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SOME MONOTONICITY PROPERTIES ASSOCIATED WITH THE ZEROS OF BESSEL FUNCTIONS

LEE LORCH

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To Professor O. Borůvka on his 90th birthday

Abstract. The k-th positive zero of a solution of the generalized Airy equation $y'' + x^{\alpha}t = 0$, $a_{\alpha k} = [c\gamma_k/(2\nu)]^{2n}$, where $c_{\nu k}$ is the k-th positive zero of the Bessel function $C_\nu(x)$ and $\nu = 1/(\alpha + 2)$. Laforgia and Muldoon [ZAMP, 39, 1988, 267–271] have studied monotonicity in α of $a_{\alpha k}$ and $(\alpha + 2)\, a_{\alpha k}$. Additional such properties are presented here.

Key words. Airy functions, Bessel functions, zeros.

AMS Classification. 33 A 40, 34 C 10.

1. INTRODUCTION

The functions $\sqrt{x}J_\nu(x)$, $\sqrt{x}Y_\nu(x)$ and, more generally, $\sqrt{x}C_\nu(x)$ all satisfy a differential equation of the form $y'' + f(x)\, y = 0$, where $C_\nu(x) = (\cos\gamma)\, J_\nu(x) - (\sin\gamma)\, Y_\nu(x)$. These are often called Sturm–Liouville equations or, in the nomenclature Professor O. Borůvka has preferred in his extensive study of their transformation properties, Jacobi equations.

The positive zeros of $J_\nu(x)$, $Y_\nu(x)$, $C_\nu(x)$ are denoted by $j_{\nu k}, y_{\nu k}, c_{\nu k} = c_{\nu k}(\gamma)$, respectively, and have been studied intensively. Various of their monotonicity properties have acquired a considerable literature. Occasionally, the generic notations j, y, c will abbreviate the foregoing.

Recently, A. Laforgia and M. E. Muldoon [7] have studied the monotonicity of

(1) $$a_{\alpha k} = [c_{\nu k}/(2\nu)]^{2\nu} \qquad \text{and of } b_{\alpha k} = a_{\alpha k}/\nu,\ 0 < \nu = 1/(\alpha + 2).$$

The quantity $a_{\alpha k}$ is the k-th positive zero of an Airy function, i.e., a solution of the generalized Airy equation $y'' + x^{\alpha}y = 0$, an equation of Sturm–Liouville (Jacobi) type.

Heie some additional monotonicity properties of $a_{\alpha k}$ and $b_{\alpha k}$ will be presented, supplementing and, in some instances, extending those in [7].

2. PRELIMINARY RESULTS

Use will be made of G. N. Watson's familiar formula [14, p. 508 (3)]

$$\frac{dc_{\gamma k}}{d\nu} = 2c_{\gamma k} \int_0^\infty K_0(2c_{\gamma k} \sinh t)\, e^{-2\nu t}\, dt, \qquad k = 1, 2, \ldots, \tag{2}$$

where $K_0(x)$ is the standard modified Bessel function, known also as Macdonald's function. $K_0(x)$ is positive and decreases as $0 < x$ increases. From this formula can be derived, with the aid of the integrals evaluated in [14, p. 388 (9)], and the inequality $\sinh t > t$

$$\frac{\nu}{c_{\nu k}} \frac{dc_{\nu k}}{d\nu} < 1, \qquad \text{for } c_{\nu k} \geqq \gamma > 0, \tag{3}$$

and

$$\frac{\nu}{c_{\nu k}} \frac{dc_{\nu k}}{d\nu} < 3^{-3/2}\pi, \qquad \text{for } c_{\nu k} \geqq 2\nu > 0. \tag{4}$$

Proof of (3):
From (2),

$$\frac{\nu}{c_{\nu k}} \frac{dc_{\nu k}}{d\nu} = 2\nu \int_0^\infty K_0(2c_{\nu k} \sinh t)\, e^{-2\nu t}\, dt < 2\nu \int_0^\infty K_0(2\nu t)\, e^{-2\nu t}\, dt = \int_0^\infty K_0(x)\, e^{-x}\, dx = 1.$$

Proof of (4):
Similarly, here

$$\frac{\nu}{c_{\nu k}} \frac{dc_{\nu k}}{d\nu} < 2\nu \int_0^\infty K_0(4\nu t)\, e^{-2\nu t}\, dt = \frac{1}{2} \int_0^\infty K_0(x)\, e^{-x/2}\, dx = 3^{-3/2}\pi = .6045 \ldots$$

In both cases the integrals are evaluated in [14, p. 388 (9) ff.].
Another lemma will come in handy:

Lemma 1. *If* $c_{\nu k} > \gamma + \pi/4$, $\nu > 0$, $\beta > 0$, *then*

$$\delta_\nu = \frac{d}{d\nu}\left\{\ln\left[\frac{c_{\nu k}}{\beta\nu}\right]^\nu\right\} \quad \textit{decreases to } -\ln\beta, \textit{ as } 0 < \nu \to \infty.$$

Proof. Clearly,

$$\delta_\nu = \ln\frac{c}{\beta\nu} + \frac{\nu}{c}\frac{dc}{d\nu} - 1.$$

Hence

$$\frac{d\delta_\nu}{d\nu} = \frac{2}{c}\frac{dc}{d\nu} - \frac{\nu}{c^2}\left(\frac{dc}{d\nu}\right)^2 - \frac{1}{\nu} + \frac{\nu}{c}\frac{d^2c}{d\nu^2}.$$

Under the assumption that $c_{\nu k} \geqq \nu + \pi/4$, Laforgia and Muldoon have proved [6, p. 473 (4.3)], that $d^2c/d\gamma^2 < 0$. (They assume $c_{\nu k} > \nu + \pi/4$; their proof is valid in the case of equality.) Hence

$$\frac{d\delta_\nu}{d\nu} < \frac{1}{\nu}\left[\frac{2\nu}{c}\frac{dc}{d\nu} - \left(\frac{\nu}{c}\frac{dc}{d\nu}\right)^2\right] - \frac{1}{\nu}.$$

The expression in brackets is of the form $2x - x^2$ which increases in $0 \leqq x \leqq 1$ to the value 1. From (3), it follows therefore that the bracketed expression is less than 1, so that $d\delta_\nu/d\nu < 0$. Thus δ_ν decreases. That it approaches $-\ln \beta$, as $\nu \to \infty$, follows from the relations

$$\lim_{\nu\to\infty} \frac{c_{\gamma k}}{\nu} = 1, \qquad \lim_{\nu\to\infty} \frac{dc}{d\nu} = 1,$$

where the second limit can be inferred from (2).

Remarks. 1. In [7], and also here for the most part, $\beta = 2$. In case $\beta = 1$, the lemma implies that $\delta_\nu > 0$, $\nu > 0$. Hence

(5) $$\left(\frac{c_{\nu k}}{\nu}\right)^\nu \quad \text{increases for } \nu > 0 \text{ if } c_{\nu k} \geqq \nu + \pi/4.$$

This contrasts with a result [8, 11, 12] that will be used below, namely

(6) $$\frac{j_{\nu k}}{\nu} \quad \text{decreases for } \nu > 0.$$

2. As Laforgia and Muldoon point out [6], the hypothesis that $c_{\nu k} > \nu + \pi/4$ is satisfied when $k = 2, 3, \ldots$, also for $c_{\nu 1} = j_{\nu 1}$, and even for any $c_{\nu 1}$ for which $0 \leqq \gamma \leqq \frac{1}{2}\pi$. These zeros are decreasing functions of γ, $0 \leqq \gamma \leqq \pi/2$, for fixed $\nu \geqq 0$, $k = 1, 2, \ldots$ [9]. The smallest (but still included of course) is $y_{\nu 1} > y_{01} = 0.89 \ldots > \frac{1}{4}\pi = .78\ldots$ The difference $y_{\nu 1} - \nu$ increases with $\nu \geqq 0$, according to [2], since $y_{01} > \frac{1}{4}$.

3. From (3) it is clear that the restriction [6, (4.3)] that $c_\nu > \nu + \pi/4$ cannot be weakened to $c_\nu \geqq \nu$.

4. Inequality (3) implies that (6) can be generalized. For fixed $k = 1, 2, \ldots$,

(6′) $$\frac{\nu}{c_{\nu k}} \quad \text{increases, } 0 < \nu < \infty, \text{ provided } c_{\nu k} \geqq \nu.$$

The proof is simple:

$$c\left(\frac{\nu}{c}\right)' = 1 - \frac{\nu}{c}\frac{dc}{d\nu} > 0.$$

When $c_{\nu k} > \nu + \pi/4$, a condition satisfied for $0 < \nu < \infty$ already for $c_{\nu k} = j_{\nu 1}$ or $c_{\nu k} = y_{\nu 1}$ (and hence for all $c_{\nu k}$ when $k = 2, 3, \ldots$, since $c_{\nu 2} > j_{\nu 1}$), there follows similarly from [6, (4.2)] the stronger result that

(6″) $$\frac{\nu + \frac{1}{2}}{c_{\nu k}} \quad \text{increases (to 1), } 0 \leqq a < \nu < \infty,$$

if $c_{\nu k} > \nu + \pi/4$, $a < \nu < \infty$.

Theorems 4.2, 4.3 and 5.1 of [6] can be employed to yield stronger monotonicity results similar to (6′) and (6″).

Moreover

(7) $$\left(\frac{c_{\nu k}}{2\nu}\right)^{2\nu} \quad \text{decreases as } \nu \text{ increases, if } 0 < \nu \leqq c_{\nu k} \leqq 2\nu,$$

whence, in particular

(8) $$\left(\frac{j_{\nu 1}}{2\nu}\right)^{2\nu} \quad \text{decreases for } \nu \geqq \frac{7}{2},$$

(9) $$\left(\frac{c_{\nu 1}}{2\nu}\right)^{2\nu} \quad \text{decreases if } c_{\nu 1} \geqq \nu \geqq \frac{7}{2},$$

and

(10) $$\left(\frac{y_{\nu 1}}{2\nu}\right)^{2\nu} \quad \text{decreases for } \nu \geqq \frac{3}{2}.$$

To prove (7), it can be noted that (3) implies that $\delta_\nu < \ln\left(\frac{c}{2\nu}\right)$, whence $\delta_\nu < 0$ under the present hypothesis. The assertions (8) and (10) follow at once, since $j_{7/2,1} = 6.98\ldots < 7$, $y_{3/2,1} = 2.79\ldots < 3$ and $j_{\nu 1} - \nu$, $y_{\nu 1} - \nu$ increase with ν. The claim (9) follows from (7), since $c_{\nu 1} \leqq j_{\nu 1}$ [9] and hence $c_{\nu 1} \leq 2\nu$, when $\nu \geqq \frac{7}{2}$.

3. FURTHER RESULTS

Actually, as will be shown below, (8) and (9) can be extended by demonstrating that the quantities in question decrease for $\nu > \nu_1$, with $1.003 < \nu_1 < 1.006$. (10) holds even for $\nu \geqq \frac{1}{2}$, but not (8); that function, on the contrary, actually increases until $\nu = \nu_1$.

The monotonicity of δ_ν implies that if $\delta_\mu \geqq 0$, then $[c_{\nu k}/(2\nu)]^{2\nu}$ increases for $0 < \nu \leqq \mu$, while $\delta_\lambda \leqq 0$ implies that this function of ν decreases for all $\nu > \lambda$, provided of course that $c_{\nu k} \geqq \nu + \pi/4$. For $c_{\nu k} = j_{\nu 1}$ or $y_{\nu 1}$ the condition is satisfied.

This choice permits the application of a formula recently discovered by M. E. H. Ismail and M. E. Muldoon [3, p. 195 (5.11)], placing in it $\nu = 1$, $k = 1$ and using only the first three terms of its infinite series.

The resulting inequality is

$$j\frac{\mathrm{d}j}{\mathrm{d}\nu} > 2(2 + 112j^{-2} - 4\,032j^{-4} + 36\,864j^{-6}),$$

where $j = j_{11} = 3.831\,705\,97$ [13, p. 2] and $\mathrm{d}j/\mathrm{d}\nu$ is evaluated at $\nu = 1$.
Hence, at $\nu = 1$

$$\frac{\mathrm{d}j_{\nu 1}}{\mathrm{d}\nu} > 1.342\,276\,7.$$

Putting $c_{\nu k} = j_{11}$ and using the resulting values, it turns out that the quantity δ_ν of the lemma exceeds .000 47, and so is positive.

On the other hand, $(j_{11}/2)^2 > (j_{3/2,1}/3)^3$, as standard tables [13] reveal.
Therefore, there exists a value $\nu = \nu_1$, $1 < \nu_1 < 3/2$, such that

(11) $$\left(\frac{j_{\nu 1}}{2\nu}\right)^{2\nu} \quad \text{increases, } 0 < \nu \leqq \nu_1.$$

and

(8′) $$\left(\frac{j_{\nu 1}}{2\nu}\right)^{2\nu}. \quad \text{decreases, } \nu \geqq \nu_1,$$

the promised extension of (8), except for the more precise determination of ν_1 to be given below in Remark 3.

Remarks. 1. The lower estimate, 1.342 276 7, for $\mathrm{d}j_\nu/\mathrm{d}\nu$ at $\gamma = 1$, derived above from the Ismail – Muldoon formula, is quite close to the actual value, since

$$\frac{\mathrm{d}j}{\mathrm{d}\nu} < \frac{\arccos(1/j_{11})}{[1 - j_{11}^{-2}]^{1/2}} = 1.353\,671\,4 \qquad \text{at } \nu = 1.$$

This follows from (2), since $\sinh t > t$, whence [14, p. 388 (9)]

$$\left.\frac{\mathrm{d}j}{\mathrm{d}\nu}\right|_{\nu=1} < 2j_{11}\int_0^\infty K_0(2j_{11}t)\,e^{-2t}\,\mathrm{d}t = \int_0^\infty K_0(x)\,e^{-x/j_{11}}\,\mathrm{d}x = \frac{\arccos(1/j_{11})}{[1 - j_{11}^{-2}]^{1/2}}.$$

2. Alternatively, (11) can be proved by employing Schläfli's formula for $\mathrm{d}j/\mathrm{d}\nu$ [14, p. 508 (2)] in δ_ν, then estimating the corresponding integral from tables of values of $J_1(t)$.

3. M. E. Muldoon, in connection with (11), has calculated values of $[j_{\nu 1}/(2\nu)]^{2\nu}$ for ν in the neighbourhood of 1. From these, it emerges that $1.003 < \nu_1 < 1.006$, i.e., that this function reaches its (unique) maximum for ν between 1.003 and 1.006, and that the maximum equals 3.670 52… Hence

$$[c_{\nu 1}/(2\nu)]^{2\nu} \leqq [j_{\nu 1}/(2\nu)]^{2\nu} < 3.670\,53, \qquad \nu > 0,$$

since $c_{\nu 1} \leqq j_{\nu 1}$ [9].

Muldoon's precise bounds for ν_1 permit rewriting (8′) and (11) as (cf. [7, Corollary 2.2])

$$(12) \qquad a_{\alpha 1} \quad \text{decreases,} \ -1.00596 < -2 + \frac{1}{\nu_1} \leqq \alpha < \infty,$$

and

$$(13) \qquad a_{\alpha 1} \quad \text{increases,} \ -2 < \alpha \leqq -2 + \frac{1}{\nu_1} < -1.003,$$

where, in both cases, $c_{\nu k} = j_{\nu 1}$.

In situations in which the order ν is kept constant but in which k or γ varies in $c_{\nu k}(\gamma)$ it is useful to recall [10, Lemma] that

$$g(x) = 2x \int_0^\infty K_0(2x \sinh t)\, e^{-2\nu t}\, dt,$$

increases as $x > 0$ increases, with ν constant; $g(c) = dc/d\nu$.

These two cases can be treated simultaneously in terms of the notation introduced in [2, § 2]. There, Elbert and Laforgia define the function $j_{\nu\varkappa}$ of the continuous variable $\varkappa > 0$ as the solution of the differential equation obtained by replacing in (2) the quantity $c_{\nu k}$ by $j = j(\nu)$, with boundary condition

$$\lim_{\nu \to -\varkappa+} j(\nu) = 0.$$

For $\varkappa = k = 1, 2, \ldots$, the zeros $j_{\nu k}$ are regained; for $k - 1 < \varkappa < k$, $j_{\nu\varkappa} = c_{\nu k}(\gamma)$ with $\gamma = (k - \varkappa)\pi$. They show, what will be used below, that $j_{\nu\varkappa}$ increases with $\varkappa > 0$ for fixed ν. This incorporates the result of [9] that $c_{\nu k}(\gamma)$ decreases as γ increases, $0 \leqq \gamma < \pi$, when $\nu \geqq 0$, $k = 1, 2, \ldots$, are fixed; in particular, $c_{\nu k} \leqq j_{\nu k}$, $\nu \geqq 0$, $k = 1, 2, \ldots$,

Lemma 2. *For $\nu > 0$ constant, $j_{\nu\varkappa} > \nu$ implies that δ_ν increases as $\varkappa > 0$ increases.*

Proof. From the definition of $\delta_\nu = \delta_\nu(\varkappa)$ in Lemma 1, with $c_{\nu k}$ rewritten as $j_{\nu\varkappa}$

$$\delta_\nu'(\varkappa) = \frac{1}{j}\frac{\partial j}{\partial \varkappa}\left[1 - \frac{\nu}{j}\frac{dj}{d\nu}\right] + \frac{\nu}{c}\frac{\partial}{\partial \varkappa}\left[\frac{dj}{d\nu}\right].$$

The first term is positive, from (3), since $j_{\nu\varkappa}$ increases with $\varkappa > 0$. The second is also positive, since $\nu \geqq 0$ is fixed, because $g(\varkappa) = dj_{\nu\varkappa}/d\nu$ increases with $\varkappa$. This proves the lemma.

One application extends (8′) and (9) to

$$(9') \qquad \left(\frac{c_{\nu 1}}{2\nu}\right)^{2\nu} \quad \text{decreases for } \nu \geqq \nu_1, \text{ if } c_{\nu 1} \geqq \nu.$$

Proof. As remarked, $c_{\nu 1} \leqq j_{\nu 1}$ [2, 9], so that Lemma 2 implies

$$\delta_\nu = \ln \frac{c_{\nu 1}}{2\nu} + \frac{\nu}{c} \frac{\mathrm{d}c}{\mathrm{d}\nu} - 1 \leqq \ln \frac{j_{\nu 1}}{2\nu} + \frac{\nu}{j} \frac{\mathrm{d}j}{\mathrm{d}\nu} - 1,$$

and the last is negative for $\nu > \nu_1$ in view of (8′).

Another application extends (11) by implying, in the context of Lemma 1, the existence of unique $\nu_\varkappa$ such that

$$\text{(11′)} \qquad \left(\frac{j_{\nu\varkappa}}{2\nu}\right)^{2\nu} \quad \text{increases, } 0 < \nu \leqq \nu_\varkappa, \text{ and decreases, } \nu \geqq \nu_\varkappa,$$

where $\nu_\varkappa$ is an increasing function of $\varkappa > 0$.

Hence (12) extends to

$$\text{(12′)} \qquad a_{\alpha\varkappa} \quad \text{decreases for } -2\nu + \frac{1}{\nu_\varkappa} \leqq \alpha < \infty, \text{ for } c_{\nu k} = j_{\nu\varkappa}.$$

Remark. The monotonicity of $\nu_\varkappa$ can be illustrated with values determined approximately from [13], especially with Muldoon's already recorded limitations on ν_1. Note that $j_{\nu, k-1/2} = y_{\nu k}$.

Then: $0 < \nu_{1/2} < 1/2$, $1.003 < \nu_1$ 1.006, $1 < \nu_{3/2} < 2$, $2 < \nu_2 < 3$, $6 < \nu_5 < 7$, $11.5 < \nu_{9,5} < 12.5$, $12.5 < \nu_{10} < 13.5$, $19 < \nu_{15} < 20$.

For a restricted range of γ, namely $\pi/2 \leqq \gamma < \pi$, decrease in (9′) commences even before ν reaches $\nu_1 > 1.003$. When $\gamma = \pi/2$, $c_{\nu 1} = y_{\gamma 1}$, and

$$\text{(9″)} \qquad \left(\frac{c_{\nu 1}}{2\nu}\right)^{2\nu} \quad \text{decreases for } \nu \geqq \frac{1}{2}, \qquad \text{if } 0 < \nu \leqq c_{\nu 1} \leqq y_{\nu 1}.$$

Proof. From Lemma 2 and [2, 9],

$$\delta_\nu < \ln \frac{y_{\nu 1}}{2\nu} + \frac{\nu}{y} \frac{\mathrm{d}y}{\mathrm{d}\nu} - 1, \qquad \pi/2 \leqq \gamma < \pi.$$

Denoting the upper bound by θ_ν, it follows from Lemma 1 that θ_ν decreases as $\nu \geqq 1/2$ increases, since $y_{1/2,1} = \pi/2 > 1/2 + \pi/4$.

Thus, $\theta_{1/2} < 0$ would imply $\theta_\nu < 0$, $\nu \geqq 1/2$, and all the more that $\delta_\nu < 0$, $\nu \geqq 1/2$, for $0 < \nu < c_{\nu 1} \leqq y_{\nu 1}$, establishing (9″).

To establish that $\theta_{1/2} < 0$, an upper bound will be inferred for $(\nu/y)\,\mathrm{d}y/\mathrm{d}\nu$ at $\nu = 1/2$. From (2) and [14, p. 388 (9)] it follows that

$$\left.\frac{\nu}{y} \frac{\mathrm{d}y}{\mathrm{d}\nu}\right|_{1/2} = \int_0^\infty K_0(\pi \sinh t)\, e^{-t}\, \mathrm{d}t < \int_0^\infty K_0(\pi t)\, e^{-t}\, \mathrm{d}t = \frac{\arccos(1/\pi)}{(\pi^2 - 1)^{1/2}} = .418\,66 \ldots$$

Hence,

$$\theta_{1/2} < \ln(\pi/2) + .418\,67 - 1 = -.129\,74 < 0,$$

completing the proof of (9″).

Finally, it will be established that, recalling (1),

$$b_1 = \left(\frac{c_{\nu 1}}{2\nu}\right)^{2\nu} \frac{1}{\nu} \quad \text{decreases for } 0 < \nu < \infty, \text{ if } c_{\nu 1} \geqq \nu. \tag{14}$$

The proof is divided into four parts, with the final part further subdivided.

(i) $0 < \nu \leqq 1/3$.

Here

$$\frac{1}{2}\frac{d}{d\nu}\{\ln b_{\alpha 1}\} = \ln\frac{c}{2\nu} - \frac{1}{2\nu} + \frac{\alpha}{c}\frac{dc}{d\nu} - 1 < \ln\frac{c}{2\nu} - \frac{1}{2\nu} \leqq \ln\frac{j_{\nu 1}}{2\nu} - \frac{1}{2\nu} < 0,$$

where the first inequality is a consequence of (3), the second of [9] and the final one from the proof of Corollary 2.5 of [7, p. 270].

(ii) $0 < c_\nu \leqq 2\nu$. Here (14) is obvious from the first inequality in (i).

(iii) $\nu \geqq 3/2$. Here (14) is obvious from the stronger result (9′).

(iv) Henceforth, therefore, it may be assumed both that $c_{\nu 1} > 2\nu$, and that $1/3 < \nu < 3/2$. This permits the use of inequality (4) from which follows a strengthening of the inequalities in (i), namely (for $c_{\nu 1} > 2\nu > 0$),

$$\frac{1}{2}\frac{d}{d\nu}\{\ln b_{\alpha 1}\} < \ln\frac{j_{\nu 1}}{2\nu} + \frac{\pi}{3^{3/2}} - 1 - \frac{1}{2\nu}. \tag{15}$$

To show that this expression is negative for the remaining ν, repeated use will be made of (6). It suffices in each interval $\lambda \leqq \nu \leqq \mu$ to prove that

$$\ln\frac{j_{\lambda 1}}{2\lambda} + \frac{\pi}{3^{3/2}} - 1 - \frac{1}{2\mu} \leqq 0, \tag{16}$$

since $j_{\nu 1}/\nu$ decreases, making the expression on the left larger than the upper bound in (15).

Tables of zeros [1, 13] permit the appropriate calculations except for the first two ν-intervals which use $j_{2/5,1} = 2.998\,849$. For this value I am indebted to Martin E. Muldoon who kindly calculated it. Later I learned that this is consistent with the value recorded in [4, p. 195].

Using these values, the inequality (16) is verified for successive closed $[\lambda, \mu]$-intervals as follows

$$\left[\frac{1}{3}, \frac{2}{5}\right], \quad \left[\frac{2}{5}, \frac{1}{2}\right], \quad \left[\frac{1}{2}, \frac{2}{3}\right], \quad \left[\frac{2}{3}, \frac{3}{4}\right], \quad \left[\frac{3}{4}, 1\right], \quad \left[1, \frac{3}{2}\right].$$

This completes the proof.

Remarks 1. For $c_{\nu k}$, $k = 2, 3, \ldots$, the analogues of (14) fail to hold for "smaller" ν, but become valid only for ever larger ν as k is fixed at larger values.

Remarks 2. Other monotonicity properties associated with Bessel function zeros, some contrasting with one another, also follow from (2). The proofs are immediate, if it is kept in mind that $K_0(x)$ is a decreasing function, that $c_{\nu k} = c_{\nu k}(\gamma)$ is an increasing function of ν when k and γ are kept fixed [14, p. 508], a decreasing function of γ when $\nu \geqq 0$ and k are fixed [9].

It is thus obvious from (2) that

(17) $$\frac{1}{c}\frac{dc}{d\nu} \quad \text{decreases,}$$

(i) as $0 < \nu$ increases (k, γ fixed), (ii) as k increases (ν, γ fixed).

Further,

(18) $$\frac{1}{c}\frac{dc}{d\nu} \quad \text{increases as } \gamma \text{ increases, } 0 \leqq \gamma < \pi, \ (\nu,\ k \text{ fixed}).$$

From the monotonicity of $g(x)$ follow

(19) $$\frac{dc}{d\nu} \quad \text{decreases as } \gamma \text{ increases, } 0 \leqq \gamma < \pi, \ (k,\ \nu \text{ fixed}),$$

and

(20) $$\frac{dc}{d\nu} \quad \text{increases as } k \text{ increases, } (\gamma,\ \nu \text{ fixed}).$$

Remarks 3. A slight extension of Corollary 2.6 of [7] concerning zeros of solution of the generalized Airy equation follow from (11′) in view of Muldoon's calculation that $1.003 < \nu_1 < 1.006$. This implies, for $k = 1, 2, \ldots$, that

(21) $$(\alpha + 2)\, a_{\alpha k} \text{ increases as } \alpha \text{ increases, } -1.0029 < \alpha < \infty.$$

The lower bound for α can be decreased further as a function of $k = 2, 3, \ldots$

4. ACKNOWLEDGEMENTS

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