František Neuman

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Archivum Mathematicum, Vol. 26 (1990), No. 2-3, 147--154

Persistent URL: http://dml.cz/dmlcz/107382

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 26, No. 2–3 (1990), 147–154

ON HALPHEN AND LAGUERRE—FORSYTH CANONICAL FORMS OF LINEAR DIFFERENTIAL EQUATIONS

FRANTIŠEK NEUMAN

(Received December 21, 1988)

Dedicated to Academician Otakar Borůvka on his 90th birthday

Abstract: Two types of canonical forms of ordinary linear differential equations were known, the so-called Halphen and Laguerre–Forsyth forms. Already in 1910 G. D. Birkhoff showed that the Laguerre–Forsyth form is local for the third order equations. The same is true for an arbitrary order as well as for the Halphen canonical forms. Recently global canonical forms of linear differential equations were intruduced and a criterion of global equivalence of those equations was derived. On the basis of this criterion we establish in this paper sufficient and necessary conditions that guarantee that an ordinary linear differential equation of an arbitrary order can be globally transformed (i.e. on its whole interval of definition) into the Halphen or the Laguerre–Forsyth canonical form.

Key words: Ordinary linear differential equations, Halphen and Laguerre–Forsyth canonical forms, global transformations.

MS Classification: 34 A 30, 34 C 20.

I. INTRODUCTION

In the mathematical literature two special forms of ordinary homogeneous linear differential equations are known, the so-called Halphen and Laguerre–Forsyth canonical forms, see e.g. [14]. Already in 1910 G. D. Birkhoff [2] pointed out that the Laguerre–Forsyth canonical form is only local for the third order equations in the sense that not every third order linear differential equation can be transformed on its whole interval of definition into an equation of that form. In [10] it was shown that the Halphen canonical form is local as well. A global canonical form was introduced in 1972 [9], and another type of a global form was derived in 1983 [10], see also a survey paper [12]. From a local point of view canonical forms of ordinary linear differential equations have recently been studied also in [1].

Here we derive sufficient and necessary conditions that guarentee that an ordinary linear differential equation of an arbitrary order can be transformed on its whole interval of definition into the Halphen or the Laguerre–Forsyth canonical form. The proofs are based on a criterion of global equivalence of linear differential equations, derived foi the second order in 1967 by O. Borůvka [3], and established for an arbitrary order in 1984 in [11].

II. NOTATION AND DEFINITIONS

Consider a linear homogeneous differential equation of the n-th order, $n \geqq 2$

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_0(x)\, y = 0 \qquad \text{on an open interval } I \subset \mathbf{R}$$

with continuous coefficients $p_i \in C_0(I)$, $i = 0, \ldots, n-1$, that we shortly denote as $P_n(y, x; I)$; $P_n^0(y, x; I)$ means $P_n(y, x; I)$ with $p_{n-1}(x) = 0$ on I. We say (see also [9]) that the equation $P_n(y, x; I)$ is globally (pointwise) transformable into an equation $Q_n(z, t; J)$

$$z^{(n)} + q_{n-1}(t)\, z^{(n-1)} + \ldots + q_0(t)\, z = 0 \qquad \text{on open } J \subset \mathbf{R},$$

$$q_i \in C^0(J), \qquad i = 0, \ldots, n-1,$$

if there exist functions f and h,

$$f\colon J \to \mathbf{R}, \qquad f \in C^n(J), \qquad f(t) \neq 0 \text{ on } J,$$

h being a C^n-diffeomorphism of J onto I, such that

$$z(t) = f(t)\, y(h(t)), \qquad t \in J \tag{1}$$

is a solution of $Q_n(z, t; J)$ whenever y is a solution of $P_n(y, x; I)$.

Let us note that this form of the transformation is the most general form of pointwise transformations between equations of the mentioned form (see [13], [14] and [4]); the global character of the transformation is guaranteed by the requirement $h(J) = I$ that expresses that solutions are transformed on their whole intervals of definition.

Consider an equation of the second order $P_2^0(u, x; I)$

$$u'' + p_0(x)\, u = 0 \qquad \text{on } I.$$

In accordance with O. Borůvka [3], this equation is

either both-side oscillatory on I,

or one-side oscillatory on I,

or of type k, k being an integer denoting the maximal number of zeros of each nontrivial solution on I in the nonoscillatory case. Moreover, in this last case when the equation is of a type k, it is of general or special kind regarding to the existence or nonexistence of two linearly independent solutions having $k - 1$ zeros on I.

Suppose that $p_0 \in C^{n-2}(I)$ in equation $P_2^0(u, x; I)$ for an integer $n > 2$. Let u_1 and u_2 be two linearly independent solutions of the equation and define

$$y_1 := u_1^{n-1},\ y_2 := u_1^{n-2}.\ u_2, \ldots, y_n := u_2^{n-1}.$$

These n functions y_i, $i = 1, \ldots, n$, form an n-tuple of linearly independent solutions of the n-th order linear differential equation

$$y^{(n)} + \binom{n+1}{3} p_0(x)\, y^{(n-2)} + 2\binom{n+1}{4} p_0'(x)\, y^{(n-3)} + \ldots = 0 \qquad \text{on } I,$$

called the iterative equation iterated from $P_2^0(u, x; I)$, see e.g. [7]. The operator on the left-hand side of the iterative equation will be denoted as $[p_0]_n\,(y, x; I)$.

The Laguerre–Forsyth canonical forms of equations of the n-th order are characterized by the vanishing of coefficients of the $(n - 1)$st and $(n - 2)$nd derivatives, i.e. equations of the form

$$y^{(n)} + p_{n-3}(x)\, y^{(n-3)} + \ldots + p_0(x)\, y = 0 \qquad \text{on } I,$$

see [8], [5] and [14].

G. H. Halphen in [6] studied the 3rd and 4th order equations that for general n can be written in the forms

$$[p_0]_n\,(y, x; I) + \varepsilon_3 y^{(n-3)} + r_{n-4}(x)\, y^{(n-4)} + r_{n-5}(x)\, y^{(n-5)} + \ldots + r_0(x)\, y = 0,$$

$$[p_0]_n\,(y, x; I) + \varepsilon_4 y^{(n-4)} + r_{n-5}(x)\, y^{(n-5)} + \ldots + r_0(x)\, y = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[p_0]_n\,(y, x; I] + \varepsilon_n y = 0,$$

$$[p_0]_n\,(y, x; I) = 0,$$

$r_i \in C^0(I)$ for $i = 0, 1, \ldots, n - 4$, ε_i being $+1$ for odd i and $+1$ or -1 for even i.

Let us note that each equation $P_n(y, x; I)$ with $p_{n-1} \in C^{n-1}(I)$ and $p_{n-2} \in C^{n-2}(I)$ can be globally transformed into an equation $Q_n(z, x; I)$ with $q_{n-1} = 0$ on I and $q_{n-2} \in C^{n-2}(I)$ by means of the transformation

$$z(x) = \exp\left\{\frac{1}{n} \int_{x_0}^{x} p_{n-1}(s)\, \mathrm{d}s\right\} y(x), \qquad x_0 \in I.$$

Hence denote

$$\mathbf{A}_n := \{P_n(y, x; I);\quad \text{open } I \subset \mathbf{R},\ p_{n-1} = 0 \text{ on } I \text{ and } p_{n-2} \in C^{n-2}(I)\}.$$

Evidently, each equation $P_n(y, x; I) \in \mathbf{A}_n$ can be written in the form

$$[p]_n (y, x; I) + r_{n-3}(x) y^{(n-3)} + \ldots + r_0(x) y = 0 \text{ on } I$$

with $r_i \in C^0(I)$, $i = 0, \ldots, n-3$, and

$$p(x) = p_{n-2}(x) \Big/ \binom{n+1}{3} \in C^{n-2}(I).$$

III. PRELIMINARY RESULTS

Borůvka's criterion of global equivalence of the second order equations [3]:
Equation $P_2^0(y, x; I)$ can be globally transformed into $Q_2^0(z, t; J)$ if and only if both equations are either both-side oscillatory, or one-side oscillatory, or of the same type k and, in this case, moreover, if they are of the same kind, i.e. both general or both special.

Criterion of global equivalence of equations for general $n \geqq 3$ [11]:
Equation $P_n(y, x; I) \in \mathbf{A}_n$:

$$[p]_n (y, x; I) + r_{n-3}(x) y^{(n-3)} + \ldots + r_0(x) y = 0 \qquad \text{on } I,$$

can be globally transformed into equation $Q_n(z, t; J) \in \mathbf{A}_n$:

$$[q]_n (z, t; J) + s_{n-3}(t) z^{(n-3)} + \ldots + s_0(t) z = 0 \qquad \text{on } J$$

by means of a global transformation (1) if and only if

(i) $h \in C^{n+1}(J)$, $\quad f = c \,.\, | h' |^{(1-n)/2}$ for a nonzero constant c,

(ii) $u'' + p(x) u = 0$ on I with $p = p_{n-2} \Big/ \binom{n+1}{3}$

is globally transformable into

$$v'' + q(t) v = 0 \quad \text{on } J \text{ w th } q = q_{n-2} \Big/ \binom{n+1}{3},$$

by means of a global transformation

$$v(t) = | h'(t) |^{-1/2} u(h(t)), \qquad \text{and}$$

(iii) the differential expression

$$r_{n-3}(x) y^{(n-3)} + \ldots + r_0(x) y$$

is obtained from

$$(s_{n-3}(t) z^{(n-3)} + \ldots + s_0(t) z) \,.\, | h'(t) |^{-(n+1)/2} (\operatorname{sign} h')^n$$

by means of the substitution $z(t) = | h'(t) |^{(1-n)/2} y(h(t))$, $h(t) = x$.

Corollary: Condition (iii) implies

$$\begin{aligned} &r_{n-3}(h(t)) \,.\, (h'(t))^3 = s_{n-3}(t) && \text{on } J, \\ &r_{n-4}(h(t)) \,.\, (h'(t))^4 = s_{n-4}(t) && \text{on } \{t \in J;\, s_{n-3}(t) = 0\}, \\ &r_{n-5}(h(t)) \,.\, (h'(t))^5 = s_{n-5}(t) && \text{on } \{t \in J;\, s_{n-3}(t) = s_{n-4}(t) = 0\}, \end{aligned}$$

etc.

IV. MAIN RESULTS

Theorem 1.

Equation $P_n(y, x; I) \in \mathbf{A}_n$, $n \geqq 2$, can be globally transformed into the Laguerre–Forsyth form if and only if the second order equation

$$u'' + p_{n-2}(x)\, u \Big/ \binom{n+1}{3} = 0 \qquad \text{on } I, \tag{2}$$

$p_{n-2} \in C^{n-2}(I)$ is of type 1, *or equivalently, if this second order equation is disconjugate on I.*

Moreover, the definition interval of the Laguerre–Forsyth form is $\mathbf{R}$ *if and only if equation* (2) *is of special kind.*

Remark. As Birkhoff [2] has shown in 1910, the Laguerre–Forsyth canonical form is not global for the third order equations. However we need not go to the third order. The Laguerre–Forsyth canonical form for the second order linear differential equations is

$$y'' = 0 \qquad \text{on } I \subset \mathbf{R}. \tag{3}$$

Since the equation $y'' + y = 0$ on any interval of the length larger than π is not of type 1, it cannot be globally transformed into the Laguerre–Forsyth form (3) even if we take the maximal interval $I = \mathbf{R}$ for which (3) is of type 1 and special. Hence also for the second order equations the Laguerre–Forsyth form is only local.

Theorem 2.

Equation $P_n(y, x; I) \in \mathbf{A}_n$, $n \geqq 3$,

$$[p]_n\,(y, x; I) + r_{n-3}(x)\, y^{(n-3)} + \ldots + r_0(x)\, y = 0 \qquad \text{on } I,$$

with $p = p_{n-2} \Big/ \binom{n+1}{3}$ *can be globally transformed into one of equations in the*

Halphen canonical form if and only if either $P_n(y, x; I)$ is an iterative equation, or the function r_{i*}, where $i*$ is the maximal of the indices $i \in \{0, \dots, n-3\}$ for which r_i is not identically zero on I, satisfies $r_{i*} \in C^n(I)$ and $r_{i*}(x) \neq 0$ for all $x \in I$.

V. PROOFS

Proof of Theorem 1.

If equation $P_n(y, x; I) \in \mathbf{A}_n$

$$y^{(n)} + p_{n-2}(x)\, y^{(n-2)} + p_{n-3}(x)\, y^{(n-3)} + \dots + p_0(x)\, y = 0$$

on I, $p_{n-2} \in C^{n-2}(I)$, $n \geqq 2$, can be globally transformed into an equation of the Laguerre–Forsyth form, i.e. into

$$z^{(n)} + q_{n-3}(t)\, z^{(n-3)} + \dots + q_0(t)\, z = 0 \qquad \text{on } J,$$

then condition (ii) of the criterion for $n \geqq 3$ implies that the second order equation (2) is globally transformable into the equation

$$v'' = 0 \qquad \text{on } J,$$

since $q_{n-2}(t) = 0$ on J. This implication is of course trivial for $n = 2$. Since $v'' = 0$ is always of type 1, and special if and only if $J = \mathbf{R}$, Borůvka's criterion asserts that equation (2) is also of type 1 and, moreover, it is special just when $J = \mathbf{R}$.

Conversely, let equation (2) with $p_{n-2} \in C^{n-2}(I)$ be of type 1. Due to Borůvka's criterion there exists a global transformation

$$v(t) = |\, h'(t) \,|^{-1/2}\, u(h(t)), \tag{4}$$

that converts equation (2) into

$$v'' = 0 \qquad \text{on } J,$$

because the last equation is always of type 1. Moreover, $J = \mathbf{R}$ if and only if equation (2) is of special kind. Hence for $n = 2$ Theorem 1 is proved. For $n \geqq 3$ solutions of (2) are of the class $C^n(I)$ (and solutions of $v'' = 0$ are $C^\omega(J)$), and relation (4) guarantees that $h \in C^{n+1}(J)$.

According to criterion for general $n \geqq 3$, the global transformation

$$z(t) = |\, h'(t) \,|^{(1-n)/2}\, y(h(t)), \qquad t \in J,$$

converts equation $P_n(y, x; I)$ into an equation whose first three coeficients are 1, 0, 0, that means, into an equation of the Laguerre–Forsyth form on the interval J.

Proof of Theorem 2.

If equation $P_n(y, x; I) \in \mathbf{A}_n$, $n \geqq 3$, can be globally transformed into the Halphen form introduced as the last in paragraph II then, according to criterion for general

$n \geqq 3$, this equation is iterative. Hence, let $P_n(y, x; I)$,

$$[p]_n(y, x; I) + r_{n-3}(x)\, y^{(n-3)} + \dots + r_0(x)\, y = 0, \qquad p = p_{n-2}\Big/\binom{n+1}{3},$$

can be globally transformed into

$$[q]_n\,(z, t; J) + \varepsilon_{n-i^*} z^{(i^*)} + s_{i^*-1}(t)\, z^{(i^*-1)} + \dots + s_0(t)\, z = 0$$

for some i^*, $0 \leqq i^* \leqq n - 3$, by means of transformation

(5) $$|\, h'(t)\,|^{(1-n)/2}\, y(h(t)) = z(t), \qquad t \in J.$$

Condition (iii) of criterion for general $n \geqq 3$ gives

$$r_i(h(t))\,.\,(h'(t))^{n-i} = 0 \qquad \text{on } J$$

for $i = n - 3, \dots, i^* + 1$, that means $r_i(x) = 0$ on I for those i. For r_{i^*} we get

$$r_{i^*}(h(t))\,.\,(h'(t))^{n-i^*} = s_{i^*}(t) = \varepsilon_{n-i^*} \qquad \text{on } J.$$

Since h is a C^{n+1}-diffeomorphism of J onto I and ε_{n-i^*} is a nonzero constant, we have $r_{i^*}(x) \neq 0$ on I and $r_{i^*} \in C^n(I)$, i^* being the maximal index $i = 0, \dots, n - 3$, for which r_i is not identically zero.

Conversely, if $P_n(y, x; I)$ is an iterative equation then, in fact, it is already in the Halphen form introduced as the last in par. II. Hence, let us assume that $P_n(y, x; I)$ is not an iterative equation and that it can be written in the form

$$[p]_n\,(y, x; I) + r_{i^*}(x)\, y^{(i^*)} + \dots + r_0(x)\, y = 0,$$

with nonvanishing r_{i^*} on I and $r_{i^*} \in C^n(I)$. According to criterion for general $n \geqq 3$, a global transformation (5) of $P_n(y, x; I)$ into an $Q_n(z, t; J)$,

$$[q]_n\,(z, t; J) + s_{n-3}(t)\, z^{(n-3)} + \dots + s_0(t)\, z = 0 \qquad \text{on } J$$

gives $s_{n-3}(t) = 0, \dots, s_{i^*+1}(t) = 0$, and

$$r_{i^*}(h(t))\,.\,(h'(t))^{n-i^*} = s_{i^*}(t),$$

on the interval $J := h^{-1}(I)$. For $s_{i^*}(t) := \varepsilon_{n-i^*}$ and $k := h^{-1}$ we have

$$r_{i^*}(x) = \varepsilon_{n-i^*}\,.\,(k'(x))^{n-i^*}, \qquad x \in I.$$

Put $\varepsilon_{n-i^*} = 1$ for odd $n - i^*$ and $\varepsilon_{n-i^*} = \operatorname{sign}(r_{i^*}(x))$ for even $n - i^*$. Then

$$k(x) = k_1 + \int_{x_0}^{x} (\varepsilon_{n-i^*}\,.\,r_{i^*}(\sigma))^{1/(n-i^*)}\, d\sigma, \qquad x_0 \in I,$$

k_1 being a constant. Evidently k is a C^{n+1}-diffeomorphism of I onto $k(I) = J$, because $r_{i^*} \in C^n(I)$ and r_{i^*} is always positive or always negative on the whole I. Hence $h = k^{-1}$ is such a C^{n+1}-diffeomorphism of J onto I for which the transformation (5) globally transforms $P_n(y, x; I)$ into an equation of the Halphen form.

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F. Neuman
Mathematical Institute of
the Czechoslovak Academy of Sciences,
branch Brno, Mendlovo nám. 1, Czechoslovakia