Horst Herrlich

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REALIZATIONS OF TOPOLOGIES AND CLOSURE OPERATORS BY SET SYSTEMS AND BY NEIGHBOURHOODS

HORST HERRLICH

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Dedicated to the memory of my friend Milan Sekanina

Abstract. Milan Sekanina and his collaborators have investigated the realizability of topologies and of closure operators by set systems. In particular they have shown that Top has precisely two [8] and Clos has no [3, 7, 2] realization by set systems. Moreover Top and Clos have precisely one realization by Conv [10]. In this paper it is shown that Top has a large (even illegitimate) collection of realizations by neighbourhoods, but Clos has only one. Moreover Clos has precisely two realizations by uniform neighbourhoods.

Key words: realizations of constructs, topological space, closure space, (uniform) neighbourhood space.

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TERMINOLOGY

Constructs are pairs (A, U) consisting of a category A and a faithful functor U: A $\to$ Set [1]. A realization of a construct (A, U) by a construct (B, V) is a full embedding E: A $\to$ B with $U = V \circ E$ [6].

Top is the construct of topological spaces and continuous maps.

Clos is the construct of closure spaces (sets with a closure operation satisfying Kuratowski's axioms except possibly the idempotency axiom) and continuous (= closure-preserving) maps.

Neigh has as objects all *neighbourhood spaces*, i.e. pairs (X, N) where N: $X \to \mathscr{P}\mathscr{P}X$ is a map, associating with any $x \in X$ a collection $N(x)$ of subsets U of X with $x \in U$; and has as morphisms f: $(X, N) \to (X', N')$ all maps f: $X \to X'$ such that $x \in X$ and $U \in N'(f(x))$ imply $f^{-1}[U] \in N(x)$.

UNeigh has as objects all *uniform neighbourhood spaces*, i.e., pairs $(X, <)$, where $<$ is a binary relation on $\mathscr{P}X$ satisfying the
conditions (1) $A < B \to A \subset B$
and (2) $A \subset B < C \subset D \to A < D$,

and has as morphisms $f\colon (X, <) \to (X', <')$ all maps $f\colon X \to X'$ such that $A <' B$ implies $f^{-1}[A] < f^{-1}[B]$.

SSet has as objects all pairs $(X, \mathscr{S})$ with $S \subset \mathscr{P}X$ and as morphisms $f\colon (X, \mathscr{S}) \to$ $\to (X', \mathscr{S}')$ all maps $f\colon X \to X'$ such that $A \in \mathscr{S}'$ implies $f^{-1}[A] \in \mathscr{S}$.

RESULTS

Proposition 1 [8]. Top *has precisely two realizations by* SSet.

Proposition 2 [3, 7, 2]. Clos *has no realization by* SSet.

Proof: Assume that E: Clos $\to$ SSet is a realization.

Notation: $E(X, \text{cl}) = (X, \mathscr{S}(\text{cl}))$. Then $\bar{E}$: Clos $\to$ SSet, defined by $\bar{E}(X, \text{cl}) =$ $= (X, \mathscr{S}(\text{cl}) \cup \{\varnothing, X\})$, is a realization too. On a 3-element set X there are precisely $4^3 = 64$ closure structures and precisely $2^{(2^3-2)} = 64$ subsets $\mathscr{S}$ of $\mathscr{P}X$ with $\{\varnothing, X\} \subset \mathscr{S}$. Hence $\bar{E}$ induces an order-isomorphism between the ordered sets F_1 of all closure structures on X and F_2 of all subsets $\mathscr{S}$ of $\mathscr{P}X$ with $\{\varnothing, X\} \subset \mathscr{S}$. Since F_1 has precisely 3 atoms and F_2 has 6, this cannot be.

Proposition 3. Top *has a proper class (even an illegitimate collection) of realizations by* Neigh.

Proof: Let C be a strongly rigid proper class of Hausdorff spaces with more than one point. (Such a class exists by [5, 4]; cf. also [11]). For every subclass Γ of C define a realization E_Γ: Top $\to$ Neigh by $E_\Gamma(X, \mathcal{O}) = (X, N_\Gamma(\mathcal{O}))$ where $U \in$ $\in N_\Gamma(\mathcal{O})(x)$ provided U is an open neighbourhood of x in $(X, \mathcal{O})$ or there exists $(X', \mathcal{O}')$ in Γ, a continuous map $f\colon (X, \mathcal{O}) \to (X', \mathcal{O}')$, and a neighbourhood V of $f(x)$ in $(X', \mathcal{O}')$ with $U = f^{-1}[V]$.

The realizations E_Γ are pairwise different, since, if $(X,\mathcal{O})$ belongs to $\Gamma \setminus \Gamma'$, then for any $x \in X$, $N_\Gamma(\mathcal{O})(x)$ consists of all neighbourhoods of x in $(X, \mathcal{O})$ and $N_{\Gamma'}(\mathcal{O})(x)$ consists of all open neighbourhoods of x in $(X, \mathcal{O})$.

Proposition 4. Clos *has precisely one realization by* Neigh.

Proof: For every closure space (X, cl) define a map $N_{\text{cl}}\colon X \to \mathscr{P}\mathscr{P}X$ by $N_{\text{cl}}(x) = \{U \subset X \mid x \notin \text{cl}(X \setminus U)\}$. Then E: Clos $\to$ Neigh, defined by $E(X, \text{cl}) =$ $= (X, N_{\text{cl}})$ is a realization.

For uniqueness, consider an arbitrary realization $\tilde{E}$: Clos $\to$ Neigh.

Notation: $\tilde{E}(X, \text{cl}) = (X, \tilde{N}_{\text{cl}})$. Let (X, cl) be a closure space. Then the following hold:

(a) $\tilde{N}_{\text{cl}}(x) \neq \emptyset$ *for every* $x \in X$.

Proof: Assume $\tilde{N}_{\text{cl}}(x_0) = \emptyset$ for some $x_0 \in X$. Let (X', cl') be an arbitrary closure space, let x be an arbitrary element of X', and let $f\colon X \to X'$ be the constant map with value x. Then continuity of f: $(X, \mathrm{cl}) \to (X', \mathrm{cl}')$ implies $\tilde{N}_{\mathrm{cl}'}(x) = \emptyset$. This in turn impliest that every map between closure spaces is a morphism. Contradiction.

(b) $X \in \tilde{N}(x)$ *for every* $x \in X$.

Proof: This follows from (a), since every constant map between closure spaces is continuous

(c) $X = \{1, 2\}$:
(c1) *if* $\mathrm{cl}\{1\} = \mathrm{cl}\{2\} = X$, *then* $\tilde{N}_{\mathrm{cl}}(1) = \tilde{N}_{\mathrm{cl}}(2) = \{X\}$,
(c2) *if* $\mathrm{cl}\{1\} = \{1\}$ *and* $\mathrm{cl}\{2\} = \{2\}$, *then* $\tilde{N}_{\mathrm{cl}}(1) = \{\{1\}, X\}$ *and* $\tilde{N}_{\mathrm{cl}}(2) = \{\{2\}, X\}$,
(c3) *if* $\mathrm{cl}\{1\} = X$ *and* $\mathrm{cl}\{2\} = \{2\}$, *then one of the following two cases holds:*
Case A: $\tilde{N}_{\mathrm{cl}}(1) = \{\{1\}, X\}$ *and* $\tilde{N}_{\mathrm{cl}}(2) = \{X\}$,
Case B: $\tilde{N}_{\mathrm{cl}}(1) = \{X\}$ *and* $\tilde{N}_{\mathrm{cl}}(2) = \{\{2\}, X)$.

Proof: follows immediately from the fact that, there are only 4 neighbourhood structures on $\{1, 2\}$, which satisfy (b).

(d) $X = \{1, 2, 3\}$: *if* $\mathrm{cl}\{1\} = \mathrm{cl}\{2\} = X$ *and* $\mathrm{cl}\{3\} = \{2, 3\}$, *then one of the following two cases holds:*
Case A: $\tilde{N}_{\mathrm{cl}}(1) = \{\{1, 2\}, X\}$ *and* $\tilde{N}_{\mathrm{cl}}(2) = \tilde{N}_{\mathrm{cl}}(3) = \{X\}$,
Case B: $\tilde{N}_{\mathrm{cl}}(1) = \tilde{N}_{\mathrm{cl}}(2) = \{X\}$ *and* $\tilde{N}_{\mathrm{cl}}(3) = \{X, \{2, 3\}\}$.

Proof: Let (X', cl') be the indiscrete closure space with underlying set $X' = \{1, 2\}$. Then the maps f: $(X', \mathrm{cl}') \to (X, \mathrm{cl})$, defined by $f(x) = x$, and g: $(X', \mathrm{cl}') \to (X, \mathrm{cl})$, defined by $g(x) = x + 1$, are continuous. Hence, by (c1), we obtain:

$$\begin{array}{ll} \text{if } U \in \tilde{N}_{\mathrm{cl}}(1), & \text{then } 2 \in U, \\ \text{if } U \in \tilde{N}_{\mathrm{cl}}(2), & \text{then } 1 \in U, \\ \text{if } U \in \tilde{N}_{\mathrm{cl}}(2), & \text{then } 3 \in U, \\ \text{if } U \in \tilde{N}_{\mathrm{cl}}(3), & \text{then } 2 \in U. \end{array}$$

Next, let $(\overline{X}, \overline{\mathrm{cl}})$ be the closure space, defined by $\overline{X} = \{1, 3\}$, $\overline{\mathrm{cl}}\{1\} = \overline{X}$ and $\overline{\mathrm{cl}}\{3\} = \{3\}$. Then the map h: $(\overline{X}, \overline{\mathrm{cl}}) \to (X, \mathrm{cl})$, defined by $h(x) = x$, is continuous. Hence, by (c3), one of the following cases must hold:
Case A: $U \in \tilde{N}_{\mathrm{cl}}(3) \to 1 \in U$,
Case B: $U \in \tilde{N}_{\mathrm{cl}}(1) \to 3 \in U$.

Since (X, cl) is not indiscrete, $\tilde{N}_{\mathrm{cl}}(1) = \tilde{N}_{\mathrm{cl}}(2) = \tilde{N}_{\mathrm{cl}}(3) = \{X\}$ cannot hold. This implies (d).

(e) Case B *cannot hold.*

Proof: Assume that case B holds. Let (X, cl) be as in (d), let (X', cl') be an arbitrary closure space, let x be an element of X', let U be a subset of X' with $x \in U$, and let f: $X' \to X$ be defined by

$$f(y) = \begin{cases} 3, & \text{if } y = x, \\ 2, & \text{if } y \in U \setminus \{x\}, \\ 1, & \text{if } y \in X' \setminus U. \end{cases}$$

Then the following conditions are equivalent:

(1) $U \in \tilde{N}_{cl'}(x)$,

(2) $f\colon (X', \tilde{N}_{cl'}) \to (X, \tilde{N}_{cl})$ is a morphism in Neigh,

(3) $f\colon (X', cl') \to (X, cl)$ is continuous,

(4) $cl'\{x\} \subset U$.

Hence in particular, if (X', cl') is a topological T_1-space, then $\tilde{N}_{cl'}(x) = \{U \subset X \mid x \in U\}$ for every $x \in X'$. Since there exist different T_1-topologies on an infinite set, $\tilde{E}$ is not injective on objects. Contradiction.

(f) $\tilde{E} = E$.

Proof: In view of (e), Case A must hold. Again, let (X, cl) be as in (d) let (X', cl') be an arbitrary closure space, let x be an element of X', let U be a subset of X' with $x \in U$, and let $f\colon X' \to Y$ be defined by

$$f(y) = \begin{cases} 1, & \text{if } y = x, \\ 2, & \text{if } y \in U \setminus \{x\}, \\ 3, & \text{if } y \in X \setminus U. \end{cases}$$

Then the following conditions are equivalent:

(1) $U \in \tilde{N}_{cl'}(x)$,

(2) $f\colon (X', \tilde{N}_{cl'}) \to (X, \tilde{N}_{cl})$ is a morphism in Neigh,

(3) $f\colon (X', cl') \to (X, cl)$ is continuous,

(4) $x \notin cl'(X \setminus U)$.

Thus $\tilde{N}_{cl} = N_{cl}$, i.e., $\tilde{E} = E$.

Proposition 5. Clos *has precisely two realizations by* UNeigh.

Proof. As in the proof of Proposition 4, two cases arise. Case A leads to the realization E_1: Clos $\to$ UNeigh, defined by $E_1(X, cl) = (X, <_1 (cl))$, where $A <_1 (cl)\, B$ iff $A \cap cl\,(X \setminus B) = \emptyset$, i.e., iff B is a neighbourhood of A in the familiar sense. Case B does not lead to a contradiction but to the realization E_2: Clos $\to$ UNeigh, defined by $E_2(X, cl) = (X, <_2 (cl)'$ where $A <_2 (cl)\, B$ iff $(X \setminus B) \cap cl\, A = \emptyset$, i.e., iff $X \setminus A$ is a neighbourhood of $X \setminus B$ in the familiar sense.

Remark. Since the construct Rere of reflexive relations has a realization E: Rere $\to$ Clos, given by

$$x \in cl\, A \leftrightarrow \exists\, a \in A\; a\varrho x,$$

since the restriction of E to objects with finite underlying sets is an isomorphism, and since the proof of Proposition 5 depends only on finite closure space, Rere has precisely two realizatioos in UNeigh (resp. in Neigh).

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Horst Herrlich
Fachbereich Mathematik und Informatik
Universität Bremen
Postfach 33 04 40
2800 Bremen 33
BRD