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ASYMPTOTIC AND INTEGRAL EQUIVALENCE OF FUNCTIONAL AND ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The main results gives conditions under a one-to-one, bicontinuous correspondence exists between g-bounded solutions of a linear differential system and such solution of perturbations of the system.

Key words. System of differential equations, functional differential equations, asymptotic equivalence.

MS Classification. 34 K 25.

The purpose of this paper is provide conditions for asymptotic equivalence and (g, p)-integral equivalence for g-bounded solutions of systems

$$u'(t) = A(t)\, u(t) + F(t, u_t) \tag{1}$$

and

$$v'(t) = A(t)\, v(t). \tag{2}$$

In the present work, we prove the existence of a homeomorphism between the sets of g-bounded solutions of (1) and (2). The asymptotic equivalence problem (1) and (2) has been studied by Hallam [4], Kenneth L. Cooke [2], Morchało [8]. The problem of integral equivalence of an ordinary and a functional differential equations has been studied by Futak [3], Haščak, Švec [5], Haščak [6], Morchało [7].

We remark that the present results extend those of Futak and Kenneth L. Cooke as we prove here the existence of a homeomorphism through the contraction mapping principle. In [3] and [2] the basic tool was Schauder's fixed point theorem.

In equations (1) and (2) u, v and F are n-dimensional vectors and A is an $n \times n$ matrix. We let $|\,.\,|$ denote any norm in n-dimensional space R^n. The letter b denotes a positive number, and C_b is the space of continuous functions mapping $\langle -b, 0\rangle$ into R^n with norm $\|\Phi\| = \sup_{-b \le s \le 0} |\Phi(s)|$. If u is any function on $\langle t_0 - b, \infty)$,

($t_0 \geqq 0$) into R^n, then for each $t \in \langle t_0, \infty)$ the symbol u_t denotes the element of C_b defined by $u_t(s) = u(t+s)$ for $-b \leqq s \leqq 0$. If u is a real valued measurable function on $R_+ = \langle 0, \infty)$, then by the symbol $u \in L_p(R_+)$, $(1 \leqq p < \infty)$ we denote that $\int_0^\infty | u(t) |^p \, dt < \infty$. Let $M_p(1 \leqq p < \infty)$ consist of all functions measurable in $t \in J = \langle t_0, \infty)$ for which

$$| z |_{M,p} = \sup_{t \in J} \Big(\int_t^{t+1} | z(s) |^p \, ds \Big)^{\frac{1}{p}} < \infty.$$

Let $g: \langle t_0 - b, \infty) \to (0, \infty)$ be a continuous function.

Definition 1. We will say a vector function $z: J \to R^n$ is g-bounded on J, if $\sup_{t \in J} | g^{-1}(t) \, z(t) | < \infty$.

Definition 2. We will say that the equations (1) and (2) are g-asymptotically equivalent if for each solution u defined on $\langle t_0 - b, \infty)$ of (1), there exists a solution v defined on J of (2) such that

$$(3) \qquad | u(t) - v(t) | = 0(g(t)) \qquad \text{as } t \to \infty$$

and conversely.

Definition 3. We will say that the equations (1) and (2) are (g, p) integrally equivalent on $J(p \geq 1)$ if for each solution u defined on $\langle t_0 - b, \infty)$ of (1) there exists a solution v defined on J of (2) such that

$$(4) \qquad | g^{-1}(t) \, [u(t) - v(t)] | \in L_p(J)$$

and conversely.

Definition 4. We will say that the equations (1) and (2) are (g, M) integrally equivalent on J, if for each solution u defined on $\langle t_0 - b, \infty)$ of (1) there exists solution v defined on J of (2) such that

$$(5) \qquad | g^{-1}(t) \, [u(t) - v(t)] | \in M_p \qquad \text{for } t \in J$$

and conversely.

Let G_g be the space of all functions z continuous and g bounded on $\langle t_0 - b, \infty)$ such that

$$| z |_g = \sup_{\langle t_0 - b, \infty)} | g^{-1}(t) \, z(t) | < \infty.$$

Let $G_{g,r} = \{z: z \in G_g, | z |_g \leqq r \text{ for all } t \in \langle t_0 - b, \infty), 0 \langle r = \text{const.}\}$.

Let $B_{g,1}$ and $B_{g,2}$ be the sets of g-bounded solutions of (1) and (2) respectively.

It is necessary to impose hypotheses upon the linear equation (2) based on the decomposition of R^n into the direct sum $R^n = X_1 \oplus X_2$, where X_i $(i = 1, 2)$ are determined in the following manner: denote $v(t, t_0, x_0)$ the solution of (2) starting from v_0 at t_0; then $v_0 \in X_1$ if and only if the solution $v(t_0, t_0, v_0)$ is bounded on $\langle t_0, \infty)$; X_2 is the direct complement of X_1. We denote by P_i $(i = 1, 2)$ the corresponding projections i.e. $P_i R^n = X_i$ $(i = 1, 2)$.

First, we assume the following:

H_1. $F(t, \Phi)$: $R_+ \times C_b \to R^n$ satisfies the Carathéodory conditions, i.e. $F(t, \Phi)$ is measurable in t for any fixed $\Phi \in C_b$ and continuous in Φ for any fixed $t \in R_+$, and for every $(t, \Phi_1), (t, \Phi_2) \in R_+ \times C_b$

$$| F(t, \Phi_1) - F(t, \Phi_2) | \leqq L(t) \, \| \Phi_1 - \Phi_2 \|,$$

where L: $R_+ \to R_+$ is continuous.

H_2. Let V be a fundamental matrix for equation (2).

H_3. $A(t)$ is an $n \times n$ matrix locally integrable on R_+.

Theorem 1. *Suppose H_1, H_2 and H_3 hold. Suppose also that:*

(i) *there exists $r, q, K (r, K > 0,\ 1 < q < \infty)$ such that*

$$\sum_{k=0}^{n} \Big(\int_{t_0+k}^{t_0+k+1} | g^{-1}(t) V(t) P_1 V^{-1}(s) |^q \mathrm{d}s \Big)^{1/q} + \\ + \sum_{k=0}^{\infty} \Big(\int_{t+k}^{t+k+1} | g^{-1}(t) V(t) P_2 V^{-1}(s) |^q \mathrm{d}s \Big)^{1/q} \leqq K < \infty,$$

(ii) $\sup\limits_{-b \leqq s \leqq 0} g(t + s) = N g_0(t)$ for $t \in J$, $0 < N = \text{const.}$

(iii) $2KN \sup\limits_{t \in J} \Big(\int_t^{t+1} (L(s)\, g_0(s))^p \, \mathrm{d}s \Big)^{1/p} \leqq \dfrac{1}{2}$, $p + q = pq$. $K \sup\limits_{t \in J} \Big(\int_t^{t+1} | F(s, 0) |^p \, \mathrm{d}s \Big)^{1/p} \leqq$ $\leqq \dfrac{r}{2}$.

Then there exists a one-to-one bicontinuous mapping Q from the set $B_{g,2}$ into the set $B_{g,1}$.

Proof. We first show that Q is well defined. Given $v \in B_{g,2} \cap G_{g,r}$, define the operator $Ru = w$, where

$$(6) \quad w(t) = \begin{cases} w(t_0) & \text{for } t \in \langle t_0 - b, t_0 \rangle, \\ v(t) + \int_{t_0}^{t} V(t) P_1 V^{-1}(s) F(s, u_s) \, \mathrm{d}s - \int_{t}^{\infty} V(t) P_2 V^{-1}(s) F(s, u_s) \, \mathrm{d}s, & t \in J. \end{cases}$$

For $u \in G_{g,2r}$, $w = Ru$ it follows from (6) that

$$| g^{-1}(t) (Ru)(t) | \leqq r + \int_{t_0}^{t} | g^{-1}(t) V(t) P_1 V^{-1}(s) | \, L(s) \, \| u_s \| \, \mathrm{d}s +$$

$$+\int_{t_0}^{t}|g^{-1}(t)V(t)P_1V^{-1}(s)|\,|F(s,0)|\,ds+\int_{t}^{\infty}|g^{-1}(t)V(t)P_2V^{-1}(s)|\,L(s)\,\|u_s\|\,ds+$$
$$+\int_{t}^{\infty}|g^{-1}(t)V(t)P_2V^{-1}(s)|\,|F(s,0)|\,ds\leqq r+2rN\int_{t_0}^{t}|g^{-1}(t)V(t)P_1V^{-1}(s)|\times$$
$$\times\ L(s)\,g_0(s)\,ds+\int_{t_0}^{t}|g^{-1}(t)V(t)P_1V^{-1}(s)|\,|F(s,0)|\,ds+$$
$$+2rN\int_{t}^{\infty}|g^{-1}(t)V(t)P_2V^{-1}(s)|\,L(s)\,g_0(s)\,ds+$$
$$+\int_{t}^{\infty}|g^{-1}(t)V(t)P_2V^{-1}(s)|\,|F(s,0)|\,ds\leqq r+$$
$$+2rN\sum_{k=0}^{n}\Big(\int_{t_0+k}^{t_0+k+1}|g^{-1}(t)V(t)P_1V^{-1}(s)|^q\,ds\Big)^{1/q}\Big(\int_{t_0+k}^{t_0+k+1}(L(s)\,g_0(s))^p\,ds\Big)^{1/p}+$$
$$+\sum_{k=0}^{n}\Big(\int_{t_0+k}^{t_0+k+1}|g^{-1}(t)V(t)P_1V^{-1}(s)|^q\,ds\Big)^{1/q}\Big(\int_{t_0+k}^{t_0+k+1}|F(s,0)|^p\,ds\Big)^{1/p}+$$
$$+2rN\sum_{k=0}^{\infty}\Big(\int_{t+k}^{t+k+1}|g^{-1}(t)V(t)P_2V^{-1}(s)|^q\,ds\Big)^{1/q}\Big(\int_{t+k}^{t+k+1}(L(s)\,g_0(s))^p\,ds\Big)^{1/p}+$$
$$+\sum_{k=0}^{\infty}\Big(\int_{t+k}^{t+k+1}|g^{-1}(t)V(t)P_2V^{-1}(s)|^q\,ds\Big)^{1/q}\Big(\int_{t+k}^{t+k+1}|F(s,0)|^p\,ds\Big)^{1/p}\leqq$$
$$\leqq r+\sup_{t\in J}\Big[2rN\Big(\int_{t}^{t+1}(L(s)\,g_0(s))^p\,ds\Big)^{1/p}+\Big(\int_{t}^{t+1}|F(s,0)|^p\,ds\Big)^{1/p}\Big]\times$$
$$\times\sum_{k=0}^{n}\Big(\int_{t_0+k}^{t_0+k+1}|g^{-1}(t)V(t)P_1V^{-1}(s)|^q\,ds\Big)^{1/q}+$$
$$+\sup_{t\in J}\Big[2rN\Big(\int_{t}^{t+1}(L(s)\,g_0(s))^p\,ds\Big)^{1/p}+\Big(\int_{t}^{t+1}|F(s,0)|^p\,ds\Big)^{1/p}\Big]\times$$
$$\times\sum_{k=0}^{\infty}\Big(\int_{t+k}^{t+k+1}|g^{-1}(t)V(t)P_2V^{-1}(s)|^q\,ds\Big)^{1/q}\leqq$$
$$\leqq r+K\Big\{2rN\sup_{t\in J}\Big(\int_{t}^{t+1}(L(s)\,g_0(s))^p\,ds\Big)^{1/p}+\sup_{t\in J}\Big(\int_{t}^{t+1}|F(s,0)|^p\,ds\Big)^{1/p}\Big\}\leqq 2r,$$

hence R maps $G_{g,2r}$ into itself. Moreover, by H_1 we have

$$|g^{-1}(t)[(Ru^1)(t)-(Ru^2)(t)]|\leqq NK\sup_{t\in J}\Big(\int_{t}^{t+1}(L(s)\,g_0(s))^p\,ds\Big)^{1/p}\,|u^1-u^2|_g$$

and hence R is a contraction in $B_{g,2r}$.

We have a well defined function Q: $Q(v)=u$ where u is a solution of (1).

Suppose $v_i\in B_{g,2}\cap G_{g,r}$ $(i=1,2)$ and $Q(v_1)=Q(v_2)$ i.e.

$$u(t)=\begin{cases}v_i(t)+\int_{t_0}^{t}V(t)P_1V^{-1}(s)F(s,u_s)\,ds-\int_{t}^{\infty}V(t)P_2V^{-1}(s)F(s,u_s)\,ds, & t\in J,\\ u(t_0)\quad\text{for } t\in\langle t_0-b,t_0\rangle.\end{cases}$$

By subtraction we find that $v_1 = v_2$ and that Q is consequently one to one. Finally, Q and Q^{-1} are continuous as is shown by the following inequalities:

$$| g^{-1}(t)\,[Q(v_1) - Q(v_2)]\,| \leqq | g^{-1}(t)\,[v_1(t) - v_2(t)]\,| +$$
$$+ \int_{t_0}^{t} | g^{-1}(t)\,V(t)\,P_1 V^{-1}(s)\,|\,L(s)\,\|\,u_s^1 - u_s^2\,\|\,ds +$$
$$+ \int_{t}^{\infty} | g^{-1}(t)\,V(t)\,P_2 V^{-1}(s)\,|\,L(s)\,\|\,u_s^1 - u_s^2\,\|\,ds \leqq$$
$$\leqq | g^{-1}(t)\,[v_1(t) - v_2(t)]\,| + N \int_{t_0}^{t} | g^{-1}(t)\,V(t)\,P_1 V^{-1}(s)\,|\,L(s)\,g_0(s) \times$$
$$\times \sup_{s \in J} | g^{-1}(s)\,[Q(v_1) - Q(v_2)]\,|\,ds + N \int_{t}^{\infty} | g^{-1}(t)\,V(t)\,P_2 V^{-1}(s)\,|\,L(s)\,g_0(s) \times$$
$$\times \sup | g^{-1}(s)\,[Q(v_1) - Q(v_2)]\,|\,ds.$$

Hence

$$| Q(v_1) - Q(v_2)\,|_g \leqq (1 - NK \sup_{t \in J} (\int_{t}^{t+1} (L(s)\,g_0(s))^p\,ds)^{1/p})^{-1}\,| v_1 - v_2\,|_g$$

and

$$| g^{-1}(t)\,[Q^{-1}(u^1) - Q^{-1}(u^2)]\,| = | g^{-1}(t)\,[v_1 - v_2]\,| \leqq$$
$$\leqq | g^{-1}(t)\,[u^1(t) - u^2(t)]| + \int_{t_0}^{t} | g^{-1}(t)\,V(t)\,P_1 V^{-1}(s)\,|\;\,| F(s, u_s^1) - F(s, u_s^2)\,|\,ds +$$
$$+ \int_{t}^{\infty} | g^{-1}(t)\,V(t)\,P_2 V^{-1}(s)\,|\;\,| F(s, u_s^1) - F(s, u_s^2)\,|\,ds \leqq$$
$$\leqq | g^{-1}(t)\,[u^1(t) - u^2(t)]\,| + NK \sup_{t \in J} (\int_{t}^{t+1} (L(s)\,g_0(s))^p\,ds)^{1/p}\,| u^1 - u^2\,|_g.$$

Hence

$$| Q^{-1}(u^1) - Q^{-1}(u^2)\,|_g \leqq [1 + NK \sup_{t \in J} (\int_{t}^{t+1} (L(s)\,g_0(s))^p ds)^{1/p}\,| u^1 - u^2\,|_g.$$

This completes the proof of the Theorem.

Theorem 2. *Let the assumptions H_1, H_2, H_3 be satisfied. Furthermore, suppose that*·

(i) $$\sup_{t_0 - b \leqq s \leqq 0} | g(t + s\,| = N g_0(t) \qquad \text{for } t \in J,\; 0 < N = \text{const.}$$

(ii) $$\sum_{k=t_0}^{t} (\int_{k}^{k+1} | g^{-1}(t)\,V(t)\,P_1 V^{-1}(s)\,|^{e}\,(L(s)\,g_0(s))^{q}\,ds)^{1/\alpha} \times$$
$$\times (\int_{k}^{k+1} | g^{-1}(t)\,V(t)\,P_1 V^{-1}(s)\,|^{a}\,ds)^{1/\beta} +$$

$$+\sum_{k=t}^{\infty}\Big(\int_k^{k+1}|g^{-1}(t)\,V(t)\,P_2V^{-1}(s)|^{c}(L(s)\,g_0(s))^{q}\,ds\Big)^{1/\alpha}\times$$

$$\times\Big(\int_k^{k+1}|g^{-1}(t)\,V(t)\,P_2V^{-1}(s)|^{a}\,ds\Big)^{1/\beta}+$$

$$+\sum_{k=0}^{n}\Big(\int_k^{k+1}|g^{-1}(t)\,V(t)\,P_1V^{-1}(s)|^{c}\,|F(s,0)|^{q}\,ds\Big)^{1/\alpha}\times$$

$$\times\Big(\int_k^{k+1}|g^{-1}(t)\,V(t)\,P_1V^{-1}(s)|^{a}\,ds\Big)^{1/\beta}+$$

$$+\sum_{k=0}^{\infty}\Big(\int_k^{k+1}|g^{-1}(t)\,V(t)\,P_2V^{-1}(s)|^{c}\,|F(s,0)|^{q}\,ds\Big)^{1/\alpha}\times$$

$$\times\Big(\int_k^{k+1}|g^{-1}(t)\,V(t)\,P_2V^{-1}(s)|^{a}\,ds\Big)^{1/\beta}\leqq K<\infty,$$

where a, c are real numbers such that $a, c\in R_+$, $1\leqq c<a<\infty$,

$$\frac{1}{q}-\left(\frac{c}{a}\right)\frac{1}{p}=1-\frac{1}{a},\qquad 1\leqq q\leqq p<\infty,$$

$$\frac{1}{p}=\frac{1}{\alpha},\qquad \frac{1}{\beta}=\frac{1}{a}-\frac{c}{ap},\qquad \frac{1}{\gamma}=\frac{1}{q}-\frac{1}{p},\qquad \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=1\right),$$

(iii) $K\sup_{t\in J}\Big(\int_t^{t+1}|F(s,0)|^{q}\,ds\Big)^{1/\gamma}\leqq\frac{r}{2},\quad 2NK\sup_{t\in J}\Big(\int_t^{t+1}(L(s)\,g_0(s))^{q}\,ds\Big)^{1/\gamma}\leqq\frac{1}{2}.$

Then there exists a one to one bicontinuous mapping Q from the set $B_{g,2}$ into the set $B_{g,1}$.

Proof. We show that $RG_{g,2r}\subset G_{g,2r}$. From (6) we obtain

$$|g^{-1}(t)\,(Ru)\,(t)|\leqq r+2rN\sum_{k=0}^{n}\int_{t_0+k}^{t_0+k+1}\big[|g^{-1}(t)\,V(t)\,P_1V^{-1}(s)|^{\frac{c}{p}}(L(s)\,g_0(s))^{\frac{q}{p}}\big]\times$$

$$\times|g^{-1}(t)\,V(t)\,P_1V^{-1}(s)|^{a\left(\frac{1}{a}-\frac{c}{ap}\right)}(L(s)\,g_0(s))^{q\left(\frac{1}{q}-\frac{1}{p}\right)}\,ds+$$

$$+\sum_{k=0}^{n}\int_{t_0+k}^{t_0+k+1}|g^{-1}(t)\,V(t)\,P_1V^{-1}(s)|^{\frac{c}{p}}\,|F(s,0)|^{\frac{p}{q}}\,|g^{-1}(t)\,V(t)\,P_1V^{-1}(s)|^{a\left(\frac{1}{a}-\frac{c}{ap}\right)}\times$$

$$\times|F(s,0)|^{q\left(\frac{1}{q}-\frac{1}{p}\right)}\,ds+2rN\sum_{k=0}^{\infty}\int_{t+k}^{t+k+1}|g^{-1}(t)\,V(t)\,P_2V^{-1}(s)|^{\frac{c}{p}}\,(L(s)\,g_0(s))^{\frac{q}{p}}\times$$

$$\times|g^{-1}(t)\,V(t)\,P_2V^{-1}(s)|^{a\left(\frac{1}{a}-\frac{c}{ap}\right)}(L(s)\,g_0(s))^{q\left(\frac{1}{q}-\frac{1}{p}\right)}\,ds+\tag{7}$$

$$+\sum_{k=0}^{\infty}\int_{t+k}^{t+k+1}|g^{-1}(t)\,V(t)\,P_2V^{-1}(s)|^{\frac{c}{p}}\,|F(s,0)|^{\frac{p}{q}}\,|g^{-1}(t)\,V(t)\,P_2V^{-1}(s)|^{a\left(\frac{1}{p}-\frac{1}{ep}\right)}\times$$

$$\times|F(s,0)|^{q\left(\frac{1}{q}-\frac{1}{p}\right)}\,ds.$$

Using Hölder's inequality (see Futak [3]) on (7) (with respect to α, β, γ), we have

$$| g^{-1}(t)\,(Ru)\,(t) | \leqq 2r.$$

Moreover we have

$$| g^{-1}(t)\,[(Ru^1)\,(t) - (Ru^2)\,(t)] | \leqq N \sup_{t \in J} \Big(\int_t^{t+1} (L(s)\, g_0(s))^q\, ds\Big)^{1/\gamma} \times$$
$$\times \Big\{ \sum_{k=0}^{n} \Big(\int_{t_0+k}^{t_0+k+1} | g^{-1}(t)\, V(t)\, P_1 V^{-1}(s) |^c (L(s)\, g_0(s))^q\, ds\Big)^{1/\alpha} \times$$
$$\times \Big(\int_{t_0+k}^{t_0+k+1} | g^{-1}(t)\, V(t)\, P_1 V^{-1}(s) |^a\, ds\Big)^{1/\beta} +$$
$$+ \sum_{k=0}^{\infty} \Big(\int_{t+k}^{t+k+1} | g^{-1}(t)\, V(t)\, P_2 V^{-1}(s) |^c (L(s)\, g_0(s))^q\, ds\Big)^{1/\alpha} \times$$
$$\times \Big(\int_{t+k}^{t+k+1} | g^{-1}(t)\, V(t)\, P_2 V^{-1}(s) |^a\, ds\Big)^{1/\beta} \Big\} | u^1 - u^2 |_g$$

and hence R is a contraction in $B_{g,2r}$. The rest of the proof follows by the similar argument as in the proof of Theorem 1 and hence we omit the details.

Theorem 3. *Under the assumptions of Theorem 1 if in additions*

1^0 $\lim_{t \to \infty} \Big(\int_t^{t+1} (L(s)\, g_0(s))^p\, ds\Big)^{1/p} = 0,$

2^0 $\lim_{t \to \infty} \Big(\int_t^{t+1} | F(s, 0) |^p\, ds\Big)^{1/p} = 0,$

3^0 $\lim_{t \to \infty} | g^{-1}(t)\, V(t)\, P_1 | = 0.$

Then for every $v \in B_{g,2}$

$$\lim_{t \to \infty} | g^{-1}(t)\,[u(t) - v(t)] | = 0,$$

where $u = Qv \in B_{g,1}$.

Proof. According to conditions $1°$, $2°$ for a given $\varepsilon > 0$, we can choose $t_2 > t_0$ such that for $t \geqq t_2$, the following relations hold:

$$2rN\Big(\int_t^{t+1} (L(s)\, g_0(s))^p\, ds\Big)^{1/p} < \frac{\varepsilon}{3k}, \qquad \Big(\int_t^{t+1} | F(s, 0) |^p\, ds\Big)^{1/p} < \frac{\varepsilon}{3k},$$

(r is defined in Theorem 1).

Hence we can choose $t_3 > t_2$, such that for $t \geq t_3$ we have

$$| g^{-1}(t)\, V(t)\, P_1 \,.\int_{t_0}^{t_2} | P_1 V^{-1}(s)\, F(s, 0) | \, ds < \frac{\varepsilon}{3}.$$

So

$$|g^{-1}(t)[u(t)-v(t)]| \leqq \int_{t_0}^{t} |g^{-1}(t)V(t)P_1V^{-1}(s)|\ |F(s,u_s)|\,ds +$$
$$+ \int_{t}^{\infty} |g^{-1}(t)V(t)P_2V^{-1}(s)|\ |F(s,u_s)|\,ds \leqq$$
$$\leqq |g^{-1}(t)V(t)P_1| \int_{t_0}^{t_2} |P_1V^{-1}(s)F(s,u_s)|\,ds +$$
$$+ 2rN\sum_{k=0}^{n}\Big(\int_{t_2+k}^{t_2+k+1} |g^{-1}(t)V(t)P_1V^{-1}(s)|^q\Big)^{1/q}\Big(\int_{t_2+k}^{t_2+k+1}(L(s)g_0(s))^p\,ds\Big)^{1/p} +$$
$$+ \sum_{k=0}^{n}\Big(\int_{t_2+k}^{t_2+k+1} |g^{-1}(t)V(t)P_1V^{-1}(s)|^q\,ds\Big)^{1/q}\Big(\int_{t_2+k}^{t_2+k+1}|F(s,0)|^p\,ds\Big)^{1/p} +$$
$$+ 2rN\sum_{k=0}^{\infty}\Big(\int_{t+k}^{t+k+1} |g^{-1}(t)V(t)P_2V^{-1}(s)|^q\,ds\Big)^{1/q}\Big(\int_{t+k}^{t+k+1}(L(s)g_0(s))^p\,ds\Big)^{1/p} +$$
$$+ \sum_{k=0}^{\infty}\Big(\int_{t+k}^{t+k+1} |g^{-1}(t)V(t)P_2V^{-1}(s)|^q\,ds\Big)^{1/q}\Big(\int_{t+k}^{t+k+1}|F(s\,.\,0)|^p\,ds\Big)^{1/p} \leqq$$
$$\leqq |g^{-1}(t)V(t)P_1| \int_{t_0}^{t_2} |P_1V^{-1}(s)F(s,u_s)|\,ds +$$
$$+ 2rN\sup_{t\geq t_2}\Big(\int_{t}^{t+1}(L(s)g_0(s))^p\,ds\Big)^{1/p} + K\sup_{t\geq t_2}\Big(\int_{t}^{t+1}|F(s,0)|^p\,ds\Big)^{1/p} < \varepsilon.$$

Therefore

$$\lim_{t\to\infty} |g^{-1}(t)[u(t)-v(t)]| = 0.$$

Theorem 4. *Under the assumption of Theorem 2 if in addition*

$$\lim_{t\to\infty}\Big(\int_{t}^{t+1}(L(s)g_0(s))^q\,ds\Big)^{1/\gamma} = 0,$$
$$\lim_{t\to\infty}\Big(\int_{t}^{t+1}|F(s,0)|^q\,ds\Big)^{1/\gamma} = 0,$$
$$\lim_{t\to\infty}|g^{-1}(t)V(t)P_1| = 0.$$

Then for every $v \in B_{g,2}$

$$\lim_{t\to\infty}|g^{-1}(t)[u(t)-v(t)]| = 0,$$

where $u \in B_{g,1}$.

Proof. [see Theorem 2 and 3].

Theorem 5. *Let the following conditions be satisfied:*

1° *The assumptions of Theorem 1 hold.*

2^0 $\int_{t_0}^{\infty} | P_1 V^{-1}(s) | L(s)\, g_0(s)\, \mathrm{d}s < \infty, \int_{t_0}^{\infty} | P_1 V^{-1}(s) | \; | F(s, 0) | \, \mathrm{d}s < \infty.$

3^0 $\int_{t_0}^{\infty} s^{1/p} L(s)\, g_0(s)\, \mathrm{d}s < \infty, \int_{t_0}^{\infty} s^{1/p} | F(s, 0) | \, \mathrm{d}s < \infty, (p \geqq 1).$

4^0 $\int_0^{\infty} \exp(-K^{-q} \int_0^t (g(s))^{-q}\, \mathrm{d}s)\, \mathrm{d}t < \infty.$

Then

$$| g^{-1}(t) [u(t) - v(t)] | \in L_p(\langle t_0, \infty)).$$

Proof. From (6) and 1° of Theorem we have

$$| g^{-1}(t) [u(t) - v(t)] | \leqq 2rN | g^{-1}(t) V(t) P_1 | \int_{t_0}^{t} | P_1 V^{-1}(s) | L(s)\, g_0(s)\, \mathrm{d}s +$$

$$+ | g^{-1}(t) V(t) P_1 | \int_{t_0}^{t} | P_1 V^{-1}(s) | \; | F(s, 0) | \, \mathrm{d}s +$$

$$+ 2rNK(\int_t^{\infty} (L(s)\, g_0(s))^p \, \mathrm{d}s)^{1/p} + K(\int_t^{\infty} | F(s, 0) |^p \, \mathrm{d}s)^{1/p}.$$

Thus from 2°, 3°, 4° of Theorem and Lemma 1 [6], Lemma 3 [7] we get that this terms belongs to $L_p(\langle t_0, \infty))$. The proof of the Theorem is complete.

Theorem 6. *Besides the conditions of Theorem 1 suppose that*

$$\int_{t_0}^{t} (\int_u^{u+1} | g^{-1}(t) V(t) P_1 V^{-1}(s) |^q \, \mathrm{d}s)^{1/q} \mathrm{d}u \leqq K,$$

$$\int_t^{\infty} (\int_u^{u+1} | g^{-1}(t) V(t) P_2 V^{-1}(s) |^q \, \mathrm{d}s)^{1/q} \mathrm{d}u \leqq K \qquad \text{for } t \geqq t_0,$$

(for convenience, all functions are assumed to vanish for all $S < t_0$). Then

$$| g^{-1}(t) [u(t) - v(t)] | \in M_p \qquad \textit{for all } t \in J.$$

Proof. From the estimates (recall that all functions vanich for $t < t_0$)

$$| g^{-1}(t) [u(t) - v(t)] | \leqq 2rN \int_{t_0}^{t} | g^{-1}(t) V(t) P_1 V^{-1}(s) | L(s)\, g_0(s)\, \mathrm{d}s +$$

$$+ \int_{t_0}^{t} | g^{-1}(t) V(t) P_1 V^{-1}(s) | \; | F(s, 0) | \, \mathrm{d}s + 2rN \int_t^{\infty} | g^{-1}(t) V(t) P_2 V^{-1}(s) | \times$$

$$\times L(s)\, g_0(s)\, \mathrm{d}s + \int_t^{\infty} | g^{-1}(t) V(t) P_2 V^{-1}(s) | \; | F(s, 0) | \, \mathrm{d}s \leqq$$

$$= 2rN \int_{t_0}^{t} | g^{-1}(t) V(t) P_1 V^{-1}(s) | L(s)\, g_0(s) \int_{s-1}^{s} \mathrm{d}u\, \mathrm{d}s +$$

$$+\int_{t_0}^{t}|g^{-1}(t)V(t)P_1V^{-1}(s)|\,|F(s,0)|\int_{s-1}^{s}\mathrm{d}u\,\mathrm{d}s+2rN\int_{t}^{\infty}|g^{-1}(t)V(t)P_2V^{-1}(s)|\times$$

$$\times(L(s)g_0(s)\int_{s-1}^{s}\mathrm{d}u\,\mathrm{d}s+\int_{t}^{\infty}|g^{-1}(t)V(t)P_2V^{-1}(s)|\,|F(s,0)|\int_{s-1}^{s}\mathrm{d}u\,\mathrm{d}s\leqq$$

$$\leqq 2rN\int_{t_0-1}^{s}\int_{u}^{u+1}|g^{-1}(t)V(t)P_1V^{-1}(s)|L(s)g_0(s)\,\mathrm{d}s\,\mathrm{d}u+$$

$$+\int_{t_0-1}^{t}\int_{u}^{u+1}|g^{-1}(t)V(t)P_1V^{-1}(s)|L(s)g_0(s)\,\mathrm{d}s\,\mathrm{d}u+$$

$$+2rN\int_{t-1}^{\infty}\int_{u}^{u+1}|g^{-1}(t)V(t)P_2V^{-1}(s)|L(s)g_0(s)\,\mathrm{d}s\,\mathrm{d}u+$$

$$+\int_{t-1}^{\infty}\int_{u}^{u+1}|g^{-1}(t)V(t)P_2V^{-1}(s)|\,|F(s,0)|\,\mathrm{d}s\,\mathrm{d}u\leqq$$

$$\leqq 2rN\int_{t_0}^{t}\Big(\int_{u}^{u+1}|g^{-1}(t)V(t)P_1V^{-1}(s)|^q\,\mathrm{d}s\Big)^{1/q}\Big(\int_{u}^{u+1}(L(s)g_0(s))^p\,\mathrm{d}s\Big)^{1/p}\mathrm{d}u+$$

$$+\int_{t_0}^{t}\Big(\int_{u}^{u+1}|g^{-1}(t)V(t)P_1V^{-1}(s)|^q\,\mathrm{d}s\Big)^{1/q}\Big(\int_{u}^{u+1}|F(s,{}_0)|^p\,\mathrm{d}s\Big)^{1/p}\mathrm{d}u+$$

$$+2rN\int_{t-1}^{\infty}\Big(\int_{u}^{u+1}|g^{-1}(t)V(t)P_2V^{-1}(s)|^q\,\mathrm{d}s\Big)^{1/q}\Big(\int_{u}^{u+1}(L(s)g_0(s))^p\,\mathrm{d}s\Big)^{1/p}\mathrm{d}u+$$

$$+\int_{t-1}^{\infty}\Big(\int_{u}^{u+1}|g^{-1}(t)V(t)P_2V^{-1}(s)|^q\,\mathrm{d}s\Big)^{1/q}\Big(\int_{u}^{u+1}|F(s,0)|^p\,\mathrm{d}s\Big)^{1/p}\mathrm{d}u,$$

we conclude that $|g^{-1}(t)[u(t)-v(t)]|\in M_p$ for $t\in J$.

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