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ORDINARY DIFFERENTIAL EQUATIONS THE SOLUTION OF WHICH ARE ACG_*-FUNCTIONS

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Dedicated to Professor Otakar Borůvka on the ocassion of his ninetieth birthday

Abstract: Ordinary differential equations are studied in the case when the concept of Perron integral is involved. An equivalent description is given for a certain class of such equations introduced recently by R. Henstock.

Key words. Ordinary differential equations, ACG_*-functions, Perron integral, convergence theorems for Perron integrable functions.

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The common concept of solution of the differential equation

$$\dot{x} = f(t, x) \tag{0.1}$$

goes back to C. Carathéodory, who in [1] linked the equation (0.1) with the concept of the Lebesgue integral. The Carathéodory solutions of (0.1) are simultaneously solutions of

$$x(t) = x(r) + (L) \int_r^t f(s, x(s))\, ds. \tag{0.2}$$

The existence of solutions of

$$x(t) = x(r) + (P) \int_r^t f(s, x(s))\, ds \tag{0.3}$$

– (P) indicates that the concept of the Perron integral is involved – was proved under various assumptions in [3], [5], [4]. In [5] a theory of (0.3) is given under the restriction that f is linear with respect to x, i.e. $f(t, x) = A(t)\, x$, $A(t)$ being an $n \times n$-matrix; this theory yields necessary and sufficient conditions for the existence and uniqueness of solutions of (0.3).

In [4] an existence theorem for the nonlinear equation (0.3) is established, however one of its assumptions is rather complicated. In Section 1 a simple form of this assumption is given; the result consists in the fact that the assumptions of the local existence theorem from [4] are fulfilled if and only if the function f can be written in the form

$$f(t, x) = g(t) + h(t, x),$$

where g is Perron integrable and h fulfils the usual Carathéodory assumptions. Section 1 is concluded by comments to the underlying convergence theorem from [4] and to a similar convergence theorem from [6]. A short information on Perron integrable functions and their indefinite integrals (primitives) can be found in Section 2.

1.

The following local existence result for the equation (0.1) was proved in [4] (Theorem 19.1).

1.1. Theorem. *Assume that the following conditions are fulfilled for* f: $[0,1] \times [0,1]^n \to R^n$:

(1.1) $f(t, .)$ *is continuous for almost all* $t \in [0,1]$,

(1.2) *the Perron integral* $\int_0^1 f(t, z)\, dt$ *exists for every* $z \in [0,1]^n$,

(1.3) *for a compact set* $S \subset R^n$, *some gauge* δ *on* $[0, 1]$ *and all* δ*-fine partitions* $D = \{\xi_0, \tau_1, \xi_1, \ldots, \xi_{k-1}, \tau_k, \xi_k\}$ *of* $[0, 1]$ *(see Section 2) and all functions* w: $[0, 1] \to [0, 1]^n$ *we have*

$$\sum_{i=1}^{k} f(\tau_i, w(\tau_i))\, (\xi_i - \xi_{i-1}) \in S.$$

Then for every $v \in (0, 1)^n$ *there is an* $\alpha > 0$ *and a* y: $[0, \alpha] \to [0, 1]^n$ *such that*

$$y(t) = v + (P) \int_0^t f(s, y(s))\, ds \tag{1.4}$$

for $t \in [0, \alpha]$.

Note 1. y is necessarily an ACG_* function on $[0, \alpha]$; see Sections 2.5, 2.6 and 2.7.

Note 2. It should be mentioned that the condition (1.2) is not stated in [4] explicitly, but it is evidently used in the proof of Theorem 19.1 in [4].

1.2. Proposition. *The function* $f\colon [0, 1]\times[0, 1]^n \to R^n$ *fulfils* (1.1), (1.2) *and* (1.3) *if and only if*

(1.5) $$f(t, x) = g(t) + h(t, x) \quad \textit{for } t \in [0, 1]\times[0, 1]^n,$$

where

(1.6) $g : [0, 1] \to R^n$ *is Perron integrable*

and $h : [0, 1]\times[0, 1]^n \to R^n$ *fulfils*

(1.7) $h(t, .)$ *is continuous for almost all* $t \in [0,1]$,

(1.8) $h(., x)$ *is measurable for* $x \in [0, 1]^n$,

(1.9) *there exists such a measurable* $m : [0, 1] \to [0, \infty]$

that $\int_0^1 m \,\mathrm{d}t < \infty$

and

$$\| h(t, x) \| \leqq m(t) \quad \textit{for } x \in [0, 1]^n \textit{ and almost all } t.$$

$$\text{(We put } \| y \|^2 = \sum_{j=1}^{n} y_j^2 \quad \text{for } y = (y_1, \ldots, y_n) \in R^n\text{).}$$

Proof. We shall prove only the "only if" part, since the converse is obvious. Let f fulfil (1.1), (1.2) and (1.3), $v \in [0, 1]^n$.

Put

$$g(t) = f(t, v),\ h(t, x) = f(t, x) - g(t) \quad \text{for } t \in [0, 1],\ x \in [0, 1]^n.$$

(1.5)–(1.8) are obviously satisfied. We have to prove (1.9). Denote $f(t, x) = (f_1(t, x), \ldots, f_n(t, x))$ with $f_j(t, x) \in R$. It follows from (1.3) that there exists such an $M > 0$ that

$$\Big| \sum_{i=1}^{k} f_j(\tau_i, w(\tau_i))(\xi_i - \xi_{i-1}) \Big| \leqq M$$

and

$$\Big| \sum_{i=1}^{k} h_j(\tau_i, w(\tau_i))(\xi_i - \xi_{i-1}) \Big| = \Big| \sum_{i=1}^{k} (f_j(\tau_i, w(\tau_i)) - f_j(\tau_i, v))(\xi_i - \xi_{i-1}) \Big| \leqq 2M$$

for $j = 1, \ldots, n$, any δ-fine partition D of $[0, 1]$ and any $w : [0, 1] \to [0, 1]^n$. (δ is the gauge given in (1.3).)

Given j, D, w, put $w^+(\tau_i) = w(\tau_i)$, $w^-(\tau_i) = v$ if $f_j(\tau_i, w(\tau_i)) > f_j(\tau_i, v)$, $w^+(\tau_i) = v$, $w^-(\tau_i) = w(\tau_i)$ otherwise. We have

$$\sum_{i=1}^{k} (f_j(\tau_i, w^+(\tau_i)) - f_j(\tau_i, v))(\xi_i - \xi_{i-1}) \leqq 2M,$$

$$\sum_{i=1}^{k} (f_j(\tau_i, v) - f_j(\tau_i, w^-(\tau_i)))(\xi_i - \xi_{i-1}) \leqq 2M$$

the summands being nonnegative. Thus

$$\sum_{i=1}^{k} | f_j(\tau_i, w(\tau_i)) - f_j(\tau_i, v) | (\xi_i - \xi_{i-1}) \leqq 4M$$

and finally we conclude that

$$\sum_{i=1}^{k} \| h(\tau_i, w(\tau_i)) \| (\xi_i - \xi_{i-1}) \leqq C = 4Mn \tag{1.10}$$

for any δ-fine partition D of $[0, 1]$ and any $w : [0, 1] \to [0, 1]^n$.

Put $M(0) = 0$ and

$$M(s) = \sup \{ \sum_{i=1}^{k} \| h(\tau_i, w(\tau_i)) \| (\xi_i - \xi_{i-1}) \}, \qquad s \in [0,1], \tag{1.11}$$

where the supremum is taken over all δ-fine partitions $\{\xi_0, \tau_1, \xi_1, \ldots, \tau_k, \xi_k\}$ of $[0, s]$ and all functions $w : [0, 1] \to [0, 1]^n$.

Since for every s_1, s_2 such that $0 \leqq s_1 < s_2 \leqq 1$ there exists a δ-fine partition of $[s_1, s_2]$ (see Section 2.3) we evidently have

$$M(s_1) \leqq M(s_2) \tag{1.12}$$

and also (cf. (1.10))

$$0 \leqq M(s) \leqq C, \qquad s \in [0, 1]. \tag{1.13}$$

Let $t, s_1, s_2 \in [0, 1]$, $s_1 < s_2$, $t - \delta(t) < s_1 \leqq t \leqq s_2 < t + \delta(t)$. If $s_1 > 0$ and if the triple (s_1, t, s_2) is added from the right to a δ-fine partition of $[0, s_1]$, a δ-fine partition of $[0, s_2]$ is obtained and therefore

$$M(s_1) + \| h(t, x) \| (s_2 - s_1) \leqq M(s_2) \qquad \text{for any } x \in [0, 1]^n,$$

i.e.

$$\| h(t, x) \| (s_2 - s_1) \leqq M(s_2) - M(s_1) \qquad \text{for } x \in [0, 1]^n. \tag{1.14}$$

Obviously the derivative $\dot{M}(t)$ exists almost everywhere. Putting

$$m(t) = \begin{cases} \dot{M}(t) & \text{if } \dot{M}(t) \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$\| h(t, x) \| \leqq m(t) \qquad \text{for } x \in [0, 1]^n \text{ and almost all } t;$$

moreover, $(L) \int_0^1 m \, \mathrm{d}t \leqq M(1) < \infty$ and (1.9) holds.

1.3. In [5] necessary and sufficient conditions for the existence and uniqueness of ACG* solutions of $\dot{x} = A(t)\, x$ were formulated as conditions on the matrix function A; the entries of A need not be Perron integrable, i.e. (1.2) need not hold.

1.4. Observe that

$$\dot{x} = g(t) + h(t, x) \tag{1.15}$$

can be transformed to the form of a Carathéodory type differential equation by using the transformation $y = x - G(t)$, G being a primitive to g, provided (1.6) – (1.9) are satisfied.

1.5. In [3] the existence theorem for the equation (0.3) is proved under the assumption that (1.1), (1.2) and

$$\| f(t, x) - f(t, y) \| \leqq L(t) \| x - y \|, \qquad t \in [0, 1], \qquad x, y \in [0, 1]^n, \tag{1.16}$$
$$\int_0^1 L(t)\, \mathrm{d}t < \infty$$

hold. Obviously (1.16) implies both (1.3) and (1.5) with (1.6) – (1.9).

1.6. In the proof of Theorem 1.1 in [4] a convergence result for Perron integrals was used (see Theorem (9.1) in [4]). A special form of this result is the following

Theorem. *Let* $\psi_j, \varphi : [0, 1] \to R$, *let* $(P) \int_0^1 \psi_j \, \mathrm{d}t$ *exist for* $j \in N$, $\psi_j(t) \to \varphi(t)$ a.e. for $j \to \infty$ *and let the following condition hold:*

(1.17) *there exist such a gauge* δ *and such* $B, C \in R$ *that*

$$B \leqq \sum_{i=1}^{k} \psi_j(t_i) (\xi_i - \xi_{i-1}) \leqq C$$

holds for every $j \in N$ *and for every* δ*-fine partition of* $[0, 1]$.

Then $(P) \int_0^1 \varphi \, \mathrm{d}t$ *exists and*

$$\lim_{r \to \infty} (P) \int_0^1 \psi_r \, \mathrm{d}t = (P) \int_0^1 \varphi \, \mathrm{d}t.$$

In a similar way as in Section 1.2 it can be proved that (1.17) is satisfied if and only if

$$| \psi_j(t) - \psi_1(t) | \leqq m(t) \qquad \text{a.e. in } [0, 1], \tag{1.18}$$

where $j \in N$ and $\int_0^1 m \, \mathrm{d}t < \infty$. In the more general case which is treated in [4] an analogous argument makes it possible to conclude the following:
the condition (9.5) from [4] holds if and only if

(1.19) there exist such a superadditive interval function Λ with values from $[0, \infty]$ and such a gauge δ on $[0, 1]$ that

$$| f_j(x) - f_1(x) | \, k(I, x) \leqq \Lambda(I), \qquad j \in N, \qquad x \in E$$

provided $x \in I \subset B(x, \delta(x))$ ($B(x, \delta)$ denotes the ball with center x and radius δ, I being an interval in E).

1.7. Another result of the same type is Theorem 5.5 from [6]. A special form of this result has the form of our Theorem 1.6 where (1.17) is replaced by

(1.20) there exist a gauge w on $[0, 1]$ and $c \in R$ such that

$$| \sum_{i=1}^{k} \int_{\xi_{i-1}}^{\xi_i} \psi_{j_i}(t) \, dt | \leqq c$$

holds for every finite sequence of positive integers $j_1, j_2, \ldots, j_k$ and for every w-fine partition of $[0, 1]$.

Again it can be proved in a similar way as in Section 1.2 that (1.20) is fulfilled if and only if (1.18) holds. In the more general case which was treated in [6] an analogous argument yields: the condition (B) of Lemma 5.4 from [6] holds if and only if

(1.21) there exist a superadditive interval function Λ with values from $[0, \infty)$ and a gauge δ such that

$$| U_j(J, t) - U_1(J, t) | \leqq \Lambda(J)$$

provided $t \in J \subset B(t, \delta(t))$ (J being an interval in K).

2

The original definition of the Perron integral relies on the concepts of major and minor functions to a given function. Here we shall give an equivalent definition, which is an immediate extension of the definition given by B. Riemann.

2.1. Let $a, b \in R$, $a < b$. A finite sequence

$$D = \{\xi_0, \tau_1, \xi_1, \ldots, \xi_{k-1}, \tau_k, \xi_k\}$$

is called a partition of $[a, b]$ if

$$a = \xi_0 \leqq \tau_1 \leqq \xi_1 \leqq \ldots \leqq \xi_{k-1} \leqq \tau_k \leqq \xi_k = b, \quad \xi_{i-1} < \xi_i \quad \text{for } i = 1, 2, \ldots, k.$$

A function $\delta : [a, b] \to (0, \infty)$ is called a gauge on $[a, b]$.

Let D be a partition of $[a, b]$ and let δ be a gauge on $[a, b]$. D is called δ-fine if $\tau_i - \delta(\tau_i) < \xi_{i-1}$, $\xi_i < \tau_i + \delta(\tau_i)$ for $i = 1, \ldots, k$.

2.2. Definition. Let $a, b \in R$, $a < b$, $u : [a, b] \to R$, $c \in R$. Then c is called the Perron integral of u (from a to b) and denoted by $(P)\int_a^b u(t)\,dt$ or $(P)\int_a^b u\,dt$ if for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\left| \sum_{i=1}^{k} u(\tau_i)(\xi_i - \xi_{i-1}) - c \right| \leqq \varepsilon$$

holds for every δ-fine partition D of $[a, b]$.
(As usual we put $(P)\int_a^a u\,dt = 0$ and $(P)\int_b^a u\,dt = -(P)\int_a^b u\,dt$.)

2.3. Definition 2.2 is meaningful, since we have the following Cousin's lemma (cf. 3.4 in [6] or [7], p. 104):

A δ-fine partition D of $[a, b]$ exists for every gauge δ on $[a, b]$.

This can be proved by an elementary supremum argument or by an argument relying on halving intervals starting with the interval $[a, b]$.

2.4. Let $(P)\int_a^b u(s)\,ds$ exist. Then for every $t \in (a, b)$ the integral $(P)\int_a^t u(s)\,ds$ exists.

This is a consequence of Cousin's lemma.

2.5. Let $(P)\int_a^b u(s)\,ds$ exist. Put $U(t) = (P)\int_a^t u(s)\,ds$ for $t \in (a, b]$, $U(a) = 0$. Then

(2.1) the derivative $\dot{U}(t)$ exists and is equal to $u(t)$ at almost every $t \in [a, b]$,

(2.2) for every set $N \subset [a, b]$ of Lebesgue measure zero and for every $\varepsilon > 0$ there exists such a function $\delta : N \to (0, \infty)$ that

$$\sum_{i=1}^{k} | U(\eta_i) - U(\xi_i) | \leqq \varepsilon$$

for every finite sequence $\xi_1, \tau_1, \eta_1, \xi_2, \tau_2, \eta_2, \ldots, \xi_k, \tau_k, \eta_k$ fulfilling $\xi_1 \leqq \tau_1 \leqq \eta_1 \leqq \xi_2 \leqq \tau_2 \leqq \eta_2 \leqq \ldots \leqq \xi_k \leqq \tau_k \leqq \eta_k$, $\tau_i \in N$, $\tau_i - \delta(\tau_i) < \xi_i, \eta_i < \tau_i + \delta(\tau_i)$, $i = 1, 2, \ldots, k$.

For the proof see [5], Theorem 3.8.

2.6. Let $U : [a, b] \to R$ fulfil (2.2) and

(2.3) the derivative $\dot{U}(t)$ exists at almost every $t \in [a, b]$.

Let $v : [a, b] \to R$, $v(t) = \dot{U}(t)$ if $\dot{U}(t)$ exists, $v(t)$ arbitrary otherwise. Then the integral $(P)\int_a^b v\,ds$ exists and is equal to $U(b) - U(a)$.

This can be proved directly from Definition 2.1; see also [5], Theorem 3.9.

2.7. The concept of an ACG_* function was introduced by A. Denjoy, A. Khintchine (see [8], p. 231) as a generalization of the concept of an absolutely continuous function. As was observed in [5], (3.19), we have an equivalent definition: $U : [a, b] \to R$ is an ACG_* function on $[a, b]$ if it fulfils (2.1) and (2.2).

2.8. Definition 2.2 and the results from Sections 2.4–2.7 can be immediately extended to functions with values in R^n. Thus by 2.5, 2.6 and 2.7 we have

(2.4) every solution x of (0.3) is an ACG_* function on every compact subinterval of its definition interval; moreover, x fulfils (0.1) almost everywhere.

(2.5) If x is an ACG_* function on every compact subinterval of its definition interval and if x fulfils (0.1) almost everywhere, then x is a solution of (0.3).

REFERENCES

[1] C. Carathéodory, *Vorlesungen über reelle Funktionen,* B. G. Teubner, Leipzig u. Berlin, 1918.

[2] A. Denjoy, *Mémoire sur la totalisation des nombres dérivés non-sommables,* Ann. Ecole Norm. 34 (1917), 181–238.

[3] N. P. Erugin, I. Z. Shtokalo et al., *Lectures on Ordinary Differential Equations* (in Russian), Golovnoe izd., Kiev, 1974.

[4] R. Henstock, *Lectures on the Theory of Integration,* World Scientific, Singapore, 1988.

[5] J. Jarník, J. Kurzweil, *A general form of the product integral and linear ordinary differential equations,* Czech. Math. Journal, 37 (112), 1987, 642–659.

[6] J. Kurzweil, *Nichtabsolut konvergente Integrale,* BSB B. G. Teubner, Leipzig, 1980.

[7] J. Mawhin, *Introduction a l'analyse,* CABAY, Libraire-éditeur, Louvain-la Neuve, 1984.

[8] S. Saks, *Theory of the Integral,* Monografie Matematyczne, Warszawa, 1937.

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