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Archivum Mathematicum, Vol. 25 (1989), No. 1-2, 5--12

Persistent URL: http://dml.cz/dmlcz/107333

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 25, No. 1 – 2 (1989), 5 – 12

TRANSITIVE TERNARY RELATIONS AND QUASIORDERINGS

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(Received January 28, 1988)

Dedicated to the memory of Milan Sekanina

Abstract. Two operators are described which enable to construct a quasiordering from a transitive ternary structure and vice versa.

Key words. Relational structure, transitive and asymmetric ternary relation, quasiordering, strong homomorphism.
MS Classification. 06 A 10, 04 A 05.

0. INTRODUCTION

Some authors have studied cyclically ordered sets, e.g. E. Čech [4] who has used a cyclic order to define an orientation of a closed curve, G. Müller [6], N. Megiddo [5], P. Alles [1] and others. A cyclic order is a nontrivial example of a relation with arity greater than 2; thus a natural question arises, which problems of the theory of ordered sets can be posed for cyclically ordered sets (e.g. dimension theory [8], completion [10], representation theory [9] a.s.o.). A great disadvantage of these investigations is the fact that there is no simple realisation of a ternary relation. This paper is an attempt to construct ternary relations from binary relations and vice versa with preservation of transitivity. The relationship between binary and ternary relations were studied in literature. So G. Birkhoff [3] posed the problem of a connection of a partial order and corresponding relation betweeness; this problem was solved by M. Altwegg [2]. M. Sekanina studied the relation betweeness in graphs [11].

1. BASIC NOTIONS

Let $G \neq \emptyset$ be a set, $n \geqq 1$ an integer and R an n-ary relation on G. The pair $\boldsymbol{G} = (G, R)$ will be called an n-ary structure. If $\boldsymbol{G} = (G, R)$ is an n-ary structure, then the set G is called a carrier of the structure $\boldsymbol{G}$ and denoted $G = c(\boldsymbol{G})$, and the set R is called a relation of the structure $\boldsymbol{G}$ and denoted $R = r(\boldsymbol{G})$.

Let $\boldsymbol{G}$ be an n-ary structure, $x \in c(\boldsymbol{G})$. We call the element x isolated, if for any $(x_1, \ldots, x_n) \in r(\boldsymbol{G})$ we have $x \neq x_i$ for all $i = 1, \ldots, n$; otherwise it is nonisolated.

Let $\boldsymbol{G}, \boldsymbol{H}$ be n-ary structures, $f : c(\boldsymbol{G}) \to c(\boldsymbol{H})$ be a mapping. f is called a homomorphism of $\boldsymbol{G}$ into $\boldsymbol{H}$ iff

$$x_1, \ldots, x_n \in c(\boldsymbol{G}), (x_1, \ldots, x_n) \in r(\boldsymbol{G}) \Rightarrow (f(x_1), \ldots, f(x_n)) \in r(\boldsymbol{H}).$$

A homomorphism f of $\boldsymbol{G}$ into $\boldsymbol{H}$ is called strong, iff it is surjective and it holds

$$y_1, \ldots, y_n \in c(\boldsymbol{H}), (y_1, \ldots, y_n) \in r(\boldsymbol{H}) \Rightarrow \text{there exist}$$
$$x_1 \in f^{-1}(y_1), \ldots, x_n \in f^{-1}(y_n) \text{ with } (x_1, \ldots, x_n) \in r(\boldsymbol{G}).$$

A bijective strong homomorphism is an isomorphism. Two n-ary structures $\boldsymbol{G}, \boldsymbol{H}$ are called isomorphic iff there exists an isomorphism of $\boldsymbol{G}$ onto $\boldsymbol{H}$.

In the sequel we shall deal only with binary and ternary structures. Recall that a binary relation which is reflexive and transitive is a quasiordering; a binary structure $\boldsymbol{G}$ in which $r(\boldsymbol{G})$ is a quasiordering is a quasiordered set. A quasiordering which is antisymmetric is an ordering; a binary structure $\boldsymbol{G}$ in which $r(\boldsymbol{G})$ is an ordering is an ordered set.

Let R be a ternary relation on a set G. We shall call this relation

transitive, iff $(x, y, z) \in R$, $(z, y, u) \in R \Rightarrow (x, y, u) \in R$,
antisymmetric, iff $(x, y, z) \in R$, $(z, y, x) \in R \Rightarrow x = z$.

A ternary structure $\boldsymbol{G}$ is called transitive, resp. antisymmetric, iff $r(\boldsymbol{G})$ is transitive, resp. antisymmetric ternary relation.

Let $\boldsymbol{G}$ be a ternary structure. Put

$D(\boldsymbol{G}) = \{(x, y, x) \in (c(\boldsymbol{G}))^3$; there exists $z \in c(\boldsymbol{G})$ with either $(x, y, z) \in r(\boldsymbol{G})$ or $(z, y, x) \in r(\boldsymbol{G})\}$,

$A(\boldsymbol{G}) = r(\boldsymbol{G}) \cup D(\boldsymbol{G})$.

In the whole paper, the symbols $D(\boldsymbol{G})$, $A(\boldsymbol{G})$ will have just this meaning.

Trivially, it holds

1.1. Lemma. *Let $\boldsymbol{G}$ be a ternary structure, $x, y \in c(\boldsymbol{G})$. If $(x, y, x) \in r(\boldsymbol{G})$, then $(x, y, x) \in D(\boldsymbol{G})$.*

Further, we prove

1.2. Lemma. *Let $\boldsymbol{G}$ be a ternary structure. If the relation $r(\boldsymbol{G})$ is transitive, then $A(\boldsymbol{G})$ is transitive.*

Proof. Let $(x, y, z) \in A(\boldsymbol{G})$, $(z, y, u) \in A(\boldsymbol{G})$. If $z \neq x$, $z \neq u$, then $(x, y, z) \in r(\boldsymbol{G})$, $(z, y, u) \in r(\boldsymbol{G})$ and $(x, y, u) \in r(\boldsymbol{G}) \subseteq A(\boldsymbol{G})$ for $r(\boldsymbol{G})$ is transitive. If $z = x$, then $(x, y, u) \in A(\boldsymbol{G})$; similarly for $z = u$. Thus $A(\boldsymbol{G})$ is transitive.

2. OPERATOR $\mathcal{Q}$

Let $\boldsymbol{G}$ be a ternary structure. Put

$B(\boldsymbol{G}) = \{((x, y, x), (z, y, z)) \in D(\boldsymbol{G}) \times D(\boldsymbol{G}); (x, y, z) \in A(\boldsymbol{G})\}$,
$\mathcal{Q}(\boldsymbol{G}) = (D(\boldsymbol{G}), B(\boldsymbol{G}))$.
Thus, $\mathcal{Q}(\boldsymbol{G})$ is a binary structure with carrier $D(\boldsymbol{G})$.

2.1. Lemma. *Let $\boldsymbol{G}$ be a ternary structure. Then the binary structure $\mathcal{Q}(\boldsymbol{G})$ is reflexive.*

Proof. Let $(x, y, x) \in D(\boldsymbol{G})$. Then $(x, y, x) \in A(\boldsymbol{G})$, thus $((x, y, x), (x, y, x)) \in \in B(\boldsymbol{G})$ and $B(\boldsymbol{G}) = r(\mathcal{Q}(\boldsymbol{G}))$ is reflexive.

2.2. Lemma. *Let $\boldsymbol{G}$ be a ternary structure. Then it holds:*

(1) *If $\boldsymbol{G}$ is transitive, then $\mathcal{Q}(\boldsymbol{G})$ is a transitive binary structure,*

(2) *If $r(\boldsymbol{G}) = A(\boldsymbol{G})$ and $\mathcal{Q}(\boldsymbol{G})$ is transitive, then $\boldsymbol{G}$ is transitive.*

Proof. (1) Let $\boldsymbol{G}$ be transitive and $(x, y, x), (z, y, z), (u, y, u) \in D(\boldsymbol{G}) = c(\mathcal{Q}(\boldsymbol{G}))$, $((x, y, x), (z, y, z)) \in B(\boldsymbol{G}) = r(\mathcal{Q}(\boldsymbol{G}))$, $((z, y, z), (u, y, u)) \in B(\boldsymbol{G})$. Then, by definition, $(x, y, z) \in A(\boldsymbol{G})$, $(z, y, u) \in A(\boldsymbol{G})$ and by 1.2. $(x, y, u) \in A(\boldsymbol{G})$. From this $((x, y, x), (u, y, u)) \in B(\boldsymbol{G})$ and $B(\boldsymbol{G})$ is transitive.

(2) Let $r(\boldsymbol{G}) = A(\boldsymbol{G})$ and $\mathcal{Q}(\boldsymbol{G})$ be transitive. Let $x, y, z, u \in c(\boldsymbol{G})$, $(x, y, z) \in r(\boldsymbol{G})$, $(z, y, u) \in r(\boldsymbol{G})$. Then (x, y, x), (z, y, z), $(u, y, u) \in D(\boldsymbol{G}) = c(\mathcal{Q}(\boldsymbol{G}))$ and $((x, y, x), (z, y, z)) \in B(\boldsymbol{G}) = r(\mathcal{Q}(\boldsymbol{G}))$, $((z, y, z), (u, y, u)) \in B(\boldsymbol{G})$. The transitivity of $B(\boldsymbol{G})$ yields $((x, y, x), (u, y, u)) \in B(\boldsymbol{G})$ which means $(x, y, u) \in A(\boldsymbol{G}) = r(\boldsymbol{G})$. Thus $r(\boldsymbol{G})$ is transitive.

From 2.1. and 2.2. it follows

2.3. Theorem. *Let $\boldsymbol{G}$ be a ternary structure. Then it holds:*

(1) *If $\boldsymbol{G}$ is transitive, then $\mathcal{Q}(\boldsymbol{G})$ is quasiordered set,*

(2) *If $r(\boldsymbol{G}) = A(\boldsymbol{G})$, then $\boldsymbol{G}$ is transitive iff $\mathcal{Q}(\boldsymbol{G})$ is a quasiordered set.*

2.4. Lemma. *Let $\boldsymbol{G}$ be a ternary structure. Then it holds:*

(1) *If the binary structure $\mathcal{Q}(\boldsymbol{G})$ is antisymmetric, then $\boldsymbol{G}$ is antisymmetric.*

(2) *If $r(\boldsymbol{G}) = A(\boldsymbol{G})$ and $\boldsymbol{G}$ is antisymmetric, then $\mathcal{Q}(\boldsymbol{G})$ is antisymmetric.*

Proof. (1) Let $\mathcal{Q}(\boldsymbol{G})$ be antisymmetric and $x, y, z \in c(\boldsymbol{G})$, $(x, y, z) \in r(\boldsymbol{G})$, $(z, y, x) \in \in r(\boldsymbol{G})$. Then $(x, y, x), (z, y, z) \in D(\boldsymbol{G}) = c(\mathcal{Q}(\boldsymbol{G}))$ and $((x, y, x), (z, y, z)) \in B(\boldsymbol{G}) = = r(\mathcal{Q}(\boldsymbol{G}))$, $((z, y, z), (x, y, x)) \in B(\boldsymbol{G})$. The antisymmetry of $B(\boldsymbol{G})$ gives $(x, y, x) = = (z, y, z)$, thus $x = z$ and $r(\boldsymbol{G})$ is antisymmetric.

(2) Let $r(\boldsymbol{G}) = A(\boldsymbol{G})$ and $\boldsymbol{G}$ be antisymmetric. Let $(x, y, x), (z, y, z) \in D(\boldsymbol{G}) = = c(\mathcal{Q}(\boldsymbol{G}))$, $((x, y, x), (z, y, z)) \in B(\boldsymbol{G}) = r(\mathcal{Q}(\boldsymbol{G}))$, $((z, y, z), (x, y, x)) \in B(\boldsymbol{G})$. Then $(x, y, z) \in A(\boldsymbol{G}) = r(\boldsymbol{G})$, $(z, y, x) \in r(\boldsymbol{G})$ and antisymmetry of $r(\boldsymbol{G})$ yields $x = z$. Thus $(x, y, x) = (z, y, z)$ and $B(\boldsymbol{G}) = r(\mathcal{Q}(\boldsymbol{G}))$ is antisymmetric.

From 2.3. and 2.4. we get immediately

2.5. Theorem. *Let G be a ternary structure with the property $r(G) = A(G)$.* Then G *is transitive and antisymmetric if and only if $\mathscr{Q}(G)$ is an ordered set.*

3. OPERATOR $\mathscr{T}$

Let G be a binary structure. Let Θ be the least equivalence on $c(G)$, containing $r(G)$ and p be the natural projection of $c(G)$ onto $c(G)/_\Theta$. Put

$$E(G)) = c(G) \cup c(G)/_\Theta,$$
$$F(G)) = \{(x, y, z); x, z \in c(G), y \in c(G)/_\Theta, (x, z) \in r(G), p(x) = p(z) = y\},$$
$$\mathscr{T}(G) = (E(G), F(G)).$$

Thus, $\mathscr{T}(G)$ is a ternary structure with carrier $E(G) = c(G) \cup c(G)/_\Theta$.

3.1. Lemma. *Let G be a binary structure. Then it holds:*

(1) *If G is reflexive, then the ternary structure $\mathscr{T}(G)$ satisfies $r(\mathscr{T}(G)) = A(\mathscr{T}(G))$,*

(2) *If G contains no isolated elements and if $r(\mathscr{T}(G)) = A(\mathscr{T}(G))$, then G is reflexive.*

Proof. (1) Assume that G is reflexive and that $A(\mathscr{T}(G)) - r(\mathscr{T}(G)) \neq \emptyset$. Let $m \in A(\mathscr{T}(G)) - r(\mathscr{T}(G))$ be any element. Then $m \in D(\mathscr{T}(G))$, thus $m = (x, y, x)$, where $x, y \in c(\mathscr{T}(G))$ and there exists $z \in c(\mathscr{T}(G))$ with either $(x, y, z) \in r(\mathscr{T}(G))$ or $(z, y, x) \in r(\mathscr{T}(G))$; say $(x, y, z) \in r(\mathscr{T}(G))$. This means $x \in c(G)$, $y = p(x)$ and as $r(G)$ is reflexive, we have $(x, x) \in r(G)$. From this it follows by definition $m = (x, y, x) \in F(G) = r(\mathscr{T}(G))$, a contradiction.

(2) Let G have no isolated elements, let $r(\mathscr{T}(G)) = A(\mathscr{T}(G))$ and assume that G is not reflexive. Then there exists an element $x \in c(G)$ with $(x, x) \bar{\in} r(G)$. Denote $p(x) = y$, thus $(x, y, x) \in F(G) = r(\mathscr{T}(G))$. As G has no isolated elements, there is an element $z \in c(G)$ satisfying either $(x, z) \in r(G)$ or $(z, x) \in r(G)$; let us say that $(x, z) \in r(G)$. Then $(x, y, z) \in F(G) = r(\mathscr{T}(G))$ and by definition it is $(x, y, x) \in D(\mathscr{T}(G)) \subseteq A(\mathscr{T}(G))$. Thus $(x, y, x) \in A(\mathscr{T}(G)) - r(\mathscr{T}(G))$, a contradiction.

3.2. Lemma. *Let G be a binary structure. Then G is transitive iff $\mathscr{T}(G)$ is a transitive ternary structure.*

Proof. 1. Let G be transitive and $x, y, z, u \in c(\mathscr{T}(G)) = E(G)$, $(x, y, z) \in r(\mathscr{T}(G)) = F(G)$, $(z, y, u) \in F(G)$. Then, by definition, $x, z, u \in c(G)$, $y \in c(G)/_\Theta$, and it holds $(x, z) \in r(G)$, $p(x) = p(z) = y$, $(z, u) \in r(G)$, $p(z) = p(u) = y$. As $r(G)$ is transitive, we have $(x, u) \in r(G)$ and $p(x) = p(u) = y$. Thus $(x, y, u) \in F(G)$ and $F(G) = r(\mathscr{T}(G))$ is transitive.

2. Let $F(G)$ be transitive ternary relation on $E(G)$ and let $x, y, z \in c(G)$, $(x, y) \in r(G)$, $(y, z) \in r(G)$. Then $(x, y) \in \Theta$, $(y, z) \in \Theta$, so that, if we denote $p(x) = u$, we have $p(y) = p(z) = u$. By definition of the relation $F(G)$ it is $(x, u, y) \in F(G)$, $(y, u, z) \in F(G)$ and transitivity of $F(G)$ yields $(x, u, z) \in F(G)$. This means $(x, z) \in r(G)$ and $r(G)$ is transitive.

From 3.1. and 3.2. we get

3.3. Theorem. *Let G be a binary structure. Then it holds:*

(1) *If G is a quasiordered set, then $\mathscr{T}(G)$ is a transitive ternary structure with the property $r(\mathscr{T}(G)) = A(\mathscr{T}(G))$.*

(2) *If G contains no isolated elements, then G is quasiordered set iff $\mathscr{T}(G)$ is a transitive ternary structure with the property $r(\mathscr{T}(G)) = A(\mathscr{T}(G))$.*

3.4. Lemma. *Let G be a binary structure. Then G is antisymmetric iff the ternary structure $\mathscr{T}(G)$ is antisymmetric.*

Proof. 1. Let G be antisymmetric and let $x, y, z \in c(\mathscr{T}(G)) = E(G)$, $(x, y, z) \in r(\mathscr{T}(G)) = F(G)$, $(z, y, x) \in F(G)$. Then $x, z \in c(G)$, $p(x) = p(z) = y$, $(x, z) \in r(G)$, $(z, x) \in r(G)$. The antisymmetry of $r(G)$ yields $x = z$ and thus $F(G) = r(\mathscr{T}(G))$ is antisymmetric.

2. Let $F(G)$ be antisymmetric and let $x, y \in c(G)$, $(x, y) \in r(G)$, $(y, x) \in r(G)$. Then $(x, y) \in \Theta$ and if we denote $p(x) = p(y) = u$, we have $(x, u, y) \in F(G)$, $(y, u, x) \in F(G)$. As $F(G)$ is antisymmetric, it is $x = y$ and thus $r(G)$ is antisymmetric.

From 3.3. and 3.4. we now get

3.5. Theorem. *Let G be a binary structure. Then it holds:*

(1) *If G is an ordered set, then $\mathscr{T}(G)$ is a transitive and antisymmetric ternary structure with the property $r(\mathscr{T}(G)) = A(\mathscr{T}(G))$,.*

(2) *If G contains no isolated elements, then G is an ordered set iff $\mathscr{T}(G)$ is a transitive and antisymmetric ternary structure with the property $r(\mathscr{T}(G)) = A(\mathscr{T}(G))$.*

4. OPERATORS $\mathscr{Q} \circ \mathscr{T}$ AND $\mathscr{T} \circ \mathscr{Q}$

4.1. Theorem. *Let G be a quasiordered set. Then the structures G and $\mathscr{Q}(\mathscr{T}(G))$ are isomorphic.*

Proof. By definition, it is $\mathscr{T}(G) = (E(G), F(G))$ where $E(G) = c(G) \cup c(G)/_\Theta$, and $\mathscr{Q}(\mathscr{T}(G)) = (D(\mathscr{T}(G)), B(\mathscr{T}(G)))$. Put for any $x \in c(G)$ $f(x) = (x, p(x), x)$. As $(x, x) \in r(G)$, it is $f(x) \in F(G) = r(\mathscr{T}(G))$ and by 1.1. $f(x) \in D(\mathscr{T}(G))$. Thus, f is a mapping of $c(G)$ into $D(\mathscr{T}(G)) = c(\mathscr{Q}(\mathscr{T}(G)))$. Let $w \in D(\mathscr{T}(G))$ be any element. Then $w = (x, y, x) \in (c(\mathscr{T}(G)))^3 = (E(G))^3$ and there exists an element $z \in E(G)$ such that either $(x, y, z) \in r(\mathscr{T}(G)) = F(G)$ or $(z, y, x) \in F(G)$. This means $x, z \in c(G)$, $y \in c(G)/_\Theta$, $p(x) = p(z) = y$ and either $(x, z) \in r(G)$ or $(z, x) \in r(G)$. But then $f(x) = (x, y, x) = w$ and the mapping f is surjective.

Let $x, y \in c(G)$, $f(x) = f(y)$. Then $(x, p(x), x) = (y, p(y), y)$, thus $x = y$. The mapping f is injective, hence a bijection of $c(G)$ onto $c(\mathscr{Q}(\mathscr{T}(G)))$. Let $x, y \in c(G)$, $(x, y) \in r(G)$. Then $(x, y) \in \Theta$, thus $p(x) = p(y) = u \in c(G)/_\Theta$ and $(x, u, y) \in F(G) = r(\mathscr{T}(G))$. By 3.1. we have $r(\mathscr{T}(G)) = A(\mathscr{T}(G))$. Further, it is $f(x) = (x, u, x) \in D(\mathscr{T}(G))$, $f(y) = (y, u, y) \in D(\mathscr{T}(G))$ and by definition we have $((x, u, x), (y, u, y)) = (f(x), f(y)) \in B(\mathscr{T}(G)) = r(\mathscr{Q}(\mathscr{T}(G)))$. Thus f is a bijective homomorphism of G onto $\mathscr{Q}(\mathscr{T}(G))$.

Let $x, y \in c(G)$ and $(f(x), f(y)) \in r(\mathcal{Q}(\mathcal{T}(G))) = B(\mathcal{T}(G))$. It is, of course, $f(x) = (x, u, x)$, $f(y) = (y, v, y)$ where $u = p(x)$, $v = p(y)$. By definition of the relation $B(\mathcal{T}(G))$ it is $u = v$, i.e. $p(x) = p(y)$, and $(x, u, y) \in A(\mathcal{T}(G))$. By 3.1. it is $A(\mathcal{T}(G)) = r((\mathcal{T}G)) = F(G)$ and this implies, by definition of the relation $F(G)$, $(x, y) \in r(G)$. Thus f is an isomorphism of G onto $\mathcal{Q}(\mathcal{T}(G))$.

4.2. Theorem. *Let G be a transitive ternary structure containing no isolated elements and such that $r(G) = A(G)$. Then there exists a strong homomorphism of $\mathcal{T}(\mathcal{Q}(G))$ onto G.*

Proof. By definition, it is $\mathcal{Q}(G) = (D(G), B(G))$, and $\mathcal{T}(\mathcal{Q}(G)) = (E(\mathcal{Q}(G)), F(\mathcal{Q}(G)))$, where $E(\mathcal{Q}(G)) = c(\mathcal{Q}(G)) \cup c(\mathcal{Q}(G))/_\Theta$; here Θ is the least equivalence on $c(\mathcal{Q}(G)) = D(G)$ containing $r(\mathcal{Q}(G)) = B(G)$.

Let $u \in E(\mathcal{Q}(G))$. If $u \in c(\mathcal{Q}(G)) = D(G)$, then $u = (x, y, x)$, where $x, y \in c(G)$ and there exists $z \in c(G)$ with either $(x, y, z) \in r(G)$ or $(z, y, x) \in r(G)$. In this case we put $f(u) = x$. Suppose that $u \in D(G)/_\Theta$. Then there exists $m \in D(G)$ such that $p(m) = u$ where p is a natural projection of $D(G)$ onto $D(G)/_\Theta$. Thus $m = (x, y, x)$ where $x, y \in c(G)$. We show that for any $n = (x', y', x') \in D(G)$ with the property $p(n) = u$ we have $y' = y$. Indeed, $p(m) = p(n)$ means $(m, n) \in \Theta$ and thus either $m = n$ or there exist a positive integer $k > 1$ and elements $m_1, \ldots, m_k \in D(G)$ such that $m_1 = m$, $m_k = n$ and $(m_i, m_{i+1}) \in B(G) \cup (B(G))^{-1}$ for all $i = 1, \ldots, k - 1$. Let $(m_i, m_{i+1}) \in B(G)$. Then $m_i = (x_i, y_i, x_i)$, $m_{i+1} = (x_{i+1}, y_{i+1}, x_{i+1})$ and by definition of the relation $B(G)$ it is $y_i = y_{i+1}$. If $(m_i, m_{i+1}) \in (B(G))^{-1}$, then $(m_{i+1}, m_i) \in B(G)$ and we have again $y_i = y_{i+1}$. Thus $y_1 = y_2 = \ldots = y_k$ and for $m = m_1 = (x_1, y_1, x_1) = (x, y, x)$, $n = m_k = (x_k, y_k, x_k) = (x', y', x')$ we have $y = y'$. Thus, any element $u \in D(G)/_\Theta$ determines just one element $y \in c(G)$ such that for some $x \in c(G)$ there is $p(x, y, x) = u$. We put $f(u) = y$. Thus, we have defined a mapping $f : c(\mathcal{T}(\mathcal{Q}(G))) \to c(G)$..

Let $x \in c(G)$ be any element. As G contains no isolated elements, there are elements $y, z \in c(G)$ such that either $(x, y, z) \in r(G)$ or $(y, x, z) \in r(G)$ or $(z, y, x) \in r(G)$. In the first and third case it is $u = (x, y, x) \in D(G) \subseteq E(\mathcal{Q}(G))$ and by definition of the mapping f we have $f(u) = x$. In the second case it is $(y, x, y) \in D(G)$, $v = p(y, x, y) \in D(G)/_\theta \subseteq E(\mathcal{Q}(G))$ and by definition we have $f(v) = x$. Thus f is a surjective mapping of $c(\mathcal{T}(\mathcal{Q}(G)))$ onto $c(G)$.

Let $u, v, w \in c(\mathcal{T}(\mathcal{Q}(G))) = E(\mathcal{Q}(G))$ and $(u, v, w) \in r(\mathcal{T}(\mathcal{Q}(G))) = F(\mathcal{Q}(G))$. Then, by definition of the relation $F(\mathcal{Q}(G))$, there is $u, w \in c(\mathcal{Q}(G)) = D(G)$, $v \in c(\mathcal{Q}(G))/_\Theta = D(G)/_\Theta$ and it holds $(u, w) \in r(\mathcal{Q}(G)) = B(G)$, $p(u) = p(w) = v$. As $u, w \in D(G)$ and $(u, w) \in B(G)$, there is $u = (x, y, x)$, $w = (z, y, z)$ for suitable $x, y, z \in c(G)$, and $(x, y, z) \in A(G) = r(G)$. By definition of the mapping f then $f(u) = x$, $f(v) = y$, $f(w) = z$ so that $(f(u), f(v), f(w)) \in r(G)$. We have proved that $f : c(\mathcal{T}(\mathcal{Q}(G))) \to c(G)$ is a surjective homomorphism of the structure $\mathcal{T}(\mathcal{Q}(G))$ onto structure G.

Let, at the end, $x, y, z \in c(G)$, $(x, y, z) \in r(G)$. If we denote $(x, y, x) = u$,

$(z, y, z) = w$, then $u, w \in D(G) = c(\mathcal{Q}(G))$ and $(u, w) \in B(G) = r(\mathcal{Q}(G))$. Thus $(u, w) \in \Theta$ so that $p(u) = p(w)$. Denote $p(u) = p(w) = v$; then $u, v, w \in E(\mathcal{Q}(G))$ and $(u, v, w) \in F(\mathcal{Q}(G)) = r(\mathcal{T}(\mathcal{Q}(G)))$. At the same time, by definition of the mapping f, it is $f(u) = x$, $f(v) = y$, $f(w) = z$, i.e. $u \in f^{-1}(x)$, $v \in f^{-1}(y)$, $w \in f^{-1}(z)$. Thus the homomorphism f of $\mathcal{T}(\mathcal{Q}(G))$ onto G is strong.

In the last theorem, the structures G and $\mathcal{T}(\mathcal{Q}(G))$ need not be isomorphic, as the following example shows.

4.3. Example. Let $G = (c(G), r(G))$ be a ternary structure with $c(G) = \{0, 1, 2\}$ and $r(G) = \{(0, 1, 2), (1, 2, 0), (2, 0, 1), (0, 1, 0), (2, 1, 2), (1, 2, 1), (0, 2, 0), (2, 0, 2), (1, 0, 1)\}$. Evidently, G is transitive and $r(G) = A(G)$. Further, $D(G) = \{(0, 1, 0), (2, 1, 2), (1, 2, 1), (0, 2, 0), (2, 0, 2), (1, 0, 1)\}$, $B(G) = \{((0, 1, 0), (2, 1, 2)), ((1, 2, 1), (0, 2, 0)), ((2, 0, 2), (1, 0, 1)), ((0, 1, 0)), (0, 1, 0)), ((2, 1, 2), (2, 1, 2)), ((1, 2, 1), (1, 2, 1)), ((0, 2, 0), (0, 2, 0)), ((2, 0, 2), (2, 0, 2)), ((1, 0, 1), (1, 0, 1))\}$, and $\mathcal{Q}(G) = (D(G), B(G))$. The least equivalence on $D(G)$ containing $B(G)$ has blocks $B_0 = \{(1, 0, 1), (2, 0, 2)\}$, $B_1 = \{(0, 1, 0), (2, 1, 2)\}$, $B_2 = \{(0, 2, 0), (1, 2, 1)\}$ so that $E(\mathcal{Q}(G)) = D(G) \cup \{B_0, B_1, B_2\}$, and $F(\mathcal{Q}(G)) = \{((0, 1, 0), B_1, (2, 1, 2)), ((1, 2, 1), B_2, (0, 2, 0)), ((2, 0, 2), B_0, (1, 0, 1)), ((0, 1, 0), B_1, (0, 1, 0)), ((2, 1, 2), B_1, (2, 1, 2)), ((1, 2, 1), B_2, (1, 2, 1)), ((0, 2, 0,) B_2, (0, 2, 0)), ((2, 0, 2), B_0, (2, 0, 2)), ((1, 0, 1), B_0, (1, 0, 1))\}$. Thus $\mathcal{T}(\mathcal{Q}(G)) = (E(\mathcal{Q}(G)), F(\mathcal{Q}(G)))$ and as $c(G)$ has 3 elements, $c((\mathcal{T}\mathcal{Q}(G))) = E(\mathcal{Q}(G))$ has 9 elements the structures G and $\mathcal{T}(\mathcal{Q}(G))$ cannot be isomorphic. If we put $f((0, 1, 0)) = f((0, 2, 0)) = 0$, $f((1, 0, 1)) = f((1, 2, 1)) = 1$, $f((2, 0, 2)) = f((2, 1, 2)) = 2$, $f(B_0) = 0$, $f(B_1) = 1$, $f(B_2) = 2$, then f is a strong homomorphism of $\mathcal{T}(\mathcal{Q}(G))$ onto G.

4.4. Remark. Denote *Quas* the category of quasiordered sets with isotonic mappings as morphisms and *Tern* the category of transitive ternary structures without isolated elements and such that $r(G) = A(G)$ with obviously defined morphisms. For morphisms $h : G \to G'$ $(G, G') \in$ *Tern* and $k : Q \to Q'$ $(Q, Q' \in$ *Quas*) define $\mathcal{Q}(h) : \mathcal{Q}(G) \to \mathcal{Q}(G')$ and $\mathcal{T}(k) : \mathcal{T}(Q) \to \mathcal{T}(Q')$ in an expected way. Then $\mathcal{Q}$: *Tern* $\to$ *Quas* and $\mathcal{T}$: *Quas* $\to$ *Tern* are covariant functors.

REFERENCES

[1] P. Alles, *Erweiterungen, Diagramme und Dimension zyklischer Ordnungen*. Doctoral Thesis, Darmstadt 1986.

[2] M. Altwegg, *Zur Axiomatik der teilweise geordneten Mengen*. Comment. Math. Helv. 24 (1950), 149–155.

[3] G. Birkhoff, *Lattice Theory* (2nd edition). New York, 1948.

[4] E. Čech, *Bodové množiny* (Point Sets). Praha, 1974.

[5] N. Megiddo, *Partial and complete cyclic order*. Bull. Am. Math. Soc. 82 (1976), 274–276.

[6] G. Müller, *Lineare und zyklische Ordnung*. Praxis Math. 16 (1974), 261–269.

[7] V. Novák, *Cyclically ordered sets.* Czech. Math. J. 32 (1982), 460–473.
[8] V. Novák, M. Novotný, *Dimension theory for cyclically and cocyclically ordered sets.* Czech. Math. J. 33 (1983), 647–653.
[9] V. Novák, M. Novotný, *Universal cyclically ordered sets.* Czech. Math. J. 35 (1985), 158–161.
[10] V. Novák, M. Novotný, *On completion of cyclically ordered sets.* Czech. Math. J. 37 (1987), 407–414.
[11] M. Sekanina, *Graphs and betweeness.* Matem. čas. 25 (1975), 41–47.

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