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ANTIMORPHISMS OF PARTIALLY ORDERED SETS

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Abstract. In this paper we prove some fixed point theorems for local antimorphisms which need not be either isotone or antitone mappings. We give, in a way necessary and sufficient conditions for the existence of fixed points on partially ordered sets. We also introduce the concepts: inf, sup-antimorphisms, and, in connection with that we also have some additional results. With such an extension, a general fixed point theorem is obtained which includes a recent result of the author, and also contains, as special cases, some results of Abian, Shmuely, Kurepa, Metcalf and Payne, and some others.

Key words. Lattices, order-reversing (antitone) mapping, fixed point posets.

MS Classification. 05 A 15.

1. INTRODUCTION AND COMMENTARY

Let P be a partially ordered set. A function f from P to P is order-preserving (or *isotone* or increasing) if for all $x, y \in P$, $x \leq y$ implies $f(x) \leq f(y)$. If f satisfies the condition that for $x, y \in P$, $x \leq y$ implies $f(y) \leq f(x)$, then f is said to be *antitone* (or decreasing). P has a fixed point under f if $f(x) = x$ for some $x \in P$. P has the fixed point property if it has a fixed point under all order-preserving functions.

An ordered set P is said to be *complete* provided any non-void subset X of P determines its own infimum $\inf X \in P$ and supremum $\sup X \in P$.

Several authors have treated the problem of characterizing posets with the fixed point property: Abian A., Abian and Brown, Davis A., Edelman, Höft H. and Höft M., Kurepa D., Rival, Smithson, Tarski, Tasković, Ward and Wong, among others.

Tarski [12], Abian and Brown [2], and others have studied fixed points of isotone mappings on partially ordered sets. In [1] and [11] fixed points of certain antitone mappings are studied.

In a poset P functions f are considered such that, for any nonempty $A \subset P$

(1) $$f(\sup A) = \inf f(A), \qquad \text{where } f(A) = \{f(a) \mid a \in A\}.$$

A function f in a poset P satisfying (1) is referred to as a *join antimorphism.* One considers also meet antimorphism satisfying, for any nonempty $A \subset P$,

$$(2) \qquad f(\inf A) = \sup f(A), \qquad \text{where } f(A) = \{f(a) \mid a \in A\}.$$

It is easily seen that every function f, defined on a complete poset (lattice) L, satisfying (1) or (2) is also antitone, that is, join or meet antimorphisms are antitone mappings. On the other hand, it is easy to construct an antitone mapping on a complete poset (lattice) which is neither a join antomorphism nor a meet antimorphism. Namely, let L be the lattice on the Figure 1 and $f: L \to L$ defined by

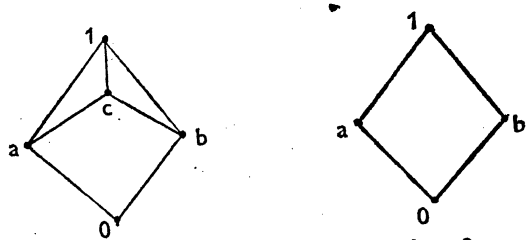

fig. 1 fig. 2

$f(0) = f(b) = f(a) = 1, f(c) = a, f(1) = 0$. Evidently, f is antitone, but $f(\sup\{a, b\}) = = f(c) = a \neq \inf\{f(a), f(b)\} = \inf\{1, 1\} = 1$. The mapping f is not a join antimorphism. If we define $g: \{0, a, b, 1\} \to \{0, a, b, 1\}$, $g(\{a, b, 1\}) = \{0\}$ and $g(0) = 1$, then g is antitone, but not a meet antimorphism (Fig. 2.).

Sufficiency for antimorphisms. Let P be a complete partially ordered set(poset) and $f: P \to P$ an antitone mapping satisfying the conditions: $f(x) \leq x$ or $f^2(x) \leq x$ for all $x \in P$. Then f is a meet antimorphism.

The analogous statement for join antimorphisms is also valid, when $x \leq f(x)$ or $x \leq f^2(x)$ for all $x \in P$ (see [11]).

Proof. Let $A \subset P$ be a nonempty set, $f(x) \leq x (x \in P)$ and $i = \inf A$. Then $f(x) \leq f(i)$ for every $x \in A$. Thus, $f(i)$ is an upper bound for $f(A)$. Let $s = \sup f(A)$, and then $s \leq f(i)$. Assume $s < f(i)$. From $f(x) \leq s$ $(x \in A)$, it follows that $s < < f(i) \leq i$ and hence $s < f(i) \leq f(s)$, i.e., $s < f(s)$-contradiction. That is $f(i) = s$, i.e., $f(\inf A) = \sup f(A)$.

When $f^2(x) \leq x$, $x \in P$ we have $s \leq f(i)$. From $f(x) \leq s$, $x \in A$, it follows that $f(s) \leq f^2(x)$, i.e., $f(s) \leq x$, $x \in A$. We conclude that $f(s)$ is a lower bound for A. Then $f(i) \leq s$, which implies $f(i) = s$, i.e., $f(\inf A) = \sup f(A)$. This completes the proof of sufficiency for antimorphisms.

In this paper we examine fixed points of mappings $f: P \to P$ which are comparable to the identity mapping $i_P: P \to P$, in the sense that for any $x \in P$, $f(x) \leq x$ or $x \leq f(x)$. For any $f: P \to P$ it is natural to consider the following sets

$$P^f := \{x \mid x \in P \wedge x \leq f(x)\}, \qquad P_f := \{x \mid x \in P \wedge f(x) \leq x\}.$$

If $f: P \to P$ is any mapping of P into P, let $I(P, f)$ be the set of all invariant points of P relative to f; i.e., $I(P,f) := \{x \mid x \in P \wedge f(x) = x\}$.

In this paper we prove some fixed point theorems for local antimorphisms which need not be either isotone or antitone mappings. We give, in a way, necessary and sufficient conditions for the existence of fixed points on partialy ordered sets.

Inf, *Sup-antimorphisms.* We also introduce the concepts: inf, sup-antimorphisms, and, in connection with that we also prove a result.

Let P be a poset. The mapping $f: P \to P$ satisfying, for nonempty sets P^f, $P_f \subset P$, the condition

$$f(\inf P^f) = \inf f(P_f), \qquad \text{where } f(P_f) = \{f(x) \mid x \in P_f\},$$

is called an *inf-antimorphism.* Similarly, if f satisfies the condition

$$f(\sup P^f) = \sup f(P_f), \qquad \text{for } \emptyset \neq P^f, P_f \subset P,$$

then such an f is said to be a *sup-antimorphism.*

2. FIXED POINTS OF LOCALLY MEET AND LOCALLY JOIN ANTIMORPHISMS

We start with a statement on join or meet antimorphisms.

Theorem 1. *Let* $(P, \leq)$ *be a partially ordered set and* f *a mapping from* P *into* P *such that:*

(A) *The set* P^f *is nonempty, the point* $s := \sup P^f$ *exists and satisfies* $f(s) \leq s$,

(B) s *is a lower bound (minorant) for the set* $f(P^f)$,

(C) (Locally join antimorphism) $f(\sup P^f) = \inf f(P^f)$.

Then:

(1.1.) *The set* $I(P,f) := I$ *is nonempty,*

(1.2.) *Neither of the conditions* (A), (B), (C) *can be deleted if* (1.1) *is to be valid.*

Dually, if

(A′) *The set* P_f *is nonempty, the point* $I_m := \inf P_f$ *exists and satisfies* $I_m \leqq \leqq f(I_m)$,

(B′) I_m *is an upper bound (majorant) for the set* $f(P_f)$,

(C′) (Locally meet antimorphism) $f(\inf P_f) = \sup f(P_f)$;

then the set $I(P,f)$ *is nonempty and neither of the conditions* (A′), (B′), (C′) *can be deleted if* (1.1.) *is to be valid.*

Proof. By the assumption (A), the set P^f is nonempty, the point $s = \sup P^f$ exists and $f(s) \leq s$. From (B), we have, for all $x \in P^f$ is $s \leq f(x)$, which using (C) implies $s \leq \inf f(P^f) = f(\sup P^f) = f(s)$. Our conclusion follows from (A) and $s \leq f(s)$, that is $f(s) = s$ and thus $s \in I(P,f)$, i.e., the set $I(P,f)$ is nonempty. This completes the proof of (1.1).

(1.2). Now we prove that the conditions (A), (B) and (C) cannot be removed. We show that by the following examples (1, 2, 3).

Example 1. (Fig. 3.) Let P be the set (interval) $[0, 2]$ and define $f: P \to P$ by $f(x) = 2$ for $x \in [0, 1]$ and $f(x) = 1$ for $x \in (1, 2]$, where P is totally ordered by the ordinary ordering $\leq$. Then conditions (B) and (C) are satisfied. Condition (A) is not satisfied ($f(s) = f(1) = 2 \geq 1$). The set I is empty.

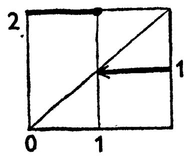

fig 3

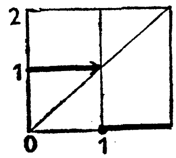

fig. 4

Example 2. (Fig. 4.) Let $P = [0, 2]$ and define $f: [0, 2] \to [0, 2]$ by $f(x) = 1$ for $x \in [0, 1)$ and $f(x) = 0$ for $x \in [1, 2]$, where P is totally ordered by the ordinary ordering $\leq$. Condition (A) is satisfied ($f(\sup P^f) = f(s) = f(1) = 0 \leq 1 = s$), condition (B) is satisfied ($s = 1$ is a minorant for the set $f(P^f) = \{1\}$), but condition (C) is not satisfied ($f(\sup P^f) = f(1) = 0 \neq 1 = \inf f(P^f)$). Again $I = \emptyset$.

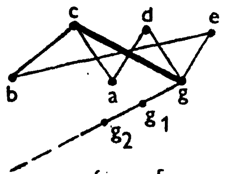

fig 5

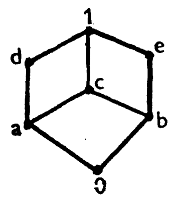

fig. 6

Example 3. (Fig. 5.) Let the poset $P = \{a, b, c, d, e, g, g_n \ (n = 1, 2, 3, \ldots)\}$ be ordered by the order relation $\leq$ so that $a \leq c$, $a \leq d$, $b \leq e$, $b \leq c$, $g \leq e$, $g \leq d$, $g \leq c$, $g_1 \leq g$, $g_{n+1} \leq g_n$ $(n \in \mathbf{N})$, and if the elements a, g, b are incomparable, then the elements c, d, e are also incomparable. Define $f: P \to P$ by $f(a) = d$, $f(b) = e$, $f(d) = f(e) = f(c) = g$, $f(g) = g_1$ and $f(g_n) = g_{n+1}$ $(n = 1, 2, \ldots)$. Condition (A) is satisfied ($P^f = \{a, b\}$, $f(\sup P^f) = f(c) = g \leq c = \sup P^f = \sup \{a, b\} = s$), condition (C) is satisfied ($f(\sup P^f) = f(c) = g = \inf f(P^f) = \inf \{d, e\} = g$), but condition (B) is not satisfied ($s = \sup P^f = c$ is not minorant for the set $f(P^f) = \{d, e\}$). Furthermore, f does not have a fixed point.

By dual considerations one proves the part of the Theorem which concerns the point $I = \inf P_f$. It sufficies to make the following substitutions: $\sup \to \inf$, $P^f \to P_f$, $\leq \to \geq$. This completes the proof of the Theorem.

Some corollaries. Now we shall apply the results above by considering the following consequence. They bring into connection the results (sufficient conditions) which were obtained in the case when the set $I(P,f)$ is nonempty.

Corollary 1. (Kurepa [6]) *Let P be a nonempty right conditionally complete set and f a decreasing selfmapping of P such that for at least one member $x \in P$ we have*

$$x \leq f(x) \quad \text{or} \quad f(x) \leq x, \quad \text{i.e.,} \quad \neg\, (\forall\, x \in P,\ x \,\|\, f(x)).$$

Let us assume that

1. $f(\sup A) = \inf f(A)$,
2. *Each point of P_f is comparable with each point of P^f,*
3. *If $s := \sup P^f \in P$ exists then $f(s) \leq s$.*

Then the set $I(P,f)$ is nonempty and $f(s) = s = \inf P_f$.

Proof. Let us prove that the sufficient conditions of this statement implies the validity of the sufficient conditions (A), (B) and (C). First, direct condition 1) implies (C), because 1) is valid for each $A \subset P$, also $A = P^f$. Otherwise, as the set P^f is bounded (from condition 2)) then according to conditional completeness; has the supremum denoted by $s := \sup P^f$, and from 3) we have $f(s) \leqq s$, i.e., our condition (A) is valid.

We prove that the condition (B) is valid, i.e. that s is a minorant for the set $f(P^f)$. From 2) also each point of P^f is comparable to each point of P_f. So, the sets P_f and P^f have minorants and majorants respectively. But, the set P is conditionally complete and so these sets of minorants and majorants have a supremum and infimum denoted by s and i. Also, from the conditions of the Corollary $f: P \to P$ is an antitone mapping, so $f(P^f) \subset P_f$, and as s is a minorant for P_f, s will be a minorant for $f(P^f)$, i.e., the condition (B) is valid. It means that Corollary 1 is the consequence of our Theorem 1.

Corollary 2. (Tasković [14]) *Let $(P, \leq)$ be a partially ordered set and f a mapping from P into P such that* (A), (C) *and*

(a) $x, y \in P^f \wedge x \leq y \Rightarrow f(y) \leq f(x)$,
(b) *P^f is a totally ordered set.*

Then the set $I(P,f)$ is nonempty.

Proof. Since $f: P \to P$ is a decreasing mapping on P^f (from (a)) and the condition (b) is valid, condition (B) is satisfied. This, with (A) and (C) proves Corollary 2.

In the following $(P, \leq)$ will denote a nonempty partially ordered set P with partial order $\leq$. A subset A of P is a *toset* (chain) just in case A is totally ordered. For $x \in P$ and $A \subset P$, define $L(x) = \{y : y \in P, y \leq x\}$, $M(x) = \{y : y \in P, x \leq y\}$, and $M(A) = \cup\{M(x) : x \in A\}$.

A partially ordered set $(P, \leq)$ is a *mod* if and only if the following hold:

(1) For all $x, y \in P$ sup $\{x, y\}$ exists,

(2) For all $x \in P$, $L(x)$ is a toset,

(3) Each nonempty subset of P which is bounded above (below) has a supremum (infimum) in P,

(4) If $x < y$, then there is a $z \in P$ such that $x < z < y$.

A function $f: P \to P$ is nonoscillatory from above if and only if for each non-maximal x and maximal toset $A \subset M(x)\backslash\{x\}$, $\cap\{f[x, u]) : u \in A\} = \{f(x)\}$. The function f is nonoscillatory from below if and only if for each nonminimal x, $\cap\{f([u, x]) : u < x\} = \{f(x)\}$.

Corollary 3. (Metcalf and Payne [7]) *Let P be a totally ordered mod. Suppose that $f: P \to P$ is a function satisfying:*

(5) *If $x \leq y$ and $f(y) \leq f(x)$, then $[f(y), f(x)] \subset f([x, y])$.*

(6) *The function f is either nonoscillatory from above or from below.*

(7) *There exists $a, b \in P$ such that $a \leq b$, $a \leq f(a)$, and $f(b) \leq b$.*

Then f has a fixed point.

Proof. First, let us prove that our condition (A) of Theorem 1 is satisfied, i.e., that $f(s) \leq s$. The whole situation we observe on the interval $[a, b]$, according to condition (3) of Corollary 3. So, let us adapt pur signs for P^f and P_f for this situation, and let the corresponding set P^f be denoted by

$$A^f := \{x : x \in [a, b] \quad \text{and} \quad t \leq f(t), \quad \text{for all } t \in [a, x]\}.$$

By the assumptions of the Corollary, the set A^f is nonempty, and the point $s := \sup A^f$ exists. It will first be shown that $f(s) \leq s$. Suppose, to the contrary, that $s < f(s)$, and let

$$A_f := \{x : s \leq x \leq f(s) \text{ and } f(x) < x\},$$

so that $s = \sup A^f = \inf A_f$. Then, for $x \in A_f$, $f(x) < x \leq f(s)$, so that condition (5) of Corollary 3 yields

$$[x, f(s)] \subset [f(x), f(s)] \subset f([s, x]), \qquad \text{for all } x \in A_f.$$

For $x \in A_f$ the sets $[x, f(s)]$ are increasing as x is decreasing, while the sets $f([s, x])$ are decreasing as x is decreasing. Thus,

$$(s, f(s)] = \bigcup_{x \in A_f} [x, f(s)] \subset \bigcap_{x \in A_f} f([s, x]);$$

however, the intersection on the right hand side has at most one element, since f is nonoscillatory from the right, which contradicts $s < f(s)$. Thus, $f(s) \leq s$, i.e., the condition (A) is satisfied. On the other hand, in an analogous proof of Corollary 1 we prove that the conditions (B) and (C) are satisfied. This proves Corollary 3.

We next demonstrate that the following condition introduced by Abian [1] is a form of continuity.

Abian's conditions. Let $f: P \to P$ where P is a mod. If $A \subset P$ is a toset, then $f(\inf A) = \sup f(A)$ and $f(\sup A) = \inf f(A)$ whenever both sides of the equalities exist.

Corollary 4. (Abian [1]) *Let $f: P \to P$ where P is a totally ordered mod. If f is decreasing and satisfied Abian's condition, then f has a fixed point.*

Proof. Since Abian's conditions imply our condition (C), because $f(\sup A) = \inf f(A)$ for all $A \subset P$, we also have $A := P^f$. In the other hand, the set P is a totally ordered set and mod, thus the point $s := \sup P^f$ exists; and as s is a minorant for $f(P^f)$, because s is a minorant also for the set P_f. Also, from Abian's conditions we have $f(s) = f(\inf P_f) = \sup f(P_f) \leq \sup P^f =: s$, i.e., $f(s) \leq s$. It means that Abian's statement is the consequence of our Theorem 1.

Corollary 5. (Tasković [13]) *Let P be a totally ordered conditionally complete set and $f: P \to P$ antitone mapping satisfiing the conditions* (4) *and* (5). *Then f has a fixed point.*

Proof. Evidently, the proof of this statement is analogous to the proof of the preceding statement of Abian.

Corollary 6. (Shmuely [11]) *Let L be a complete atomic lattice and let $f: L \to L$ be an antitone mapping satisfying the conditions:*

(a) $x \leq f^2(x)$ *for every* $x \in L$,
(b) $a \leq f(a)$ *for each atom* $a \in L$.

Then f has a fixed point.

Proof. Since $f: P \to P$ is an antitone mapping and $x \leq f(x)$ for every $x \in L$, we have from sufficiency for antimorphism, that our condition (C) is satisfied. In this paper $P^f(A)$ denotes the family of all subsets A of L satisfying $\sup A \leq f(\sup A)$. Notice that $\{0\} \in P^f(A)$ and $P^f(A)$ is ordered by set inclusion. Here we use the following statement of Shmuely [11]:

Lemma (Shmuely [11]) *Under the assumption of Corollary* 6, *$P^f(A)$ ordered by inclusion, has a maximal element.*

Now, let $A_0 \subset L$ be a maximal element of $P^f(A)$ and put $s := \sup A_0 (= \sup P^f(A))$. Obviously, $s \leq f(s)$. Assuming $s < f(s)$ we can find an atom $r \in L$ and $r \notin A_0$, such that $r \leq f(s)$. Also, $s \leq f(r)$, because f is antitone and (a) is valid. This together with $r \leq f(r)$ yields, from (C),

$$f(\sup \{r, s\}) = \inf \{f(r), f(s)\} \geqslant \sup \{r, s\},$$

contradicting the maximality of A_0. Thus $f(s) = s$, i.e., obviously $f(s) \leq s$, and s is minorant for the set $f(P^f)$, because $s = f(s) = f(\sup P^f(A)) = \inf f(P^f(A))$. It means, the conditions (A) and (B) are satisfied and thus Shmuely's statement is the consequence of Theorem 1.

3. FIXED POINTS OF INF, SUP-ANTIMORPHISMS

Theorem 2. *Let $(P, \leq)$ be a partially ordered set and f a mapping from P into P such that:*

D) *The set P^f is nonempty, the point $s := \sup P^f$ exists and satisfies $s \leq f(s)$,*

(E) *s is an upper bound (majorant) for the set $f(P_f)$,*

(F) (Sup-antimorphism) $f(\sup P^f) = \sup f(P_f)$.

Then: (2.1) *The set $I(P,f)$ is nonempty,* (2.2) *Neither of the conditions* (D), (E), (F) *can be deleted if* (2.1) *is to be valid,*

Dually, if

(D′) *The infimum of the set P_f defined by $I_m = \inf P_f$* exists and $f(I_m) \leqq I_m$,

(E′) *I_m is a lower bound (minorant) for the set $f(P^f)$,*

(F′) (Inf-antimorphism) $f(\inf P_f) = \inf f(P^f)$,

then set $I(P,f)$ is nonempty and neither of the conditions (D′), (E′), (F′) *can be deleted if* (2.1) *is to be valid.*

Proof. The set P^f being, by assumption, nonempty, the point $s = \sup P^f$ exists, and from (D), $s \leq f(s)$. From (E), it follows that s is an upper bound for the set $f(P_f)$ and thus we have $x \in P_f, f(x) \leq s$, i.e., $\sup f(P_f) \leq s$, which implies $f(s) = f(\sup P^f) = \sup f(P_f) \leq s$. Our conclusion follows from (D) and $f(s) \leq s$, that is $f(s) = s$ and thus $s \in I(P,f)$, i.e., the set $I(P,f)$ is nonempty. This completes the proof of (2.1).

(2.2). Now we prove that the conditions (D), (E) and (F) not be removed. We show that by the following examples.

Example 4. (Figure 6) Let P be the lattice (poset) on the Figure 6 and let $f: P \to P$ be defined by $f(a) = f(b) = f(c) = 1$, $f(d) = f(1) = a$, $f(0) = a$, $f(e) = 0$. Conditions (D) and (E) are satisfied ($P^f = \{0, a, b, c\}$, $s = \sup P^f = c \leq f(s) = f(c) = 1$, $s = c$ is majorant for the set $f(P_f) = f(\{d, e, 1\}) = \{f(d), f(e), f(1)\} = \{0, a\}$) but condition (F) is not satisfied ($f(\sup P^f) = f(c) = 1 \neq a = \sup f(P_f)$). Furthemore, f does not have a fixed point.

Example 5. Also, neither of the conditions (D) and (E) can be deleted if (2.1) is to be valid, which is illustrated by the following examples for $P = [0, 2]$ and $f: P \to P$, defined geometrically by

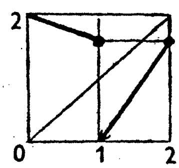

fig. 7

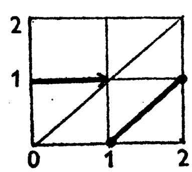

fig. 8

In Figure 7., conditions (D) and (F) are satisfied but condition (E) is not. Further, in Figure 8., conditions (E) and (F) are fulfilled, but condition (D) is not, and f (Fig. 7. and 8.) has not fixed point.

(2′) By dual considerations one proves the part of Theorem 2. ((2.1), (2.2)) which concerns the point $I_m = \inf P_f$; it sufficies to make the following substitutions: $s \to I_m$, $P^f \to P_f$, $\sup \to \inf$, $\leq \to \geqslant$.

REFERENCES

[1] A. Abian, *A fixed point theorem for nonincreasing mappings*, Boll. Un. Mat. Ital. 2 (1969), 200–201.

[2] S. Abian, A. Brown, *A theorem on partially ordered sets with applications to fixed point theorem*, Canad. J. Math. 13 (1961), 78–82.

[3] A. Davis, *A characterization of complete lattice*, Pacific J. Math. 5 (1955), 311–319.

[4] P. H. Edelman, *On a fixed point theorem for partially ordered set*, Discrete Math. 15 (1979), 117–119.

[5] Dj. Kurepa, *Fixpoints of monotone mapping of oredered sets*, Glasnik Mat. fiz. astr. 19 (1964), 167–173.

[6] Dj. Kurepa, *Fixpoints of decreasing mapping of ordered sets*, Publ. Inst. Math. Beograd (N. S.) 18 (32) (1975), 111–116.

[7] F. Metcalf, T. H. Payne, *On the existence of fixed points in a totally ordered set*, Proc. Amer. Math. Soc. 31 (1972), 441–444.

[8] H. and M. Höft, *Some fixed point theorems for partially ordered sets*, Canad. J. Math. 28 (1976), 992–997.

[9] I. Rival, *A fixed point theorem for finite partially ordered sets*, J. Combin. Theory Ser. A 21 (1976), 309–318.

[10] R. Smithson, *Fixed points in partially ordered sets*, Pacific J. Math. 45 (1973), 363–367.

[11] Z. Shmuely, *Fixed points of antitone mappings*, Proc. Amer. Math. Soc. 52 (1975), 503–505.

[12] A. Tarski, *A lattice theoretical fixpoint theorem and its applications*, Pacific J. Math. 5 (1955), 285–309.

[13] M. Taskovič, *Banach's mappings of fixed points on spaces and ordered sets*, Thesis, Math. Balcanica 9 (1979), p. 130.

[14] M. Taskovič, *Partially ordered sets and some fixed point theorems*, Publ. Inst. Math. Beograd (N. S.) 27 (41) (1980), 241–247.

[15] L. E. Ward, *Completeness in semilattices*, Canad. J. Math. 9 (1957), 578–582.

[16] W. S. Wong, *Common fixed points of commuting monotone mappings*, Canad. J. Math. 19 (1967), 617–620.

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