Archivum Mathematicum

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Archivum Mathematicum, Vol. 26 (1990), No. 1, 7--17

Persistent URL: http://dml.cz/dmlcz/107364

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 26, No. 1 (1990), 7–18

THE EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF CERTAIN CLASS OF THE INTEGRO DIFFERENTIAL EQUATIONS

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(Received May 19, 1986)

Abstract. This paper is concerned with the existence and asymptotic behavior of solutions of the integro-differential equations

$$g(x)\,y' = y + \int_{0+}^{x} [\sum_{i+j=1}^{N} u_{ij}(x) \,.\, v_{ij}(s)\, y^i(x) \,.\, y^i(s)]\,ds$$

in a neighbourhood of the singular point $[0^+, 0]$.

Key words. Wazewski's topological method, strict egress point, strict ingress point, retract.

MS Classification. 45 J 05.

1. INTRODUCTION AND BASIC NOTIONS

The asymptotic behaviour of solutions of the ordinary differential equations in a neighbourhood of the singular point were studied by Kiguradze [1], Čečik [2], Diblík [3, 4], Konjuchova [5, 6]. The integro-differential equations have different properties from ordinary differential equations even in the simplest cases (see [7]).

Therefore known qualitative methods of investigation of the ordinary differential equations, e.g. Wazewski's topological method, cannot be applied to integro-differential equations. This paper generalizes the theorem of the existence and asymptotic behavior of solutions of the integro differential equation

$$g(x)\,y' = y + \int_{0+}^{x} [\sum_{i+j=1}^{N} u_{ij}(x)\, v_{ij}(s)\, y^i(x)\, y^j(s)]\,ds \tag{1}$$

in a neighbourhood of the singular point $[0^+, 0]$ – see [8]. This theorem was proved on behalf of the strong assumption of the existence of a region through each point of which only one solution of the equation (1) goes. We can find this region only in the linear case.

Notation

(i) $f(x) = O(g(x))$ for $x \to x_0^+$ *denotes that there exists* $K > 0$ *such that* $\left|\frac{f(x)}{g(x)}\right| < K$ *on some right hand neighbourhood of the point* x_0.

(ii) $f(x) = o(g(x))$ for $x \to x_0^+$ *denotes that* $\lim\limits_{x \to x_0^+} \frac{f(x)}{g(x)} = 0$.

(iii) $f(x) \sim g(x)$ for $x \to x_0^+$ *denotes that* $\lim\limits_{x \to x_0^+} \frac{f(x)}{g(x)} = 1$.

(iv) *Let* $\{\varphi_n(x)\}_1^\infty$ *be a sequence of functions such that* $\varphi_{i+1}(x) = o(\varphi_i(x))$ *for* $x \to x_0^+$, $i > 1$, *then* $f(x) \approx \sum\limits_{i=1}^{n} a_i\varphi_i(x)$ *for* $x \to x_0^+$ *denotes* $\left[f(x) - \sum\limits_{i=1}^{n} a_i \cdot \varphi_i(x)\right] = O(\varphi_{n+1}(x))$ *for* $x \to x_0^+$, $n \in N$, $a_i = \text{const}$.

Definition. Every function $y(x) \in C^1(0, x_0]$ satisfying (1) for each $x \in (0, x_0]$ will be called *a solution of the equation* (1).

If the conditions of existence and uniqueness of solution of the equation (1) are not fulfilled in a point $[x_0, y_0]$ then this point will be called *the singular point of the equation* (1).

2. THE CONSTRUCTION OF A FORMAL SERIES SATISFYING THE EQUATION (1)

We shall seek the solution of (1) in the form of an one parametric series

$$y(x, C) = \sum_{h=1}^{\infty} f_h(x)\, \varphi^h(x, C), \tag{2}$$

where $\varphi(x, C)$ is a general solution of the equation $g(x)\, y' = y$ so that

$$\varphi(x, C) = C \exp\left[\int_{x_0}^{x} \frac{dt}{g(t)}\right],$$

$f_1(x) = 1$, $f_h(x)$, $h \geq 2$ are unknown function, $C \neq 0 = \text{const}$. Denote

$$y_n(x) = \sum_{h=1}^{n} f_h(x)\, \varphi^h(x, C),$$

$$K_h(x, s) \equiv \sum_{i+j=2}^{N} u_{ij}(x)\, v_{ij}(s) \left[\sum_{\substack{\alpha+\beta=h \\ \alpha \geq i,\, \beta \geq j}} \left(\sum_{\sum_{k=1}^{i} w_k = \alpha} \prod_{s=1}^{i} f_{w_s}(x)\, \varphi^{w_s}(x, C) \right) \times \left(\sum_{\sum_{k=1}^{j} v_s = \beta} \prod_{r=1}^{j} f_{v_r}(s)\, \varphi^{v_r}(s, C) \right) \right],$$

where

$$\sum_{\sum_{k=1}^{0} w_k = \alpha} \prod_{s=1}^{0} f_{w_s}(x)\, \varphi^{w_s}(x, C) = \begin{cases} 0 & \text{for } \alpha \neq 0,\ i + j < h, \\ 1 & \text{in the other cases,} \end{cases}$$

$$\sum_{\sum_{k=1}^{0} v_k = \beta} \prod_{r=1}^{0} f_{v_r}(s)\, \varphi^{v_r}(s, C) = \begin{cases} 0 & \text{for } \beta \neq 0,\ i + j < h, \\ 1 & \text{in the other cases,} \end{cases}$$

f_{w_s}, f_{v_r} are functions from (2), $h \geq 2$.

Put

$$\Sigma_h = \Sigma_h(f_1, f_2, \dots, f_{h-1}) \equiv \varphi^{-h}(x, C) \int_{0+}^{x} K_h(x, s)\, \mathrm{d}s, \qquad h \geqq 2,$$

$$T_2 = \Sigma_2(f_1),$$

$$T_h \equiv \frac{1}{h-1} \Sigma_h(f_1, T_2, \dots, T_{h-1}) \qquad \text{for } h \geqq 3.$$

We formally differentiate the series (2) and substitute into (1). Comparing the coefficients of equal powers of $\varphi(x, C)$, we obtain for unknown functions $f_h(x)$ the system of the recurrence equations

$$(3_h) \qquad g(x)\, f'_h = (1 - h)\, f_h + \varphi^{-h}(x, C) \int_{0+}^{x} K_h(x, s)\, \mathrm{d}s, \qquad h \geqq 2.$$

Consider the following assumptions:

A_1, $g(x) \in C^1(0, x_0]$, $g(x) > 0$, $\lim_{x \to 0^+} g(x) = 0$, $x_0 > 0$, $g'(x) \sim \psi_1(x) \cdot g^{\lambda_1}(x)$ *for* $x \to 0^+$, $\lambda_1 > 0$, $\lim_{x \to 0^+} \psi_1(x) \cdot g^{\tau}(x) = 0$, τ *is here and in the sequel any positive number.*

A_2, $T_h \in C^0(0, x_0]$, $T_h = b_{0h}(x) \cdot g^{\lambda_h}(x) + O(b_{1h}(x) \cdot g^{\lambda_h + \varepsilon_h}(x))$, $\varepsilon_h > 0$, $\lim_{x \to 0^+} b_{ih}(x) \cdot g^{\tau}(x) = 0$, $i = 0, 1$, $b_{0h}(x) \in C^1(0, x_0]$, $b_{0h}(x) \neq 0$, $b'_{0h}(x) \sim \psi_{2h}(x) \cdot g^{\lambda_{2h}}(x)$ for $x \to 0^+$, $\lambda_{2h} + 1 > 0$, $\lim_{x \to 0^+} \psi_{2h}(x) \cdot g^{\tau}(x) = 0$, $\lim_{x \to 0^+} [b_{0h}(x)]^{-1} \cdot g^{\tau}(x) = 0$, $h \geq 2$.

A_3, *There exists constants* $v_h \in (\lambda_h, \lambda_h + \min\{\lambda_1, \lambda_{2h} + 1, \varepsilon_h - \Delta^*_{h-1}\})$, *where* $\Delta^*_{h-1} = \max(\Delta_1, \dots, \Delta_{h-1})$, $\Delta_{h-1} = \lambda_{h-1} + \varepsilon_{h-1} - v_{h-1}$, $\Delta_1 = 0$, $h \geq 2$.

A_4, $u_{ij}(x), v_{ij}(x) \in C^0(0, x_0]$, $\lim_{x \to 0^+} u_{ij}(x) \cdot \varphi(x, C) = 0$, $\lim_{x \to 0^+} \int_x^{x_0} | v_{ij}(s) |\, \mathrm{d}s < \infty$.

A_5, $p(x) \in C^0(0, x_0]$, $p(x) = b_0(x) \cdot g^n(x) + O(b_1(x) \cdot g^{n+\varepsilon}(x))$, $\varepsilon > 0$, $\lim_{x\to 0+} b_i(x) \cdot$ $\cdot\, g^{\iota}(x) = 0$, $i = 0, 1$, $b_0(x) \in C^1(0, x_0]$, $b_0(x) \neq 0$, $b_0'(x) \sim \psi_2(x) \cdot g^{\lambda_2}(x)$ *for* $x \to 0^+$, $\lambda_2 + 1 > 0$, $\lim_{x\to 0+} \psi_2(x) \cdot g^{\tau}(x) = 0$, $\lim_{x\to 0+} g^{\tau}(x) \cdot [b_0(x)]_-^1 = 0$.

In the sequel we shall use the results of the papers [4], [8], which we can formulate for our purposes in the following way:

Lemma 2.1. *Suppose that* (A_1), $(A_5])$ *hold. Let* q *be a constant,* $q < 0$. *Then the equation*

$$g(x) \cdot y' = q \cdot y + p(x)$$

has a unique solution on an interval $(0, x_0]$ *satisfying the relations*

$$y(x) = \frac{-1}{q} b_0(x)\, g^{\lambda}(x) + O(g^{\nu}(x)), \qquad y'(x) = O(g^{\nu-1}(x)),$$

on an interval $(0, x_\nu]$, $0 < x_\nu < x_0$, $\nu \in (\lambda, \lambda + \min\{\lambda_1, \lambda_2 + 1, \varepsilon\})$. For the proof see [4].

Theorem 2.1. *Let assumptions* (A_1), (A_2), (A_3) *hold. Then coefficients* $f_h(x)$ *of the series* (2) *possess the asymptotic form*

$$f_h(x) = b_{0h}(x)\, g^{\lambda_h}(x) + O(g^{\nu h}(x)), \qquad f_h'(x) = O(g^{\nu_h - 1}(x)), \tag{4_h}$$

on an interval $(0, x_{\nu_h}]$, $0 < x_{\nu_h} < x_0$.

Moreover, the functions $f_h(x)$ *are uniquely defined as solutions of the recurrence equations* (3_h) *and*

$$f_h(x) = \int_{0+}^{x} \left\{\left[\exp \int_x^s \frac{h-1}{g(t)}\, dt\right] \frac{\varphi^{-h}(s, C)}{g(s)} \int_{0+}^{s} K_h(s, u)\, du\right\} ds \tag{5_h}$$

hold for $x \in (0, x_0]$.

For the proof see [8].

3. THE EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE EQUATION (1)

The technique used for the existence and asymptotic behaviour of solutions of the equation (1) is based on the well-known Schauder's fixed point theorem and Wazewski,s topological method (see [9]).

The Schauder's theorem. *Let E be a Banach space and S its nonempty convex and closed subset. If P is a continuous mapping of S into itself and PS is relatively compact then the mapping P has at least one fixed point.*

Theorem 3.1. *Let assumptions* (A_i), $i = 1, \ldots, 4$ *hold*

Then for each value of a parameter $C \neq 0$ *there exists a solution* $y(x, C)$ *of the equation (1) such that*

$$(6) \qquad | y^{(i)}(x, C) - y_{n-1}^{(i)}(x, C) | \leqq \delta \, | (f_n(x) \, \varphi^n(x, C))^{(i)} | \qquad i = 0, 1$$

on an interval $(0, x_0]$, $\delta > 1$ *a constant,* x_0 *depends on* δ, C, n *and* $f_n(x)$ *has the form of* (5_n).

Proof.

1. We shall define the Banach space E and a subset S of E with required properties.

2. We construct a mapping P of S into itself.

3. We prove the continuity of a mapping P and relative compactness of PS.

1. The concrete Banach space which appears in the following is the space $C^0[0, x_0]$ of all continuous functions on the interval $(0, x_0]$, $x_0 > 0$, with the usual norm

$$h(x) = \max_{x \in [0, x_0]} | h(x) | .$$

A subset S of the Banach space $C^0[0, x_0]$ will be the set of all functions $h(x)$ from $C^0[0, x_0]$ satisfying the inequality

$$(7) \qquad | h(x) - y_{n-1}(x, C) | < \delta \, . \, | f_n(x) \, . \, \varphi^n(x, C) | .$$

The set S is obviously nonempty and, as it is easy to see, is convex and closed.

2. Now we shall construct the mapping P. Let $h_0(x) \in S$ is an arbitrary function. Substituting $h_0(s)$ instead of $y(s)$ into (1) we obtain the differential equation

$$(8) \qquad g(x) \, y' = y + \int_{0+}^{x} [\sum_{i+j=2}^{N} u_{ij}(x) \, v_{ij}(s) \, y^i(x) \, h_0^j(s)] \, ds .$$

Set

$$(9_1) \qquad y(x) = y_{n-1}(x, C) + \varphi^{n-1}(x, C) \, . \, Y_0 ,$$

$$(9_2) \qquad y'(x) = y'_{n-1}(x, C) + \frac{1}{n-1} [\varphi^{n-1}(x, C)]' \, Y_1 ,$$

where new variables satisfy the differential equation

$$(10) \qquad g(x) \, Y'_0 = (1 - n) \, . \, Y_0 + Y_1 .$$

It follows from (7) that

$$(11) \qquad h_0(x) = y_{n-1}(x, C) + H_0(x), \qquad H_0(x) < \delta \, | f_n(x) \, . \, \varphi^n(x, C) | .$$

Substituting (9_1), (9_2), (11) into the equation (8) and by means of (3_h), we get

$$(12) \qquad Y_1 = Y_0 + \varphi(x, C) \, \Sigma_n + \varphi^{1-n}(x, C) \int_{0+}^{x} Q_n(x, s, Y_0(x), H_0(s)) \, ds ,$$

where

(13) $$Q_n = \sum_{I(4)=n+1}^{2n} P_{i(4)}(x)\,\varphi^{i_1}(x, C)\,Y_0^{i_2}(x)\,R_{i(4)}(s)\,\varphi^{i_3}(s, C)\,H_0^{i_4}(s),$$

$I(4) = \sum_{n=1}^{4} i_r$, $i(4) = (i_1, \dots, i_4)$, $P_{i(4)}(x)$ is the polynomial with respect to arguments $u_{ij}(x)$, $f_h(x)$, $R_{i(4)}(s)$ is the polynomial with respect to arguments $v_{ij}(s)$, $f_h(s)$, $h = 1, \dots, n-1$, $i + j = 2, \dots, N$.

Substituting (12) into (10) we obtain

(14) $$g(x)\,Y_0' = (2 - n)\,Y_0 + \varphi(x, C)\,\Sigma_n + \varphi^{1-n}(x, C) \int_{0+}^{x} Q_n(x, s, Y_0(x), H_0(s))\,ds.$$

We shall prove that (14) has at most one solution satisfying conditions $Y_0(0^+) = 0$ and $|Y_0(x)| < \delta \,.\, |f_n(x) \,.\, \varphi(x, C)|$ on an interval $(0, x_0]$.

In view of (9_1), (9_2) it is obvious that solution of (14) determine the solution of (8).

In the sequel we shall use the Wazewski's topological method. Investigate the behavior of integral curves of (14) with respect to the boundary of the set

$$\Omega_0 = \{(x, Y_0) : 0 < x < x_0,\, u_0(x, Y_0) < 0,\, u_0(x, Y_0) = Y_0^2 - [\delta \,.\, f_n(x) \,.\, \varphi(x, C)]^2\}.$$

We calculate the derivative $\dot{u}_0(x, Y_0)$ along the trajectories of (14) on the set

$$U_0 = \{(x, Y_0) : 0 < x < x_0,\, u_0(x, Y_0) = 0\}.$$

We obtain from the definition $u_0(x, Y_0)$ and (14) that

$$\dot{u}_0(x, y_0) = \frac{2}{g(x)}\big[(2 - n)\,Y_0^2 + Y_0\varphi(x, C)\,\Sigma_n + Y_0\varphi^{1-n}(x, C) \times$$
$$\times \int_{0+}^{x} Q_n(x, s, Y_0(x), H_0(s))\,ds - \delta^2 f_n(x)\,\varphi^2(x, C)\,(g(x)\,f_n' + f_n)\big].$$

The relation $g(x) \,.\, f_n' = (1 - n) \,.\, f_n + \Sigma_n$ yields

$$\dot{u}_0(x, y_0) = \frac{2}{g(x)}\big[y_0\varphi(x, C)\,\Sigma_n - \delta^2 f_n(x)\,\varphi^2(x, C)\,\Sigma_n +$$

(15) $$+ Y_0\varphi^{1-n}(x, C) \int_{0+}^{x} Q_n(x, s, y_0(x), H_0(s))\,ds\big].$$

By L'Hospital's rule and from the fact that $g'(x) \sim \psi_1(x) \,.\, g^{\lambda_1}(x)$ for $x \to 0^+$ we obtain that $\lim_{x \to 0+} \dfrac{\varphi^{\tau}(x, C)}{g^{\sigma}(x)} = 0$, σ is arbitrary real number.

The assumptions of the theorem 3.1. and the relation $\lim_{x \to 0+} \dfrac{\varphi^{\tau}(x, C)}{g^{\sigma}(x)} = 0$ imply that the powers of the function $\varphi(x, C)$ influence in decisive way the convergence to zero of single terms of (15). It is obvious that the first two terms in (15) are of the second order with respect to $\varphi(x, C)$. Estimate the integral terms in (15). Using the relation (11) and the definition Y_0, $\varphi(x, C)$, Q_n, we obtain

$$| Y_0 \varphi^{1-n}(x, C) \int_{0+}^{x} Q_n(x, s, Y_0(x), H_0(s)) \, ds | =$$

$$= | Y_0 \varphi^{1-n}(x, C) \int_{0+}^{x} \sum_{I(4)=n+1}^{2n} P_{i(4)}(x) \varphi^{i_1}(x, C) Y_0^{i_2}(x) R_{i(4)}(s) \varphi^{i_3}(s, C) H_0^{i_4}(s) \, ds | \leqq$$

$$\leqq | \delta | f_n(x) | \varphi^{2-n}(x, C) \sum_{I(4)=n+1}^{2n} \varphi^{-(4)}(x, C) P_{i(4)}(x) | \delta f_n(x)^{i_2} \int_{0+}^{x} R_{i(4)}(s) | \delta f_n(s)|^{i_4} \, ds | .$$

The last term is of the third order at least with respect to $\varphi(x, C)$.

Hence, because of $f_n \,.\, \Sigma_n \sim (n-1) \,.\, b_{0n}^2(x) \,.\, g^{2\lambda_n}(x)$ for $x \to 0^+$, we have

$$\operatorname{sgn} \dot{u}_0(x, Y_0) = \operatorname{sgn}(-f_n(x) \,.\, \Sigma_n) = -1 \tag{16}$$

for sufficiently small x_0 depending on C, δ, n.

The relation (16) implies that each point of the set U_0 is a strict ingress point with respect to the equation (14). Change the orientation of the axis x into opposite. Now each point of the set U_0 is a strict egress point with respect to the new system of co-ordinates. Let Ω_{0e} be a set of all strict egress point.

Let $Z(x) \subseteq \Omega_0 \cap \Omega_{0e}$,

$$Z(x) = \{(x, Y_0) : x = x_{00}, 0 < x_{00} < x_0, u_0(x, Y_0) \leqq 0\}.$$

Then $U_0 \cap Z(x)$ is not a retract of $Z(x)$. For, if there exists a retraction $\pi : Z(x) \to U_0 \cap Z(x)$, then there exists a continuous map of $Z(x)$ into itself, $x \to -\pi(x)$, without fixed points. This contradicts the Brouwer's fixed point theorem. But $U_0 \cap Z(x)$ is a retract of U_0, because we can choose a retraction $\pi(x, Y_0) = (x_{00}, Y_0^*)$, where $u_0(x_{00}, Y_0^*) = 0$, $\operatorname{sgn} Y_0 = \operatorname{sgn} Y_0^*$.

By Wazewski's topological method there exists at least one point $(x_{00}, Y_0) \in Z(x) \cap \Omega_0$ such that the solution of (14) remains in Ω_0 on its maximal interval of existence to the left of x_{00}. It is obvious that this assertion remains true for arbitrary function $h_0(x) \in S$.

Now we shall prove a uniqueness of the solution of (14) satisfying the given conditions.

Let $\bar{Y}_0$ be a solution of (14). Put $Z_0 = Y_0 - \bar{Y}_0$.

Substituting into (14) we obtain

$$g(x) Y_0' = (2 - n) Z_0 + \varphi^{1-n}(x, C) \int_{0+}^{x} [Q_n(x, s, Z_0(x) + \bar{Y}_0(x, H_0(s))- \\ - Q_n(x, s, \bar{Y}_0(x), H_0(s))] \, ds. \tag{17}$$

Let Ω_1 be a region $\Omega_1 = \{(x, Z_0): 0 < x < x_0, u_1(x, Z_0) < 0\}$, where $u_1(x, Z_0) = Z_0 - [\delta . f_n(x) . \varphi^{1-\alpha}(x, C)]^2$, $\alpha > 0$ is sufficiently small constant. Investigate the behavior of integral curves of (17) with respect to the boundary of Ω_1.

Using the same method as above, we have

$$\operatorname{sgn} \dot{u}_1(x, Z_0) = -1 \tag{18}$$

for sufficiently small x_0. It is obvious that $\Omega_0 \subseteq \Omega_1$. Let $Z_0(x)$ be any nonzero solution of (17) lying in Ω_0 and $(x_1, Z_0(x_1) \in \Omega_0$ for $0 < x_1 < x_0$.

Let $\bar{\delta}_1 > 0$ be a constant $\bar{\delta}_1 < \delta$ such that $(x_1, Z_0(x_1)) \in \partial\Omega_1(\bar{\delta}_1)$. If the solution $Z_0(x)$ lay in $\Omega_1(\bar{\delta}_1)$ for $0 < x < x_1$, it would have to be valid that $(x_1, Z_0))$ is a strict egress point of $\partial\Omega_1(\bar{\delta}_1)$. This contradicts the relation (18).

Hence, in Ω_0 there is only the trivial solution of (17). The uniqueness is proved. From (9_1) we obtain

$$| y_0(x) - y_{n-1}(x, C) | = | \varphi^{n-1}(x, C) . Y_0 | < \delta . | f_n(x) \varphi^n(x, C) |, \tag{19}$$

where $y_0(x)$ is a solution of (8).

Similarly from (9_2) we have

$$| y_0'(x) - y_{n-1}'(x, C) | = \left| \frac{\varphi^{n-1}(x, C)}{g(x)} Y_1 \right| = \frac{\varphi^{n-1}(x, C)}{g(x)} | \bar{y}_1 |.$$

Since $| Y_1 | < \delta . \varphi(x, C)$, $| f_n(x) + \Sigma_n | = \delta . g(x) . \varphi^{1-n}(x, C) . | (f_n(x) . \varphi^n(x, C))' |$, it follows that

$$| y_0'(x) - y_{n-1}'(x, C) | < \delta . | (f_n(x) . \varphi^n(x, C))' |. \tag{20}$$

We enlarge the solution $y_0(x)$ continuously to the point $x = 0$. Introduce a mapping P as follows:

$$P : h_0(x) \to y_0(x).$$

Evidently that P maps S into itself and $PS \subseteq S$.

3. It remains to prove that PS is relatively compact and P is a continuous mapping.

It is easy to see, from inequalities (19), (20), that PS is the set of uniformly bounded and equicontinuous functions for $x \in [0, x_0]$.

By Ascoli's theorem (see [10]) PS is relatively compact. Let $\{h_k(x)\}^+$ be an arbitrary sequence in S such that

$$\| h_k(x) - h_0(x) \| = \varepsilon_k, \qquad \lim_{k\to\infty} \varepsilon_k = 0, \qquad h_0(x) \in S.$$

It is obvious that the solution $Y_k(x)$ of the equation

$$g(x)\, Y_0' = (2 - n)\, Y_0 + \varphi(x, C)\, \Sigma_n + \varphi^{1-n}(x, C) \int_{0+}^{x} Q_n(x, s, Y_0(x), H_k(s))\, ds, \tag{21}$$

corresponds to the function $h_k(x)$ and $\bar{Y}_k(x) \in \Omega_0$. Similarly, the solution $\bar{Y}_0(x)$ of (14) corresponds to the function $h_0(x)$.

We shall show that $|\bar{Y}_k(x) - \bar{Y}_0(x)| \to 0$ uniformly on $[0, x_0]$. Consider the region

$$\Omega_0^k = \{(x, Y_0)\,;\, 0 < x < x_0, u_0^k(x, Y_0) < 0\},$$

where $u_0^k(x, Y_0) = (Y_0 - \bar{Y}_0)^2 - [\varepsilon_k \cdot f_n(x) \cdot \varphi^{1-\alpha}(x, C)]^2$, $\alpha > 0$ is sufficiently small constant, $k \geq 1$. Evidently, $\Omega_0 \subseteq \Omega_0^k$ for any k and sufficiently small x_0. Investigate the behaviour of integral curves of (21) with respect to the boundary of Ω_0^k.

Using the same method, as above, we obtain for trajectory derivatives

$$\operatorname{sgn} \dot{u}_0^k(x, Y_0) = -1$$

for sufficiently small x_0 and any k.

By Wazewski's topological method there exists at least one solution of (21) lying in Ω_0^k, where, of course, the solution $\bar{Y}_k(x)$ of (21) lies.

Hence, it follows that

$$|\bar{Y}_k(x) - \bar{Y}_0(x)| < \varepsilon_k \cdot |f_n(x) \cdot \varphi^{1-\alpha}(x, C)| \leqq M \cdot \varepsilon_k,$$

$M > 0$ is a constant depending on n, x_0.

From (9_1) we obtain

$$|y_k(x) - y_0(x)| \leqq \varphi^{n-1}(x, C) \cdot |\bar{Y}_k(x) - \bar{Y}_0(x)| \leqq \varepsilon_k, M \cdot \varphi^{n-1}(x, C) \leqq \varepsilon_k \cdot m,$$

on an interval $[0, x_0]$.

This estimate implies that P is continuous. We have thus proved that the mapping P satisfies the assumptions of the Schauder's theorem and hence there exists a function $h(x) \in S$ with $h(x) = P(h(x))$, namely $h(x) = y(x)$.

The proof is complete.

Theorem 3.1. *Let assumptions (A_i), $i = 1, \dots, 4$ are fulfilled. Then the next asymptotic estimates of the solution $y(x, C)$ of (1) hold.*

$$y^{(i)}(x, C) \approx \sum_{k=1}^{n-1} [f_k(x)\, \varphi^k(x, C)]^{(i)} \qquad \text{for } x \to 0^+, i = 0, 1,$$

where $f_k(x)$ is function of (4_k).

Proof. The assertions of the theorems (2.1), (3.1.) remain true, therefore it is sufficient to show that

$$\lim_{x \to 0^+} \frac{[f_h(x)\, \varphi^h(x, C)]^{(i)}}{[f_{h-1}(x)\, \varphi^{h-1}(x, C)]^{(i)}} = 0 \qquad \text{for } i = 0, 1, h = 2, \dots, n-1.$$

Using the theorem 2.1 and the relations $\lim\limits_{x \to 0+} \dfrac{\varphi^{\tau}(x, C)}{g^{\sigma}(x)} = 0$ and $\lim\limits_{x \to 0+} [b_{0h}(x)]^k \,.$ $. \, g^{\tau}(x) = 0$, $k = \pm 1$, $h = 2, \ldots, n - 1$, we get

$$\lim_{x \to 0+} \frac{f_h(x)\, \varphi^h(x, C)}{f_{h-1}(x)\, \varphi^{h-1}(x, C)} = \lim_{x \to 0+} \frac{b_{0h}(x)\, g^{\lambda_h}(x)}{b_{0h-1}(x)\, g^{\lambda_h+1}(x)}\, \varphi(x, C) = 0.$$

From (4_h) we obtain

$$\lim_{x \to 0+} \frac{(f_h(x)\, \varphi^h(x, C))'}{(f_{h-1}(x)\, \varphi^{h-1}(x, C))'} = \lim_{x \to 0+} \frac{g(x)\, f_h' + h f_h(x)}{g(x)\, f_{h-1}' + (h - 1)\, f_h(x)}\, \varphi(x, C) =$$

$$= \lim_{x \to 0+} \frac{h b_{0h}(x)\, g^{\lambda_h}(x)}{(h - 1)\, b_{0_h - 1}(x)\, g^{\lambda_h - 1}(x)}\, \varphi(x, C) = 0.$$

The theorem is proved.

Example. Consider the equation

$$x^2 y' = y + \int_{0+}^{x} y(x)\, s y(s)\, \mathrm{d}s.$$

In this case $g(x) = x^2$, $u_{11}(x) = 1$, $v_{11}(s) = s$, $N = 2$, $u_{ij} = v_{ij} = 0$,

$$\varphi(x, C) = C \exp\left(\frac{1}{x_0} - \frac{1}{x}\right).$$

It is easy to see that assumptions theorems 2.1., 3.1., 3.2. fulfilled, e.g. for $n = 4$.

The recurrence equation (3_h) have the form

$$\varphi : 1 = 1,$$

$$\varphi^2 : x^2 f_2' = -f_2 + \varphi^{-2}(x, C) \int_{0+}^{x} s \varphi(x, C)\, \varphi(s, C)\, \mathrm{d}s,$$

$$\varphi^3 : x^2 f_3' = -2 f_3 + \varphi^{-3}(x, C) \int_{0+}^{x} s[f_2(x)\, \varphi^2(x, C)\, \varphi(s, C) + f_2(s)\, \varphi^3(s, C)\, \varphi(s, C)]\, \mathrm{d}s,$$

$$\varphi^4 : x^2 f_4' = -3 f_4 + \varphi^{-4}(x, C) \int_{0+}^{x} s[f_2(x)\, f_2(s)\, \varphi^2(x, C)\, \varphi^2(s, C) +$$
$$+ f_3(x) \,.\, \varphi^3(x, C) \,.\, \varphi(s, C) + f_3(s) \,.\, \varphi^3(s, C) \,.\, \varphi(x, C)]\, \mathrm{d}s.$$

Hence

$$f_1(x) = 1,$$

$$f_2(x) = x^3 + O(x^{2\nu_2}), \qquad f_2' = O(x^{2\nu_2 - 2}), \qquad \nu_2 \in \left(\frac{3}{2}, 2\right),$$

$$f_3(x) = \frac{3}{4} x^6 + O(x^{2\nu_3}), \qquad f_3' = O(x^{2\nu_3 - 2}), \qquad \nu_3 \in \left(3, \frac{7}{2}\right),$$

$$f_4(x) = \frac{22}{9} x^9 + O(x^{2\nu_4}), \qquad f_4' = O(x^{2\nu_4 - 2}), \qquad \nu_4 \in \left(\frac{9}{2}, 5\right).$$

By theorem 3.2.,

$$y(x, C) \approx \varphi + (x^3 + O(x^{2\nu_2}))\, \varphi^2 + \left(\frac{3}{4} x^6 + O(x^{2\nu_3})\right) \varphi^3.$$

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