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ON EFFECTIVE CRITERIA OF SOLVABILITY OF THE BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS OF THE *n*-th ORDER

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Abstract. By the method of I. T. Kiguradze effective criteria of the existence and uniqueness of solution is obtained of certain boundary value problem for ordinary differential equation of the n-th order.

Key words. Boundary value problems with functional condition, ordinary differential equations of the n-th order, existence and uniqueness, effective criteria.

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In the paper the boundary value problem is investigated

$$u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t)), \tag{f}$$

$$u^{(i-1)}(t_i) = \varphi_i(u, u', \dots, u^{(n-1)}) \qquad (i = 1, \dots, n), \tag{φ}$$

where the function $f\colon \langle a, b\rangle \times R^n \to R$ satisfies the local Carathéodory conditions, $t_i \in \langle a, b\rangle$ and $\varphi_i : C^{n-1}\langle a, b\rangle \to R$ $(i = 1, \dots, n)$ – are continuous functionals. Here $\langle a, b\rangle$ – is a segment, R^n is n-dimensional vector real space with points $x = (x_i)_{i=1}^n$ normed by $\| x \| = \sum_{j=1}^{n} | x_j |$, $R^n_+ = \{x \in R^n : x \geqq 0\}$, $C^{n-1}\langle a, b\rangle$ – the space of functions continuous together with their derivatives up to the order $n - 1$ with the norm

$$\| u \|_{C^{n-1}} = \max \{ \sum_{i=1}^{n} | u^{(i-1)}(t) | ; a \leqq t \leqq b\},$$

$L^p\langle a, b\rangle$ – space of function integrable on $\langle a, b\rangle$ in p-th power with a norm

$$\| u \|_{L^p} = \begin{cases} [\int_a^b | u(t) |^p \, dt]^{1/p} & \text{for } 1 \leqq p < +\infty, \\ \text{vrai max} \{| x(t) |; a \leqq t \leqq b\} & \text{for } p = +\infty, \end{cases}$$

$$l(q, q_0) = \begin{cases} \left(\frac{q_0}{q} - 1\right)^{-1/q_0} \left[\frac{q_0}{q\pi} \sin \frac{q\pi}{q_0}\right] & \text{when } 1 \leqq q < q_0 < +\infty, \\ 1 \quad \text{when } 1 \leqq q \text{ and } q = q_0 \text{ or } q_0 = +\infty. \end{cases}$$

Under the solution (f, φ) we understand a function with absolutely continuous $n - 1$ derivative on $\langle a, b\rangle$ which satisfies the equation (f) for almost all $t \in \langle a, b\rangle$ and satisfies the boundary condition (φ). The method applied is analogous to that of I. T. Kiguradze in the paper [1] dedicated to the Cauchy–Nicoletti problem. The problem (f, φ) is for the case $n = 2$ investigated in [2].

Theorem 1. *Let on $\langle a, b\rangle \times R^n$ satisfy the inequality*

$$(1) \quad f(t, x_1, \ldots, x_n) \operatorname{sign} [(t - t_n) x_n] \leqq h(t) | x_n | + \sum_{j=1}^{n} h_j(t) | x_j | + \omega(t, \sum_{j=1}^{n} | x_j |),$$

and in $C^{n-1}\langle a, b\rangle$ the inequalities

$$(2) \qquad | \varphi(u, u', \ldots, u^{(n-1)}) | \leqq \sum_{j=1}^{n} r_{ij} \| u^{(j-1)} \|_{L^{q_0}} + c_i \qquad (i = 1, \ldots, n),$$

where

a) $h_j \in L^{p_j}\langle a, b\rangle$ $(j = 1, \ldots, n)$ *are nonnegative functions,* $p_j \geqq 1$, $\frac{1}{p_j} + \frac{1}{q_j} = 1$, $q_j \leqq q_0$ $(j = 1, \ldots, n)$, $h \in L^p\langle a, b\rangle$, $p \geqq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $q \leqq q_0$;

b) $\omega : \langle a, b\rangle \times R_+ \to R_+$ *is not decreasing on the second argument,* $\omega(., \varrho) \in L\langle a, b\rangle$ *for all* $\varrho \in (0, +\infty)$ *and*

$$(3) \qquad \lim_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho) \, dt = 0;$$

c) $r_{ij}, c_i \in R_+$ $(i, j = 1, \ldots, n)$,

$$s_i = \sum_{k=1}^{n} \{\beta(b - a)^{1/q_0} \sum_{j=i}^{n} [(b - a) \, l(q, q_0)]^{j-i} r_{jk} + [(b - a) \, l(q, q_0)]^{n-i} d_k\} < 1$$

$$(i = 1, \ldots, n),$$

when

$$\beta = \max \{\exp \int_{t_n}^{t} h(s) \operatorname{sign} (s - t_n) \, ds, \quad a \leqq t \leqq b\}$$

and either

$$(4_1)\quad d_k=(b-a)^{1/q_k}\beta l(q_k,q_0)\,[\int_a^b h_k^{p_k}(t)\exp(-p_k\int_{t_n}^t h(s)\operatorname{sign}(s-t_n)\,ds)\,dt]^{1/p_k}$$

$$(k=1,\dots,n)$$

or

$$(4_2)\quad d_k=(b-a)^{\frac{1}{q_k}-\frac{1}{q_0}}\|\,|\int_{t_n}^t h_k^{p_k}(\tau)\exp(p_k\int_\tau^t h(s)\operatorname{sign}(s-\tau)\,ds)\,d\tau|^{1/p_k}\|_{L^{q_0}}.$$

$$(k=1,\dots,n).$$

Then the problem (f,φ) *has at least one solution.*
To prove theorem 1 the following lemma is suitable.

Lemma 1. *Let the conditions a)–c) of theorem 1 be satisfied. Then there exists a nonnegative constant* c_0 *such that*

$$(5)\qquad \|u\|_{C^{n-1}}\leqq c_0,$$

for any function $u\in\tilde C^{n-1}\langle a,b\rangle$, *satisfying the differential inequalities*

$$(6)\quad u^{(n)}(t)\operatorname{sign}[(t-t_n)u^{(n-1)}(t)]\leqq h(t)|u^{(n-1)}(t)|+\sum_{j=1}^n h_j(t)|u^{(j-1)}(t)|+$$
$$+\omega(t,\sum_{j=1}^n|u^{(j-1)}(t)|)\qquad\text{if } a\leqq t\leqq b$$

and the conditions

$$(7)\qquad |u^{(i-1)}(t_i)|\leqq\sum_{j=1}^n r_{ij}\|u^{(j-1)}\|_{L^{q_0}}+c_i\qquad(i=1,\dots,n).$$

Proof. Let $u\in\tilde C^{n-1}\langle a,b\rangle$ be the function satisfying the presumption of lemma. We shall prove the correctness of the estimate (5).

When integrating the inequalities (6) and putting $\|u\|_{C^{n-1}}=\varrho$, we shall get

$$(8)\quad |u^{(n-1)}(t)|\leqq\beta[|u^{(n-1)}(t_n)|+\|\omega(\cdot,\varrho)\|_{L^1}]+\sum_{j=1}^n|\int_{t_n}^t h_j(\tau)\times$$
$$\times\exp(-\int_{t_n}^\tau h(s)\operatorname{sign}(s-t_n)\,ds)|u^{(j-1)}(\tau)|\,d\tau|\exp(\int_{t_n}^t h(s)\operatorname{sign}(s-t_n)\,ds).$$

From here by means of the inequalities of Helder and Levin (see [1] lemma 4.7) we shall find

$$\|u^{(n-1)}\|_{L^{q_0}}\leqq\beta(b-a)^{1/q_0}[|u^{(n-1)}(t_n)|+\|\omega(\cdot,\varrho)\|_{L^1}]+\sum_{j=1}^n|\int_{t_n}^t h_j^{p_j}(\tau)\times$$
$$\times\exp[p_j(\int_{t_n}^t h(s)\operatorname{sign}(s-t_n)\,ds-\int_{t_n}^\tau h(s)\operatorname{sign}(s-t_n)\,ds)]\,d\tau|^{1/p_j}.|\int_{t_n}^t|u^{(j-1)}(\tau)|^{q_j}\,d\tau|^{1/q_j}$$

and consequently,

$$(9)\quad \| u^{(n-1)} \|_{L^{q_0}} \leqq \beta(b-a)^{1/q_0}[\,| u^{(n-1)}(t_n) | + \| \omega(.,\varrho) \|_{L^1}] + \sum_{j=1}^{n} d_j \| u^{(j-1)} \|_{L^{q_0}}.$$

Analogously, from the inequality

$$(10)\qquad | u^{(i-1)}(t) | \leqq | u^{(i-1)}(t_i) | + \left| \int_{t_i}^{t} | u^{(i)}(\tau) | \, d\tau \right| \qquad (i = 1, \ldots, n),$$

we have

$$(11)\qquad \| u^{(i-1)} \|_{L^{q_0}} \leqq (b-a)^{1/q_0} | u^{(i-1)}(t_i) | + \left\| \left| \int_{t_i}^{t} | u^{(i)}(\tau) \, d\tau \right| \right\|_{L^{q_0}} \leqq$$

$$\leqq (b-a)^{1/q_0} | u^{(i-1)}(t_i) | + (b-a)\, l(q, q_0) \| u^{(i)} \|_{L^{q_0}} \qquad (i = 1, \ldots, n-1).$$

Designating

$$\xi_i = \| u^{(i-1)} \|_{L^{q_0}}, \qquad \xi_{0i} = | u^{(i-1)}(t_i) | \qquad (i = 1, \ldots, n)$$

from the inequalities (10), (11) we shall get

$$(12)\qquad \xi_n \leqq \beta(b-a)^{1/q_0} \xi_{0n} + \sum_{j=1}^{n} d_j \xi_j + \beta(b-a)^{1/q_0} \| \omega(.,\varrho) \|_{L^1}$$

and

$$(13)\quad \xi_i \leqq \beta(b-a)^{1/q_0} \sum_{j=i}^{n-1} [(b-a)\, l(q, q_0)]^{j-i} \xi_{0j} + [(b-a)\, l(q, q_0)]^{n-i} \xi_n$$

$$(i = 1, \ldots, n-1).$$

Taking the conditions (7) into consideration we shall get

$$(14)\qquad \xi_n \leqq \sum_{j=1}^{n} [\beta(b-a)^{1/q_0} r_{nj} + d_j]\, \xi_j + \beta(b-a)^{1/q_0} [c_n + \| \omega(.,\varrho) \|_{L^1}]$$

and

$$(15)\qquad \xi_i \leqq \beta(b-a)^{1/q_0} \sum_{k=1}^{n} \sum_{j=i}^{n-1} r_{jk} [(b-a)\, l(q, q_0)]^{j-i} \xi_k +$$

$$+ \beta(b-a)^{1/q_0} \sum_{j=i}^{n-1} [(b-a)\, l(q, q_0)]^{j-i} c_j + [(b-a)\, l(q, q_0)]^{n-i} \xi_n$$

$$(i = 1, \ldots, n-1).$$

Due to (14) and (15)

$$(16)\ \xi_i \leqq \sum_{k=1}^{n} \{\beta(b-a)^{1/q_0} \sum_{j=i}^{n} [(b-a)\, l(q, q_0)]^{j-i} r_{jk} + [(b-a)\, l(q, q_0)]^{n-i} d_k\}\, \xi_k +$$

$$+ \beta(b-a)^{1/q_0} \{ \sum_{j=i}^{n} [(b-a)\, l(q, q_0)]^{j-i} c_j + \| \omega(.,\varrho) \|_{L^1}\} \qquad (i = 1, \ldots, n).$$

Putting $\xi_0 = \max\{\xi_k : k = 1, \ldots, n\}$, $s_0 = \max\{s_k : k = 1, \ldots, n\}$, from the inequality (16) we find

$$\xi_0 \leqq c[r + \| \omega(., \varrho) \|_{L^1}],$$

where

$$(17)\quad c = \beta(b-a)^{1/q_0}(1-s)^{-1}, \quad r = \max\{\sum_{j=i}^{n} [(b-a)\, l(q, q_0)]^{j-i} c_j, \, i = 1, \ldots, n\}.$$

Consequently

$$(18)\qquad \| u^{(i-1)} \|_{L^{q_0}} \leqq c \cdot [r + \| \omega(., \varrho) \|_{L^1}] \qquad (i = 1, \ldots, n).$$

From the other hand, according to (8), (10) and (18) we have

$$(19)\qquad \varrho \leqq c_1^* + c_2^* \| \omega(., \varrho) \|_{L^1},$$

where $c_{1,2}^*$ are sufficiently large nonnegative constants, not depending on u.

Let $K = c_1^* + c_2^*$. Then from (3) it follows the existence of the constant $\eta_0 > K$ such, that for all $\eta \geqq \eta_0$

$$(20)\qquad \| \omega(., \eta) \|_{L^1} < \frac{\eta}{K}.$$

If $\| u \|_{C^{n-1}} > \eta_0$, then from (20) it follows, that

$$\varrho < c_1^* + c_2^* \frac{\varrho}{K} = \frac{c_1^* K + c_2^* \varrho}{K} < \frac{c_1^* + c_2^*}{K} \varrho = \varrho.$$

The obtained contradiction proves the correctness of the estimate (5).

Proof of theorem 1: Let c_0 be the constant, chosen according to lemma. We put

$$\chi(s) = \begin{cases} 1 & \text{for } s \leqq c_0, \\ 2 - s/c_0 & \text{for } c_0 \leqq s \leqq 2c_0, \\ 0 & \text{for } s \geqq 2c_0, \end{cases}$$

$\tilde{f}(t, x_1, \ldots, x_n) = \chi(\| x \|)\, f(t, x_1, \ldots, x_n)$,
$\tilde{\varphi}_i(u, u', \ldots, u^{(n-1)}) = \chi(\| u \|_{C^{n-1}})\, \varphi_i(u, u', \ldots, u^{(n-1)}) \quad (i = 1, \ldots, n)$.
and define the operator $A = A_1 : C^{n-1}\langle a, b\rangle \to C^{n-1}\langle a, b\rangle$,

$$(A_n u)(t) = \tilde{\varphi}_n(u, u', \ldots, u^{(n-1)}) + \int_{t_n}^{t} \tilde{f}(\tau, u(\tau), \ldots, u^{(n-1)}(\tau))\, d\tau,$$

$$(A_{n-i} u)(t) = \tilde{\varphi}_{n-i}(u, u', \ldots, u^{(n-1)}) + \int_{t_{n-i}}^{t} (A_{n-i+1} u)(\tau)\, d\tau \qquad (i = 1, \ldots, n-1).$$

On the basis of the properties of the function f of the functionals φ_i and due to functions $\tilde{f}$ and functionals $\tilde{\varphi}_i$ the existence of the function f_0 and the constant r_0 is evident, such that

$$|\tilde{f}(t, x_1, \ldots, x_n)| \leqq f_0(t) \in L\langle a, b\rangle,$$
$$|\tilde{\varphi}_i(u, u', \ldots, u^{(n-1)})| \leqq r_0 \in R_+ \qquad (i = 1, \ldots, n).$$

Therefore, due to theorem of Schauder there exists a fixed point of the operator A, i.e. there exists a solution of the problem $(\tilde{f}, \tilde{\varphi})$. The solution of the problem satisfies the inequalities

$$u^{(n)} \operatorname{sign}\left[(t - t_n)\, u^{(n-1)}(t)\right] = \tilde{f}(t, u, \ldots, u^{(n-1)}) \operatorname{sign}\left[(t - t_n)\, u^{(n-1)}(t)\right] \leqq$$
$$\leqq f(t, u, \ldots, u^{(n-1)}) \operatorname{sign}\left[(t - t_n)\, u^{(n-1)}(t)\right] \leqq h(t)\,|u^{(n-1)}(t)| +$$
$$+ \sum_{j=1}^{n} h_j(t)\,|u^{(j-1)}(t)| + \omega(t, \sum_{j=1}^{n} |u^{(j-1)}(t)|)$$

and

$$|u^{(j-1)}(t_i)| = |\varphi_i(u, u', \ldots, u^{(n-1)})| = \chi(\|u\|_{C^{n-1}})\,|\varphi_i(u, u', \ldots, u^{(n-1)})| \leqq$$
$$\leqq |\varphi_i(u, u', \ldots, u^{(n-1)})| \leqq \sum_{j=1}^{n} r_{ij}\,\|u^{(j-1)}\|_{L^{q_0}} + c_i \qquad (i = 1, \ldots, n),$$

i.e. $u(t)$ satisfies at the same time the presumptions of lemma. So, then the estimate (5) holds and

$$\chi(\|u(t)\|) = 1, \qquad \chi(\|u\|_{C^{n-1}}) = 1,$$

i.e. u is a solution of the problem (f, φ).

Theorem 2. *Let for almost all $t \in \langle a, b\rangle$ and $x_l \in R^n$ $(l = 1, 2)$*

$$\left[f(t, x_{11}, \ldots, x_{1n}) - f(t, x_{21}, \ldots, x_{2n})\right] \operatorname{sign}\left[(t - t_n)(x_{1n} - x_{2n})\right] \leqq$$
$$\leqq -h(t)\,|x_{1n} - x_{2n}| + \sum_{j=1}^{n} h_j(t)\,|x_{1j} - x_{2j}| \tag{21}$$

and in $C^{n-1}\langle a, b\rangle$

$$|\varphi_i(u, u', \ldots, u^{(n-1)}) - \varphi_i(v, v', \ldots, v^{(n-1)})| \leqq \sum_{j=1}^{n} r_{ij}\,\|u^{(j-1)} - v^{(j-1)}\|_{L^{q_0}} \quad (i = 1, \ldots, n), \tag{22}$$

where h, h_i, r_{ij} $(i, j = 1, \ldots, n)$ satisfy the presumptions of theorem 1. Then the problem (f, φ) has not more than one solution.

Proof. If the existence of two solutions u, v of the problem (f, φ) is admitted and if designated $z = u - v$, then from the inequalities (21), (22) we obtain (5), (6), where $\omega = 0$, $c_j = 0$ $(j = 1, \ldots, n)$. Then from the inequalities (8)–(11) it follows that $c_0 = 0$, i.e. $z(t) \equiv 0$.

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