* $\{\xi_t,\ t \in J\}$ *of the random boundary value problem* (1) *has the form*

$$P_{\xi_t}(\Sigma) = P_\eta(\Sigma'), \qquad \Sigma \in \mathscr{B}, \tag{11}$$

where P_η *is the distribution of the random element* η *and* Σ' *is the range of the operator* $[B + C\Phi(1, 0)]^{-1}\, \Phi^{-1}(t, 0)$ *restricted to the domine* Σ.

Proof. For arbitrary $\Sigma \in \mathscr{B}$ we have

$$P_{\xi_t}(\Sigma) = P(\xi_t \in \Sigma) = P(\Phi(t, 0)\,[B + C\Phi(1, 0)]^{-1}\, \eta \in \Sigma) = P_\eta(\Sigma').$$

Remark 4. If $\mathbf{B}$ is a reflexive Banach space then the assumption (vi) may be replaced (see [1]) by the following (vi') η is a weak first order element.

Remark 5. The same as in Remark 4 is true if η is weak first order and separable valued random element in $\mathbf{B}$ (i.e. $\eta(\Omega)$ is separable set in $\mathbf{B}$).

Remark 6. In the finite dimensional case ($\mathbf{B} = \mathbf{R}^n$) from Theorem 2 we obtain result which is given in [7] i.e. the distribution of solution is given by

$$\begin{aligned} P_{\xi_t}(B_1, \ldots, B_n) = P(h_1(B_1, \ldots, B_n), \ldots, h_n(B_1, \ldots, B_n)) \times \\ \times |\det[\text{matrix } \Psi(t)]^{-1}|, \end{aligned} \tag{12}$$

where $B_1, \ldots, B_n$ are Borel sets in $\mathbf{R}^n$.

Now let us remind that the expected value $E\xi$ of a random element $\xi\colon \Omega \to \mathbf{B}$ is defined as a Pettis integral

$$E\xi = \int_\Omega \xi(\omega)\, dP(\omega), \tag{13}$$

i.e. the expected value, if exists, then it is an element of **B** such that the Lebesgue integral

$$\int_{\Omega} \langle \xi(\omega), x^* \rangle \, dP(\omega) = \langle E\xi, x^* \rangle \tag{14}$$

for every $x^* \in \mathbf{B}^*$.

However the cross-corelation operator

$$K_{\xi\eta} : \mathbf{B}^* \to \mathbf{B}^{**}$$

of random elements ξ and η with weak second order in **B** is defined by the equality

$$k_{\xi,\eta}(x^*, y^*) = \langle x^*, K_{\xi,\eta}, y^* \rangle, \tag{15}$$

where

$$k_{\xi,\eta}(x^*, y^*) = E\langle \xi, x^* \rangle \langle \eta, y^* \rangle \tag{16}$$

is a bilinear form.

If the random elements ξ and η are separably-valued then

$$K_{\xi,\eta}(y^*) = E\langle \eta, y^* \rangle \, \xi, \tag{17}$$

(see [1]).

Theorem 3. *Under the assumptions of Theorem 1, if*

(vii) *there exists the expected value $E\eta$ of the random element η, then there exists the expected value of the solution of the problem* (1) *and has the form*

$$m(t) = \Phi(0, t) [B + C\Phi(1, 0)]^{-1} E\eta, \qquad t \in J \tag{18}$$

and

$$| m(t) | \leqslant \| \Phi(0, t) [B + C\Phi(1, 0)]^{-1} \| E | \eta |, \qquad t \in J. \tag{19}$$

Proof. First of all let us remark that by assumption (v) the operator $B + C\Phi(1, 0)$ has a continuous inverse defined on its range. Moreover $[B + C\Phi(1, 0)]^{-1} \in [\mathbf{B}]$. Hence $\Phi(0, t) [B + C\Phi(1, 0)]^{-1} \in \mathbf{B}]$ for every $t \in J$. Therefore if $E\eta$ is expected value of the random element η then the random element $\Phi(0, t) [B + C\Phi(1, 0)]^{-1} \eta$ is Pettis integrable and

$$J \in t \mapsto m(t) = E\Phi(0, t) [B + C\Phi(1, 0)] \eta.$$

Hence (18) and (19) hold.

Theorem 4. *Under the assumptions of Theorem 3, if*

(vii) ***η is a weak second order random element in* B *and***

(ix) *the operator* $\Psi(t)$ *has its adjoint operator* $\Psi^*(t)$ *and its second adjoint operator* $\Psi^{**}(t)$ *for every* $t \in J$, *then the cross-corelation operator of the solution of the problem* (1) *has the form*

$$K_{\xi_s \xi_t} = \Psi^{**}(s)\, k_\eta \Psi^*(t), \qquad s, t \in J, \tag{20}$$

where K_η *is the corelation operator of the random element* η.

Proof. By assumptions (vii)–(ix) the bilinear form

$$\begin{aligned} k_{\xi_s \xi_t}(x^*, y^*) &= E\langle \Psi(s)\,\eta, x^*\rangle \langle \Psi(t)\,\eta, \quad y^*\rangle = \\ &= E\langle \eta, \Psi^*(s)\, x^*\rangle \langle \eta, \Psi^*(t)\, y^*\rangle = \\ &= k_{\eta\eta}(\Psi^*(s)\, x^*, \quad \Psi^*(t)\, y^*) = \langle \Psi^*(s)\, x^*, K_\eta \Psi^*(t)\, y^*\rangle = \\ &= \langle x^*, \Psi^{**}(s)\, K_\eta \Psi^*(t)\, y^*\rangle \quad \text{for every} \quad x^*, y^* \in \mathbf{B}^*. \end{aligned}$$

Hence

$$\langle x^*, K_{\xi_s \xi_t} y^*\rangle = \langle x^*, \Psi^{**}(s)\, K_\eta \Psi^*(t)\, y^*\rangle$$

and (20) holds.

REFERENCES

[1] Н. Н. Вахания, В. И. Тариеладзе, С. А. Чобаян, *Вероятностные распределения в Банаховых пространствах*, Наука, Москва, 1985.

[2] Ю. Л. Далецкий, М. Г. Креин, *Устойчивость решений уравнений в Банаховом пространстве*, Наука, Москва, 1970

[3] А. А. Дезин, *Общие вопросы теории граничных задач*, Москва, 1980

[4] С. Г. Крейн, *Линейные дифференциальные уравнения в Банаховом пространстве*, Изд. Наука, Москва 1967

[5] G. E. Ladas and V. Lakshmikantham, *Differential Equations in Abstract Spaces*, Academic Press, N. York 1972.

[6] И. В. Мельникова, А. Г. Кудрявцев, *Краевая задача для уравнения первого порядка в Банаховом пространстве*, Изв. Выш. уч. Зав. 3 (274) (1985), 52—56

[7] Ningg-Mao Xia, W. E. Boyce, M. R. Barry, *Two point Boundary Value Problems Containing a Finite Number of Random Variables*, Stoch.. Analysis and Appl. 1 1 1983, 117–137.

[8] A. For W. Walter, *Existence Theorems for a Two-Point Boundary Value Problem in Banach Space*, Math. Ann. 244 §979, 55–64.

[9] З. А. Штейнвиль, *О корректности обтих краевых задач для параболических уравнений*, Винити, № 1417—1418, 1978

Dobieslaw Bobrowski
Mathematical Institute
Technical University
Poznań
Poland