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MONO-UNARY ALGEBRAS IN THE WORK OF CZECHOSLOVAK MATHEMATICIANS

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To Otakar Borůvka for his 90th birthday

Abstract. In 1950, O. Borůvka formulated a problem that led to study homomorphisms of mono-unary algebras. His problem stimulated the investigation of these algebras in Czechoslovakia. This article includes a brief survey of typical results obtained by Czechoslovak mathematicians in this area.

Key words. Mono-unary algebra, component, connected mono-unary algebra, cycle, grade, partial mono-unary algebra, machine, machine homomorphism, simulation of a machine, Pawlak machine.

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1. BORŮVKA'S PROBLÉM

After the World War II, **O. Borůvka** held lectures on Algebra at the Faculty of Sciences of the Masaryk University in Brno, and, consequently, met some methodological problems. One of them concerned commuting matrices, particularly, how to find all matrices commuting with a given matrix, i.e., how to find all linear mappings of an n-dimensional vector space V into V that commute with a given linear mapping of V into V. Borůvka was persuaded that the problem may be methodically simplified if neglecting the linear structure of the vector space and of all linear mappings. Thus, in 1950, he formulated a problem that may be simply presented in terms of mono-unary algebras.

A part of Borůvka's lectures appeared later in the form of a book (cf. [2]).

2. MONO-UNARY ALGEBRAS AND THEIR HOMOMORPHISMS

Let A be a set and f a mapping of A into A. Then the ordered pair (A, f) is said to be a *mono-unary algebra*. In what follows, we define iterations of f: $f^0 = \mathrm{id}_A$, $f^{n+1} = f \circ f^n$ for any $n \in \mathrm{N}$ where N is the set of all non-negative integers.

Let (A, f), (B, g) be mono-unary algebras and h be a mapping of A into B. Then h is said to be a *homomorphism* of (A, f) into (B, g) if $h \circ f = g \circ h$. Now, we obtain

Problem. *For any mono-unary algebras (A, f), (B, g) find all homomorphisms of (A, f) into (B, g).*

For solving this problem some definitions are needed.

Let (A, f) be a mono-unary algebra.

We set $e = \{(x, y) \in A \times A;$ there exist $m \in \mathbf{N}$, $n \in \mathbf{N}$ such that $f^m(x) = f^n(y)\}$. Then e is an equivalence on A. Any block of e defines a subalgebra of (A, f) that is called a *component* of (A, f). A mono-unary algebra with exactly one component is said to be *connected*.

An element $x \in A$ is said to have property p if there exists a sequence $(x_i)_{i \in \mathbf{N}}$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for any $i \in \mathbf{N}$. We denote by B_∞ the set of all elements in A that have property p. Furthermore, we put $B_0 = \{x \in A; f^{-1}(x) = \emptyset\}$; let $\alpha > 0$ be an ordinal and suppose that B_λ has been defined for any $\lambda < \alpha$. Then we put $B_\alpha = \{x \in (A - B_\infty) - \bigcup_{\lambda < \alpha} B_\lambda; f^{-1}(x) \subseteq \bigcup_{\lambda < \alpha} B_\lambda\}$. Then there exists a least ϑ such that $B_\vartheta = \emptyset$. Clearly, the family of all B_α is disjoint and forms a decomposition of A. We set $S(x) = \alpha$ if $x \in B_\alpha$. Then $S(x)$ is said to be the *grade* of x; clearly, $S(x)$ is an ordinal or ∞. We suppose that $\alpha < \infty$ for any ordinal α and that the class Ord of ordinals is ordered in the usual way.

Let (A, f) be a connected mono-unary algebra. An element $x \in A$ is said to be *cyclic* if there exists $n \in \mathbf{N}$, $n > 0$, such that $f^n(x) = x$. Let Z be the set of all cyclic elements in A; then Z is called the *cycle* of (A, f). We denote by R the cardinality of Z.

If we need to distinguish between symbols defined for different algebras, we add symbols of algebras as subscripts. Hence, we write $S_{(A,f)}$, $Z_{(A,f)}$, $R_{(A,f)}$ for S, Z, R defined above.

Let (A, f), (B, g) be connected mono-unary algebras. The algebra (B, g) is said to be *admissible* to (A, f) if one of the following conditions is satisfied.

(i) $R_{(B,g)} \neq 0$ and $R_{(B,g)}$ divides $R_{(A,f)}$.

(ii) $R_{(B,g)} = 0 = R_{(A,f)}$ and there exist $x_0 \in A$, $x_0' \in B$ such that $S_{(A,f)}(f^n(x_0)) \leqq S_{(B,g)}(g^n(x_0'))$ for any $n \in \mathbf{N}$.

If (i) holds, then we choose $x_0 \in A$ and $x_0' \in Z_{(B,g)}$ arbitrarily and $S_{(A,f)}(f^n(x_0)) \leqq \infty = S_{(B,g)}(g^n(x_0'))$ holds for any $n \in \mathbf{N}$.

Let (B, g) be admissible to (A, f). We choose x_0, x_0' as above and define $P_0 = \{f^n(x_0); n \in N\}$, $P_{i+1} = f^{-1}(P_i) - \bigcup_{k \leqq i} P_k$ for any $i \in \mathbf{N}$. Clearly, $A = \bigcup_{i \in \mathbf{N}} P_i$ with disjoint summands.

If $x \in P_{i+1}$, then $f(x) \in P_i$ holds. Furthermore, if $x \in A$, $x' \in B$ and $S_{(A,f)}(x) \leqq S_{(B,g)}(x')$, then for any $t \in f^{-1}(x)$ there exists $t' \in g^{-1}(x')$ such that $S_{(A,f)}(t) \leqq S_{(B,g)}(t')$. Using these facts, we describe

Construction C. Let (A, f), (B, g) be connected mono-unary algebras where (B, g) is admissible to (A, f). Let $x_0 \in A$, $x'_0 \in B$ be such that $S_{(A,f)}(f^n(x_0)) \leqq \leqq S_{(B,g)}(g^n(x'_0))$ for any $n \in \mathrm{N}$ where $x'_0 \in Z_{(B,g)}$ if $R_{(B,g)} \neq 0$.

We put $h(f^n(x_0)) = g^n(x'_0)$ for any $n \in \mathrm{N}$ and, therefore, $h(x)$ is defined for any $x \in P_0$ in such a way that $S_{(A,f)}(x) \leqq S_{(B,g)}(h(x))$. Suppose that $h(t)$ has been defined for any $t \in \bigcup_{0 \leqq j \leqq i} P_j$ in such a way that $S_{(A,f)}(t) \leqq S_{(B,g)}(h(t))$. If $x \in P_{i+1}$, ther $t = f(x) \in P_i$ and $t' = h(t)$ has been defined. We take an element $x' \in g^{-1}(t')$ such that $S_{(A,f)}(x) \leqq S_{(B,g)}(x')$ and put $h(x) = x'$. In this way, h may be defined on the set P_{i+1}.

By induction, a mapping h of A into B is defined. □

The condition "$x'_0 \in Z_{(B,g)}$ if $R_{(B,g)} \neq 0$" was ommitted, by mistake, in [45], [46]. This mistake was corrected in [38]. Using Construction C, we describe

Construction K. Let (A, f), (B, g) be mono-unary algebras. Let F be a mapping assigning to any component (A_i, f_i) of (A, f) an admissible component $F(A_i, f_i) = = (B_j, g_j)$ of (B, g). Let h_i be a mapping of A_i into B_j obtained by Construction C. Define h to be the union of all mappings h_i. □

The main result is as follows.

Theorem (cf. [45], [46]). *Let (A, f), (B, g) be mono-unary algebras, h a mapping of A into B. Then the following assertions are equivalent.*

(i) *h is a homomorphism of (A, f) into (B, g).*

(ii) *h is obtained by Construction K.* □

Hence, Construction K provides exactly all homomorphisms of one mono-unary algebra into another one.

This Theorem solves, particularly, the problem of finding all mappings commuting with a permutation (cf. [55], [46]). Similarly, it may be considered to provide all solutions of the functional equation $h(f(x)) = g(h(x))$ where f, g are given functions (see [53], [46]).

3. APPLICATIONS

Our Theorem provides solutions of further problems (cf. [45]).

Let A be a set, m a mapping of A into A. Put $M(a, b) = (m(a), m(b))$ for any $(a, b) \in A \times A$. Then the set of all binary operations H on A such that m is an endomorphism of (A, H) coincides with the set of all homomorphisms of $(A \times A, M)$ into (A, m) that may be constructed on the basis of Theorem. For a generalization see [26].

Interest in mono-unary algebras revived simultaneously with the study of Pawlak machines (cf. [51], [1], [47], [48]). A *machine* is an ordered triple (A, f, r) where A is a set, f a partial unary operation on A, and r a unary relation on A such that $r \subseteq A - \operatorname{dom} f$. A homomorphism of a machine (A, f, r) into a machine (B, g, s) is a mapping h of A into B such that

(i) $x \in \operatorname{dom} f$ implies $h(x) \in \operatorname{dom} g$ and $h(f(x)) = g(h(x))$ for any $x \in A$;

(ii) $x \in r$ implies $h(x) \in s$ for any $x \in A$.

Consider also the following condition.

(i') for any $x \in \operatorname{dom} f$ there exists $n(x) \geqq 1$ such that $g^{n(x)}(h(x))$ exists and $h(f(x)) = g^{n(x)}(h(x))$ holds.

If a mapping h of A into B satisfies (i'), (ii), then it is said to be a *simulation* of the machine (A, f, r) in (B, g, s).

Another generalization of homomorphism is studied in [37].

A machine (A, f, r) with $r = A - \operatorname{dom} f$ is said to be a *Pawlak machine*; the machine $(A, f, \emptyset)$ coincides with the partial mono-unary algebra (A, f).

A Pawlak machine (A, f, r) imitates the activity of a programmed computer. If a state $x \in A$ is given, then the machine goes to the state $f(x)$, then to $f^2(x), \ldots;$ either this sequence – called computation – is infinite or it is finite because $f^n(x) \notin \operatorname{dom} f$ for some $n \geqq 0$.

The methods of [45], [46] could be transferred to machines. Successively, all homomorphisms of partial mono-unary algebras (see [26]) and of Pawlak machines (cf. [27], [30]) and all homomorphisms and simulations of machines (see [48]) were constructed. An algorithm for finding all homomorphisms of a finite partial mono-unary algebra into another one was described in [13], an application of this algorithm to Mealy automata was presented in [15] and [16].

A problem that may be solved on the basis of Theorem is the following. For arbitrary mono-unary algebras (A, f), (B, g) decide whether at least one homomorphism of (A, f) into (B, g) exists. This inspires the study of the category of mono-unary algebras. Particularly, to any algebra (A, f) an element $\psi(A, f)$ of a preordered class is assigned in such a way that a homomorphism of (A, f) into (B, g) exists if and only if $\psi(A, f) \leqq \psi(B, g)$; the mapping ψ is then said to be a *characterization mapping* of the category. Such a characterization mapping was constructed for connected partial mono-unary algebras in [28] and [32], for partial mono-unary algebras in [41], for connected machines in [50]. Necessary and sufficient conditions of a similar type for the existence of a simulation of a Pawlak machine in another one are formulated in [39]. Some sufficient conditions for the existence of an injective homomorphism of a partial mono-unary algebra into another one are presented in [29].

Clearly, the identity on the class of all mono-unary algebras may be considered to be a characterization mapping if we put $(A, f) \leqq (B, g)$ whenever there exists at least one homomorphism of (A, f) into (B, g). In [31] all sets of connected mono-unary algebras are found that are ordered with respect to the above introduced preordering $\leqq$.

There exists a set with the cardinality of continuum consisting of connected mono-unary algebras such that it is an antichain with respect to the relation $\leqq$ (see [14]); a similar result holds for the so called weakly rigid infinitely countable connected mono-unary algebras if $(A, f) \leqq (B, g)$ means the existence of an isomorphism of (A, f) into (B, g) (cf. [19]). If considering Pawlak machines and their simulations in a similar way, an antichain of machines whose cardinality exceeds n can be found for any $n \in \mathrm{N}$ (see [39]).

4. ALGEBRAIC PROBLEMS CONCERNING MONO-UNARY ALGEBRAS

The above mentioned results provoked further problems.

In [35], necessary and sufficient conditions for a number $R \in \mathrm{N}$ and a function S of the set A into the class $\mathrm{Ord} \cup \{\infty\}$ are formulated for the existence of a mapping f of A into A such that $R_{(A,f)} = R$, $S_{(A,f)} = S$.

If an algebra (A, f) is given, how many operations g on A exist such that the algebras (A, f), (A, g) have the same endomorphisms, congruences, components, convex subsets? If (A, f) is complete and connected, then there is only a finite number of complete operations g such that (A, f), (A, g) have the same endomorphisms. If (A, f) is only complete, then the number of such complete operations g is less or equal to c where c is the cardinality of the continuum (see [18]). The last result holds also for partial mono-unary algebras (cf. [23]). Two complete mono-unary algebras (A, f), (A, g) with the same endomorphisms have the same components (see [18]); the same holds for partial mono-unary algebras if two exceptional types of algebras are excluded (cf. [22]). If we exclude complete mono-unary algebras where any equivalence is a congruence, then the number of operations g on A such that (A, f), (A, g) have the same congruences is less or equal to c; a similar result holds for partial mono-unary algebras (see [21]). The cardinal c may be replaced by 4 under a further condition, namely $f^{-1}(A - \mathrm{dom} f) \neq \emptyset$ (cf. [20]). Pairs of partial mono-unary algebras (A, f), (A, g) having the same convex subsets are investigated in [25].

If A is an uncountable set and f is an arbitrary unary operation on A, then the cardinality of the system of endomorphisms, of the system of congruences, and of the system of subalgebras of the algebra (A, f) equals $2^{\mathrm{card}\, A}$ (see [33], [34]).

Particular properties of the endomorphism monoid of a mono-unary algebra (A, f) influence the operation f; e.g., there exists only a finite number of mono-unary algebras with corregular endomorphism monoids (cf. [11]). Mono-unary algebras with commutative endomorphism monoids are characterized in [54].

A monoid T of transformations of a set A is said to be realizable if there is a universal algebra (A, F) such that its endomorphism monoid coincides with T. For a mono-unary algebra (A, f) the realizability of the monoid $(\{f^n;\ n \in \mathrm{N}\}, \circ)$ is studied in [17]. For any mono-unary algebra (A, f) there exists a mono-unary algebra (B, g) such that the group of its weak automorphisms is isomorphic to the automorphism group of (A, f) (see [52]).

Mono-unary algebras permit a cardinal arithmetic similarly as ordered sets: the sum of two disjoint algebras is obtained by forming the union of their carriers and the union of their operations, the product is the usual direct product, and the power is the set of all homomorphisms of the exponent into the basis provided with a naturally defined operation. In [40], [42], and [43] the arithmetic of the class of mono-unary algebras is investigated where any algebra is a union of a finite number of cycles. Basic rules of this arithmetic are deduced in [40]; particularly, canonical expressions for types of these algebras are given and rules for obtaining products and powers are presented. In [42] explicit formulas for algebras of automorphisms and of injective homomorphisms are given. Canonical forms of automorphisms are studied in [43] and canonical forms of inverse automorphisms and of the composition of two automorphisms are deduced. Sets of these types with addition and multiplication were completely characterized in [44].

Clearly, any partial mono-unary algebra can be embedded into a complete mono-unary algebra that is said to be its completion. To any class $\mathscr{A}$ of partial mono-unary algebras both the class $\mathscr{A}^*$ of their completions and the variety $\mathscr{V}(\mathscr{A})$ generated by $\mathscr{A}$ are assigned. In [24] the commutativity of the operators * and $\mathscr{V}$ is studied.

5. MONO-UNARY ALGEBRAS WITH A FURTHER STRUCTURE

Theorem of Section 2 provides all mappings of an n-dimensional vector space V into V that commute with a given linear transformation of V. The original Borůvka's question requested not all but only all linear mappings of V into V that commute with a given one. Such construction is presented in [49]. If f is a linear transformation of V into V, then any λ such that $(f - \lambda\, \mathrm{id}_V)\, x = o$ for some $x \neq o$ is said to be a root of f. To any root λ of f, there exists a subspace $V_\lambda = \{x \in V; (f - \lambda\, \mathrm{id}_V)^n\, x = o$ for some $n \in \mathrm{N}\}$. Then $a_\lambda = f - \lambda\, \mathrm{id}_A \restriction V_\lambda$ is a linear transformation of V_λ into V_λ, i.e., (V_λ, a_λ) is a mono-unary algebra. According to the ideas of [56], a basis B_λ of V_λ exists such that $(B_\lambda \cup \{o\}, a_\lambda \restriction B_\lambda \cup \{o\})$ is a mono-unary algebra. Then any homomorphism of $(B_\lambda \cup \{o\}, a_\lambda \restriction B_\lambda \cup \{o\})$ into (V_λ, a_λ) (constructed on the basis of Theorem) may be linearly extended to a linear transformation b_λ of V_λ into V_λ. Linear transformations b_λ constructed in this way to any root λ of f define a linear transformation b of V. Then b commutes with f and all linear transformations of V commuting with f can be constructed in this way. This construction yields all matrices commuting with a given one as it is described in [36].

Hence, we investigated a mono-unary algebra (A, f) where further operations on A were defined; we were interested in the homomorphisms of this algebra into another algebra of the same type, i.e., in all homomorphisms between the mono-unary algebras that preserve the additional operations.

Another problem of a similar type is the following. The set A is a carrier of a mono-unary algebra and of a topological space. Find conditions under which the system of all continuous closed transformations of the space into itself coincides with the system of all endomorphisms of the algebra. Such algebra is said to *correspond* to the topology.

This was solved in [3] where mono-unary algebras corresponding to quasi-discrete topologies were described. Similarly, mono-unary algebras corresponding to T_C-topologies are described in [4]. In [6] sufficient conditions are given for the endomorphism monoid of a connected mono-unary algebra to be regular; these conditions mean the existence of a certain topology such that the algebra is corresponding to it.

The papers [5] include several characterizations of mono-unary algebras whose subalgebras form a chain with respect to inclusion and several characterizations of the so called reduced algebras. Characterization of a mono-unary algebra whose components have either idempotent ($f^2 = f$) or involutory ($f^2 = \text{id}$) operations is expressed by means of certain binary relations in [7]. Mono-unary algebras with idempotent operations are characterized by means of the existence of exactly one T_0 quasi-discrete topology to which the algebra is corresponding in [8]. In [9] a mono-unary algebra is characterized such that the endomorphism monoid coincides with the system of all strongly isotone mappings of the carrier into itself where strong isotony is related to a suitable preordering of the carrier. If $\mathbf{R}$ is the set of all real numbers and $q(x) = x^2$, then for $(\mathbf{R}, q)$ there exists no quasipseudometric on $\mathbf{R}$ such that the algebra $(\mathbf{R}, q)$ is corresponding to the topology defined by means of quasipseudometric, but for $(\mathbf{R} - \{-1\}, q \restriction \mathbf{R} - \{-1\})$ such quasipseudometric always exists (cf. [10], [12]).

6. CONCLUSION

The above mentioned papers prove that there are non-trivial results concerning mono-unary algebras though these algebras are very transparent. Naturally, the solved problems are specific for mono-unary algebras and for other algebras either have no sense or are too difficult.

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