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MEDIAN GROUPS

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Dedicated to the memory of Milan Sekanina

Abstract. Some properties of groups endoved with a special ternary operation are investigated. Such groups are a natural generalization of lattice ordered groups.

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1. INTRODUCTION

By a *median algebra* is meant an algebra with one ternary operation satisfying the identities

$$(a, a, b) = a, \tag{1}$$

$$((a, d, c), b, c) = ((b, c, d), a, c). \tag{2}$$

Such algebras were investigated (under various names) by several authors. A survey of these algebras is e.g. in [1].

An important example of median algebras is derived from distributive lattices. Given a distributive lattice $\mathscr{L}$ and the operation

$$(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \tag{3}$$

then $\mathrm{M}(\mathscr{L}) = (L; (, ,))$ is a median algebra. According to [10] each median algebra is isomorphic to a subalgebra of an algebra $\mathrm{M}(\mathscr{L})$.

In an l-group $\mathscr{G} = (G; +, -, 0, \wedge, \vee)$ the operations (3) and $+$ are related by the identity

$$u + (a, b, c) + v = (u + a + v, u + b + v, u + c + v). \tag{4}$$

Definition. *By a median group (m-group) there is meant an algebra $(G; +, -, 0, (, ,))$ where $(G; +, -, 0)$ is a group, $(G; (, ,))$ is a median algebra and the identity (4) holds.*

If $\mathscr{G}$ is an l-group then the m-group $(G; +, -, 0, (, ,))$, where the ternary operation is given by (3), is said to be *associated with* $\mathscr{G}$.

The class of m-groups is much larger than that of m-groups associated with l-groups. Nevertheless some results which are valid for l-groups can be applied (possibly in a modified form) to m-groups. The present paper contains some examples of such results.

Some fundamental properties of m-groups were announced in [7]. Several interesting results on m-groups and their important classes are contained in [9].

2. SOME PROPERTIES OF MEDIAN ALGEBRAS

2.1. Fundamental notions and properties. Let $\mathscr{A} = (A; (\ ,\ ,\ ))$ be a median algebra. If $a, b, c \in A$ and $(a, b, c) = b$, we say that b is between a and c (in symbols, abc). If $a_1, \ldots, a_n \in A$ and $a_i a_j a_k$ holds for $1 \leqq i \leqq j \leqq k \leqq n$, we denote this by $a_1 a_2, \ldots, a_n$. (a, b) will denote the set $\{x \in A\colon axb\}$. $((a, b); \wedge, \vee)$ is a distributive lattice where $x \wedge y = (a, x, y)$ and $x \vee y = (b, x, y)$ [6]. aub, buc and cua imply $(a, b, c) = u$ [10]. Call a mapping $\varphi\colon A \to B$ between two median algebras betweenness-preserving if abc implies $(\varphi a)(\varphi b)(\varphi c)$. Let $\mathscr{C} = (C; \leqq)$ be a linearly ordered set and let abc mean that $a \leqq b \leqq c$ or $c \leqq b \leqq a$. If φ is a betweenness-preserving injective mapping from C to a median algebra $\mathscr{A}$, the set $\{\varphi c\colon c \in C\}$ will be called a *line* (in $\mathscr{A}$). A subset K of A is said to be convex if $a, b \in K$, $u \in A$ and aub imply $u \in K$. One can easily check that K forms a subalgebra of $\mathscr{A}$.

N will denote the set of positive integers.

2.2. The following identities hold in an m-algebra [8, Th. 2].

(5) $$(a, b, c) = (b, a, c) = (b, c, a),$$

(6) $$((a, b, c), d, e) = ((a, d, e), b, (c, d, e)).$$

The following relations are easy to prove.

(7) abc implies cba,

(8) $a(a, b, c)\, b$,

(9) [10] abc and buc imply $abuc$,

(10) abc and acb imply $b = c$.

These identities and relations are used freely in what follows.

2.3. We say that the elements a, b, c, d of a median algebra form a *cyclic quadruple* (a, b, c, d) whenever abc, bcd, cda and dab hold. It can be easily shown that the element d is uniquely determined by the elements a, b, c.

2.4 [3, Proposition 2]. A subset L of a median algebra with $\operatorname{card} L \neq 4$ is a line iff for any $a, b, c \in L$ one of the relations abc, bca, cab holds. Obviously a subset of a line is a line. If a is an element of a line L such that for each $b, c \in L$ either abc or acb holds, we say that a is an end element of L.

2.5. *Let A be a line in a median algebra and* $0, a \in A$, $a \neq 0$. *Denote* $A' = \{x \in A: x0a\}$, $A_a = A - A'$. Then $A = A' \cup A_a$ *and* $x \in A'$ *together with* $y \in A_a$ *imply* $x0y$. Routine proof omitted.

2.6 [4]. A subset C of a median algebra $\mathscr{A}$ is called a *Čebyšev set* if for each $a \in A$ an element $a_C \in C$ exists such that aa_Ct holds for any $t \in C$. a_C will be said to be the *projection* of a into C. It can be easily shown that the element a_C is uniquely determined and that C is a convex subalgebra of $\mathscr{A}$. The mapping $x \to x_C$ is a homomorphism of $\mathscr{A}$ into C [4, 5.8].

2.7. *A maximal line in a median algebra $\mathscr{A}$, which is convex, is a Čebyšev set.*

Proof. Let C be such a line and $x \in A$. If for each $a, b \in C$ either abx or bax holds then $C \cup \{x\}$ is a line, hence $x \in C$ and $x_C = x$ is the projection. Consider the opposite case. If card $C = 4$ then x_C is the element (u, x, v) where u, v are the end elements of C. Suppose card $C \neq 4$. Then $u, v \in C$ exist such that neither uvx nor vux holds. We shall show that $t = (u, x, v)$ is the projection x_C. $t \in C$ and $t \notin \{u, v\}$. Let $a \in C$ and $s = (x, t, a)$. There are three possibilities: i) auv, ii) avu, iii) uav, and it suffices to consider i) and iii). The case i) yields $autv$. This together with ast implies that either $asut$ or $aust$ holds. In the first case xst implies xut which together with xtu yields $u = t$ – a contradiction. In the second case we have $ustx$ and tsx, hence $t = s = (x, t, a)$. In the case iii) either $uatv$ or $utav$ holds. In both cases this implies xta (e.g. the first possibility yields tau and xtu).

2.8. Let $\mathscr{A}$ be a median algebra and $0 \in A$. The algebra $(A; \wedge)$ where $a \wedge b = (a, 0, b)$, is a semilattice [10]. The corresponding order relation will be denoted by $\leqq$ (i.e. $a \leqq b$ means $0ab$). $a \vee b$ will denote sup $\{a, b\}$ if exists. In such a case $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for any $c \in A$ (see [10, 8] and [12, 3]).

3. ELEMENTARY PROPERTIES OF MEDIAN GROUPS

3.1. Examples of median groups

a) To any l-group $\mathscr{G}$ there is its associated m-group $\mathrm{M}(\mathscr{G})$. Such m-groups satisfy the identity

$$(*) \qquad (x, 0, -x) = 0.$$

T. Marcisová [9] has shown that an m-group satisfies (*) iff it satisfies the identity

$$(**) \qquad -(x, y, z) = (-x, -y, -z)$$

and it does not contain any non-zero element of a finite order.

The following examples show that there are finite m-groups.

b) Let $\mathscr{B} = (B; \wedge, \vee, ', 0, 1)$ be a Boolean algebra. Define $a + b = (a \wedge b')\vee \vee (a \wedge b)$, $-a = a$, and take the operation (3). Then $(B; +, -, 0, (, ,))$ is an m-group.

c) Let $\mathscr{C}_4$ be a (cyclic) group with the elements 0, 1, 2, 3, and addition mod 4. Take the distributive lattice with the same elements as $\mathscr{C}_4$, where 0 is the least element and 1, 3 are atoms. The group $\mathscr{C}_4$ with the operation (3) is an m-group different from that in b) with the four-element Boolean algebra $\mathscr{B}$.

In what follows $\mathscr{G}$ denotes an m-group.

3.2. *$a \leqq c$ and $b \leqq c$ imply $(a, b, c) = a\vee b$.*

The proof is straightforward.

3.3. *Let the elements $a, b \in G$ satisfy*

(i) $a\wedge b = 0 = a\wedge (-b) = (-a)\wedge b$.

Then a) $(-a)\wedge (-b) = 0$, b) $a + b = a\vee b = b + a$.

Proof. a) Denote $(-a)\wedge (-b) = (-a, 0, -b) = u$. Then $0u(-a)$ and $0u(-b)$ hold. Since $a0b$, $a0(-b)$ and $(-a)0b$, we get successively $(-a-b)(-b)0$, $(-a-b)(-b)u$, $(-a)0(u+b)$, $u0(u+b)$, $0(-u)b$, and symmetrically $0(-u)a$. This together with $b0a$ yields $b0(-u)$. Since $0(-u)b$, we get $u = 0$.

b) $(0, b, a + b) = (-b, 0, a) + b = b$, hence $b \leqq a + b$, and similarly $a \leqq a + + b$. Using a) and 3.2 we get $a + b = a + (-b, -a, 0) + b = a\vee b$.

3.4. Elements $a, b \in G$ satisfying (i) in 3.3 will be said to be *orthogonal* (in symbols, $a \perp b$).

In l-groups the relation $a \perp b$ is defined to mean (ii) $|a|\wedge |b| = 0$, where $|a| = a\vee (-a)$ [2, 3.1]. It can be readily proved that in an m-group associated with an l-group, (i) and (ii) are equivalent.

3.5. *$a \perp b$* iff *$(0, a, a + b, b)$* is *a cyclic quadruple.*

We omit the easy proof.

3.6. *Given $a, b \in G$, $a + b = a\vee b$ iff $a \perp b$.*

Proof. $a+b = a\vee b$ implies $a=a\wedge (a+b)=(a, 0, a+b)=a+(0, -a, b)$, hence $(-a)\wedge b = 0$, and similarly, $a\wedge (-b) = 0$. Using 3.2 we get $a + b = = (a, a + b, b) = a + (-b, 0, -a) + b$, hence $(-a)\wedge (-b) = 0$ and $a \perp b$. The converse implication was proved in 3.3b).

3.7. *Let $a, x, y \in G$ and $a \perp x$, $a \perp y$. Then $a\wedge (x + y) = 0$.*

Proof. First $(a, 0, x+y)=((a, 0, x), a, x+y)=((a, a, x+y), 0, (x, a, x+y))= = (a, 0, (x, a, x + y))$. Now $(x, a, x + y) = x + (0, -x + a, y)$. Since $-x \perp a$, $-x + a = (-x)\vee a$ ((3.3b)), hence $(0, -x + a, y) = ((-x)\vee a)\wedge y = ((-x)\wedge \wedge y)\vee (a\wedge y) = (-x)\wedge y$ (see 2.8). Using this we get $(x, a, x + y) = x + + (-x, 0, y) = (0, x, x + y)$ hence $(a, 0, x + y) = (a, 0, (0, x, x + y)) = = ((a, 0, 0), (a, 0, x), x + y) = (0, 0, x + y) = 0$.

3.8. *$a \perp x$ and $a \perp y$ imply $a \perp x + y$.*

Proof. From the suppositions it follows $-a \perp x$, $-a \perp y$, $a \perp -x$ and $a \perp -y$. Using 3.7 we get $(-a) \wedge (x + y) = 0 = a \wedge (-(x + y))$.

3.9. *Let $a \in G$. The set $H = \{x \in G: a \perp x\}$ forms a subgroup of K.*

Proof. From the definition of the orthogonality it follows that $x \in H$ impliex $-x \in H$. This together with 3.8 proves the assertion.

4. CONVEX MAXIMAL LINES

If (i) $\varphi : \mathcal{G} \cong \mathcal{A} \times \mathcal{B}$ is a direct decomposition of an m-group $\mathcal{G}$ then the m-group $\mathcal{A}$ is isomorphic to the m-subgroup of $\mathcal{G}$, whose elements are $\varphi^{-1}(a, 0)$, $a \in A$. An analogous assertion holds for $\mathcal{B}$. In what follows we shall suppose that in the direct decomposition (i), $\mathcal{A}$ and $\mathcal{B}$ are m-subgroups of $\mathcal{G}$.

We shall deal with direct decompositions (i) in which $\mathcal{A}$ is a line. In this case convex maximal lines (i.e. maximal lines which are convex) prove to be important.

4.1. Theorem. *Let an m-group $\mathcal{G}$ satisfy the identity* (*) *and let A be a line in* M($\mathcal{G}$) *such that* $0 \in A$. *The following are equivalent.*

(a) *A forms a subgroup of* $(G; +, -, 0)$ *and a direct factor of $\mathcal{G}$.*

(b) *A is a convex maximal line in* M($\mathcal{G}$).

Corollary [5, Th. 1]. *Let $\mathcal{G}$ be an l-group. A maximal chain in $\mathcal{G}$ which is convex and contains* 0, *is a direct factor of $\mathcal{G}$.*

Remark. The following assertion is easy to prove. Let $\mathcal{G}$ be an l-group. A convex line in M($\mathcal{G}$) containing 0 is a (convex) chain.

The proof of Theorem 4.1 is divided into a sequence of lemmas. $\mathcal{G}$ is supposed to satisfy (*) unless other is said.

4.2. *Let $\mathcal{G} = \mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ is a non-singleton line. Then $A_1 = \{(a, 0): a \in A\}$ is a maximal line in $\mathcal{G}$ and it is convex.*

Corollary. *Let $\mathcal{G}$ be an l-group and let $\mathcal{G} = \mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ is a non-singleton chain. Then $\{(a, 0): a \in A\}$ is a convex maximal chain* (and contains (0, 0)).

Proof. Obviously A_1 is convex. Let $c = (u, v) \in A \times B$ and $A_1 \cup \{c\}$ be a line. Then either (i) c is an end element of $A_1 \cup \{c\}$ or (ii) c is between some two elements of A_1. The case (ii) yields $c \in A_1$ immediately. In the case (i) recall that 0 and $2u$ belong to A hence either $u0(2u)$ or $u(2u)\,0$ holds. Combining this with $0u(2u)$ (a consequence of $(-u)\,0u$) we get $u = 0$. Then for any $a \in A$ either $0a(-a)$ or $0(-a)\,a$ holds. Because of $a0(-a)$ this gives $a = 0$. This contradiction shows that the case (i) is not possible.

4.3. *Suppose $\mathscr{G}$ satisfies* (**). *Then* $0xy$ *implies* $0(y-x)\,y$ *and* $0(-x+y)\,y$.

Proof. From the supposition we get $0(-x)\,(-y)$, hence $y - x = y + (0, -x, -y) = (y, y - x, 0)$. The proof of the second relation is similar.

4.4. *Let $\mathscr{G}$ satisfy* (**). *Let A, B be convex lines in $\mathscr{G}$ with the end element* 0. *If $p \in A - B$ and $q \in B - A$ then $p \wedge q = 0$.*

Proof. Denote $p \wedge q = r$. $(r, p, q) = ((p, 0, q), p, q) = r$ and $0rp$, $0rq$. For the elements $p' = p - r$ and $q' = q - r$ we get $p' \wedge q' = (0, p - r, q - r) = (r, p, q) - r = 0$. According to 4.3, $0p'p$ and $0q'q$. Hence there hold

either a1) $0p'r$ or a2) $rp'p$, and

either b1) $0q'r$ or b2) $rq'q$.

a1) and b1) yield $0p'q'$ or $0q'p'$, hence $p' = p' \wedge q' = 0$ i.e. $p = r$ or $q' = 0$ i.e. $q = r$. This is a contradiction, since $r \in A \cap B$. a1) and b2) yield $0p'q'$ – a contradiction as above. The case a2) and b1) is symmetric. If a2) and b2) hold then $0 = p' \wedge q' = r$ (since $r \leqq p' \leqq p$, $r \leqq q' \leqq q$ and $p \wedge q = r$).

4.5. *Let $\mathscr{G}$, A, B be as in 4.4. If neither $A \subset B$ nor $B \subset A$ then $a \wedge b = 0$ for each $a \in A$ and each $b \in B$.*

Proof. Let p, q have the same meaning as in 4.4. Then $p \wedge q = 0$. For $a \in A$, $b \in B$ there hold either a1) $0ap$ or a2) $0pa$, and either b1) $0bq$ or b2) $0qb$. a2) and b2) yield $a \in A - B$ and $b \in B - A$ hence $a \wedge b = 0$ according to 4.4. a1) and b2) yield $a \wedge b = a \wedge p \wedge b$. But $p \wedge b = 0$ by 4.4. The case a2) and b1) is symmetric and a1) and b1) give $a \wedge b = a \wedge p \wedge b \wedge q = 0$.

4.6. *Let $\mathscr{G}$ be arbitrary, $u \in G$, and let A be a line in $\mathscr{G}$. Then $u + A = \{u + a : a \in A\}$ is a line. If A is a maximal line (convex line) then so is $u + A$.*

Routine proof omitted.

4.7. (see [9]). *For each $a \in G$ and each $n \in N$, $0a(na)$ holds.*

4.8. *If $a \in G$ and $m, n \in N$, $m < n$, then $0(ma)\,(na)$.*

The proof proceeds by induction on m. For $m = 1$ the assertion holds by 4.7. Assume $0((m-1)\,a)\,(na)$ $(m < n)$. According to 4.7 $0a((n + 1 - m)\,a)$ hence $((m-1)\,a)\,(ma)\,(na)$ which, together with the assumption, yields $0(ma)\,(na)$.

4.9. *Let $a \in G$. If $(0, a)$ is a line then $(-a, a)$ is a line too.*

Proof. Since $(-a, 0) = -a + (0, a)$, $(-a, 0)$ is a line by 4.6. Because of $(-a)\,0a$, $(-a, 0) \cup (0, a)$ is a line. It suffices to show:

(i) $t \in (-a, a)$ implies $t \in (-a, 0)$ or $t \in (0, a)$.

Denote $u = (0, t, -a)$, $v = (0, t, a)$. Then $-v = (0, -t, -a)$ and $(0, u, -v) = (0, (0, t, -a), (0, -t, -a)) = ((0, t, -t), 0, -a) = (0, 0, -a) = 0$. From $0va$ we get $0(-v)\,(-a)$ and, because of $0u(-a)$, either $0u(-v)$ or $0(-v)\,u$ holds. In the first case $u = (0, u, -v) = 0$, in the second $v = 0$. $u = 0$ implies $t0(-a)$ which together with $at(-a)$ yields $at0$ i.e. $t \in (0, a)$. Similarly, if $v = 0$ then $t \in (0, -a)$.

4.10. *If $a \in G$ and $(0, a)$ is a line then $(-na, na)$ is a line for each $n \in N$.*

Proof. For $n = 1$ this holds by 4.9. Suppose $(-(n-1)a, (n-1)a) = B$ is a line ($n > 1$). According to 4.6, $a + B = ((-n+2)a, na)$ is a line. By 4.8 $0((n-2)a)(na)$ hence $0((-n+2)a)(-na)$ which together with $(na)0(-na)$ yields $0 \in (na, (-n+2)a)$ hence $(0, na)$ is a line. Now 4.9 is applicable.

4.11. *Let A be a convex maximal line in $\mathscr{G}$, containing 0. Then*

i) *For each $a \in A$ and $n \in N$, na belongs to A.*

ii) *$a \in A$ implies $-a \in A$.*

iii) *$a, b \in A$ imply $a + b \in A$.*

Proof. i) If $a = 0$ the assertion is trivial. Suppose $a \neq 0$. Let A' and A_a be as in 2.5. We apply 4.5 to the convex lines A_a and $(0, na)$ (see 4.10). There are three possibilities.

a) $(0, na) \subset A_a$, b) $A_a \subset (0, na)$,

c) $x \wedge y = 0$ for each $x \in A_a$ and each $y \in (0, na)$.

The case a) yields $na \in A$ immediately. In the case c) we get, using the relation $0a(na)$ (4.7), that $a = a \wedge na = 0$ – a contradiction. Consider the case b). Take $y \in A'$ and set $(y, 0, na) = u$. Then $yu0a$. Since a and u belong to $(0, na)$, either $0au$ or $0ua$ holds. The first case together with $a0u$ yields $a = 0$ – a contradiction. In the second case $u = 0$, so that $y0(na)$. Hence $y0(na)$ for each $y \in A'$ and $A' \cup (0, na)$ is a line and $A = A' \cup A_a \subset A' \cup (0, na)$. The maximality of A implies $na \in A$.

ii) The set $B = -a + A$ is a convex maximal line (4.6) and contains the elements 0 and $-a$. By i) $-2a \in B$, hence $-a \in a + B = A$.

iii) There are three possibilities: 1. $0ab$, 2. $0ba$, and 3. $a0b$. In the case 1. $b(a+b)(2b)$. Using i) we get $a + b \in A$. The case 2. is similar. In the third case $(-a)0(-b)$, hence $0a(a-b)$ and $b(a+b)a$ so that again $a + b \in A$.

4.12. *Let A be as in* 4.11 *and $b \in G$. If $0 \neq a \in A$ and $b \perp a$ then $b_A = 0$.*

Proof. $b \perp a$ implies $b0a$ and $b0(-a)$. This together with $a0(-a)$ yields $(b, a, -a) = 0$. For the element $t = b_A$ there hold $t \in A$, bta and $bt(-a)$. There are three possibilities: $a(-a)t$, $(-a)at$, and $(-a)ta$. In the first case we get $a(-a)tb$, hence $0 = (b, -a, a) = -a$ – a contradiction. Similarly the second case is not possible. In the last case we get $(b, a, -a) = t$ hence $t = 0$.

4.13. *Let A be as in* 4.11. *If $0 \neq a \in A$ and $b \perp a$ then $b \perp x$ for each $x \in A$.*

Proof. By 4.12 and 4.11, $b0x$ and $b0(-x)$. Since $-b \perp a$ too, $(-b)0x$. Hence $b \perp x$.

4.14. *Let A be as in* 4.11. *Denote $B = \{-x_A + x: x \in G\}$, $x_1 = x_A$ and $x_2 = -x_1 + x$. Then $x = x_1 + x_2$, $x_1 \in A$, $x_2 \in B$* and $x_1 \perp x_2$.

Note that $x_1 + x_2 = x_2 + x_1$ (see 3.3).

Proof. The element $u = (x_1, 0, x_2)$ belongs to A. By 4.11 iii) $x_1 + u \in A$, hence $(x, x_1, x_1 + u) = x_1$, so that $0 = -x_1 + (x, x_1, x_1 + u) = (x_2, 0, u) = u$ i.e.

$(x_1, 0, x_2) = 0$. Further $(-x_1, 0, x_2) = -(x_1, 0, -x_1 + x) = -x_1 + (0, x_1, x) =$ $= -x_1 + x_1 = 0$ and $(x_1, 0, -x_2) = 0$ by (**).

In what follows we shall use the notations from 4.14.

4.15. *For each* $x \in G$, $(-x)_A = -x_A$.

Proof. According to 4.14 $x_1 \perp x_2$. We have to show that $(-x)(-x_1)\,a$ for each $a \in A$. Since $(-x, -x_1, a) = (-x_2 - x_1, -x_1, a) = (-x_2, 0, a + x_1) - x_1$, it suffices to show

(i) $(-x_2, 0, a + x_1) = 0$.

If $x_1 = 0$ then $x0t$ for each $t \in A$, hence $x_2 0(-a)$ and, according to (**), $(-x_2)\,0a$, which yields (i). If $x_1 \neq 0$ then $x_2 \perp x_1$ by 4.14 and according to 4.12, $(x_2)_A = 0$, hence $x_2 0(-x_1 - a)$ and $(-x_2)\,0(a + x_1)$ i.e. (i) holds.

4.16. If $a_A = 0$ *then* $a \perp t$ *for each* $t \in A$.

Proof. By 4.15 $(-a)_A = 0$ hence for each $t \in A$ there holds $(-a)\,0t$ (and $a0t$, $a0(-t)$).

4.17. *The mapping* $x \to x_A$ *is a homomorphism from the group* $(G; +, -, 0)$ *onto its subgroup* A.

Proof. It suffices to show that for each $a \in A$ and $x, y \in G$, $(x + y, x_1 + y_1, a) =$ $= x_1 + y_1$. Since $(x + y, x_1 + y_1, a) = x_1 + (x_2 + y_2, 0, -x_1 + a - y_1) + y_1$ it suffices to prove $x_2 + y_2 \perp -x_1 + a - y_1$, i.e., according to 3.9,

(i) $x_2 \perp -x_1 + a - y_1$,

(ii) $y_2 \perp -x_1 + a - y_1$.

If $x_1 = 0$, (i) holds by 4.16. If $x_1 \neq 0$, (i) follows from $x_2 \perp x_1$ by 4.13. The proof of (ii) is similar.

4.18. *The following holds for any* $x \in G$.

(i) *The representation* $x = a + b$, $a \in A$, $b \in B$, *is unique.*

(ii) *If* $a \in A$ *then* $a_1 = a$ *and* $a_2 = 0$. *If* $b \in B$ *then* $b_1 = 0$ *and* $b_2 = b$.

(iii) $a \in A$ *and* $b \in B$ *imply* $(a + b)_1 = a$, $(a + b)_2 = b$.

(iv) $a \in A$ *and* $b \in B$ *imply* $a \perp b$.

(v) $a + b = b + a$ *for each* $a \in A$ *and* $b \in B$.

Proof. (i) $b = -y_A + y$ for some $y \in G$ (see 4.14). $x = a + b$ hence $y - x =$ $= y_A - a$, and using 4.11 and 4.17 we get $y_A - a = (y - x)_A = y_A - x_A$. Thus $a = x_A$ and $b = -x_A + x$.

(ii) The assertion on a is obvious. If $b \in B$, there is $y \in G$ such that $b = -y_1 + y$. According to 4.17, $b_1 = b_A = -y_1 + y_1 = 0$ aod $b_2 = b$. (iii) follows from 4.17 and (ii), (iv) follows from (ii) and 4.16, and (v) follows from (iv) and 3.3.

4.19. *If* $x = x_1 + x_2$ *and* $y = y_1 + y_2$ *are the representations in* 4.18 (i) *then* $x + y = (x_1 + y_1) + (x_2 + y_2)$, $-x = (-x_1) + (-x_2)$ *where* $x_1 + y_1$ *and* $-x_1$ *belong to* A *and* $x_2 + y_2$, $-x_2$ *belong to* B.

Proof. By 4.17, $(x+y)_1 = x_1 + y_1$. Further $(x+y)_2 = -(x+y)_1 + (x+y) = x_2 + y_2$. This proves the first assertion. The proof of the second assertion is similar.

4.20. *B forms a subgroup of the group $\mathscr{G}$.*

Proof. The assertion follows from 4.18 and 4.19.

4.21. *B is a Čebyšev set and $x \to -x_A + x$ is the corresponding projection.*

Proof. Let $x \in G$ and let $x = x_1 + x_2$ be the representation in 4.18 (i). For each $b \in B$ we get $(x, x_2, b) = (x_1 + x_2, x_2, b) = (x_1, 0, b - x_2) + x_2$. $b - x_2$ belongs to B (4.20) and by 4.18 (iv), $x_1 \perp b - x_2$, so that $(x_1, 0, b - x_2) = 0$. Hence x_2 is the desired projection.

The following theorem, together with 4.2, completes the proof of the theorem 4.1.

4.22. *The mapping φ: $x \to (x_1, x_2)$ is an isomorphism of m-groups $\mathscr{G}$ and $\mathscr{A} \times \mathscr{B}$ where $\mathscr{A}$ and $\mathscr{B}$ are m-subgroups of $\mathscr{G}$ with carriers A and B respectively.*

Proof. According to [4, 5.8] the projection into a Čebyšev subset is a homomorphism of median algebras. This together with 4.18 and 4.19 implies that φ is a homomorphism of *m*-groups. Consider the mapping ψ: $A \times B \to G$ with $\psi(a, b) = a + b$. From the definition of φ it follows that $\psi \circ \varphi = \mathrm{id}_G$ and by 4.18 $\varphi \circ \psi = \mathrm{id}_{A \times B}$, hence φ is a bijection.

4.23. **Theorem.** *Let $\mathscr{G}$ be an m-group satisfying (*) and let A be a convex maximal line in $\mathscr{G}$. If $a \in A$ then $-a + A$ is a direct factor of K.*

The theorem follows from 4.6 and 4.1.

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