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FINITENESS OF THE SET OF SOLUTIONS OF SOME BOUNDARY-VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. It is shown that for second order ordinary differential equations with analytical nonlinearities and Dirichlet or Neumann boundary conditions, there exist a most finitely many solutions if they are a priori bounded. Similar results hold for first order ordinary differential equations with periodic boundary conditions.

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I. INTRODUCTION

The obtention of existence results for nonlinear boundary value problems by topological degree methods require in general the obtention of a priori bounds for all possible solutions of a family of problems containing the original one (see e.g. [1, 5, 6] for a survey and the literature therein).

In this note, we show that for second order ordinary differential equations with analytical nonlinearities and Dirichlet or Neumann boundary conditions, a priori bounds for their possible solutions imply the existence of at most finitely many solutions. This is a consequence of a shooting type argument and of the properties of zeros of real analytic functions. This argument fails in the case of periodic boundary conditions because the corresponding Poincaré mapping is an analytic mapping of two variables. However, the method can be applied to the periodic problem for a first-order scalar differential equation and provides, as special case, a very easy proof of a result of Pliss [9] for polynomial right-hand sides.

Finally, we can mention that our results can be motivated by a special averaging method given in [2].

II. FINITE NUMBERS OF SOLUTIONS FOR THE DIRICHLET AND NEUMANN PROBLEMS

Let us consider the second order differential equation

$$x'' + f(t, x, x') = 0 \tag{1}$$

with the boundary conditions

$$x(0) = x(\pi) = 0 \tag{2}$$

or

$$x'(0) = x'(\pi) = 0. \tag{3}$$

In the following we assume that

i) $f : [0, \pi] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous

and

ii) for each $t \in [0, \pi]$, $f(t, ., .)$ is analytic on $\mathbf{R}^2$.

By classical results on the Cauchy problem, the initial value problem

$$\begin{gathered} x'' + f(t, x, x') = 0, \\ x(0) = 0, \qquad x'(0) = y \end{gathered} \tag{4}$$

has for each $y \in \mathbf{R}$ a unique local solution $\xi(t, y)$ such that $\xi(t, .)$ is analytic for each t for which the solution exists. Moreover, the set Ω_f of $y \in \mathbf{R}$ such that $\xi(., y)$ is defined at least over $[0, \pi]$ is a (possibly empty) open subset of $\mathbf{R}$.

Theorem 1. *Assume that there exists a compact subset K of* $\mathbf{R}$ *contained in Ω_f and such that each possible solution x of* (1), (2) *satisfies the condition*

$$x'(0) \in K.$$

Then the problem (1), (2) *has at most finitely many solutions.*

Proof. It is well known that x is a solution of (1), (2) if and only if $y = x'(0) \in \Omega_f$ and y satisfies the equation

$$\xi(\pi, y) = 0.$$

If Ω_f is empty, we are done, and if not, the assumption $y \in K \subset \Omega_f$ and the properties of the set Z of zeros of the real analytic function $\xi(\pi, .)$ implies that Z is at most finite and the same is true for the set of solutions of (1), (2).

Corollary 1. *Assume that $\Omega_f = \mathbf{R}$ and that there exists $R > 0$ such that each possible solution x of* (1), (2) *satisfies the inequality*

$$| x'(0) | \leq R. \tag{5}$$

Then the conclusion of Theorem 1 holds.

Proof. Take $K = [-R, R]$.

For $u : [0, \pi] \to \mathbf{R}$ continuous on $[0, \pi]$, let us write

$$\| u \|_\infty = \max_{t \in [0, \pi]} | u(t) |.$$

Corollary 2. *Assume that there exists $M > 0$ such that each possible solution x of (1), (2) satisfies the inequality*

$$\| x' \|_\infty \leqq M. \tag{6}$$

Then the conclusion of Theorem 1 holds.

Proof. Notice first that (6) and (2) imply that $\| x \|_\infty \leqq \int_0^\pi | x'(s) | \, ds \leqq \pi M$. Let us show now that the set Z of zeros of $\xi(\pi, .)$ is closed in $\mathbf{R}$. Let (y_n) be a sequence in Z which converges to y. Then each $y_n \in \Omega_f$ and $\xi(., y_n)$ is a solution of (1), (2) so that we have

$$| \xi(t, y_n) | + | \xi_t'(t, y_n) | \leqq (1 + \pi) M, \qquad t \in [0, \pi], n \in \mathbf{N}$$

by condition (6). If $y \notin \Omega_f$, it follows from classical results on the extendability of solutions of the Cauchy problem that $\xi(., y)$ is defined on $[0, b[$ for some $b \in \,]0, \pi[$ and that

$$\lim_{t \to b-} [| \xi(t, y) | + | \xi_t'(t, y) |] = +\infty.$$

Hence there exists $b' \in \,]0, b[$ such that

$$| \xi(b', y) | + | \xi_t'(b', y) | > (1 + \pi) M.$$

Now, $(\xi(., z), \xi_t'(., z))$ is defined on $[0, b']$ for all z in a neighborhood V of y, on which $(\xi(b', .), \xi_t'(b', .))$ is continuous. Thus,

$$(1 + \pi) M \geqq \lim_{n \to \infty} [| \xi(b', y_n) | + | \xi_t'(b', y_n) |] =$$
$$= | \xi(b', y) | + | \xi_t'(b', y) | > (1 + \pi) M,$$

a contradiction. Thus Z is closed in $\mathbf{R}$. Now, by (6) we have

$$| x'(0) | \leqq M$$

for all possible solutions x of (1), (2), so that $Z \subset \Omega_f \cap [-M, M]$. Thus Z is a compact subset of $\mathbf{R}$ contained in Ω_f and the result follows from Theorem 1.

Remark 1. Condition (5) of Corollary 1 holds in particular when a constant $R > 0$ exists such that

$$\int_0^\pi | x''(t) | \, dt \leqq R$$

whenever x is solution of (1), (2). This can be seen immediately by Rolle's theorem which implies, because of (2), that $x'(\tau) = 0$ for some $\tau \in \,]0, \pi[$, and hence

$$|x'(0)| = |x'(\tau) - x'(0)| = \left|\int_0^\tau x''(t)\,\mathrm{d}t\right| \leqq \int_0^\pi |x''(t)|\,\mathrm{d}t \leqq R.$$

Remark 2. Condition (6) of Corollary 2 holds in particular when there exists a constant M_0 such that

$$\|x\|_\infty \leqq M_0 \tag{7}$$

for all possible solutions of (1), (2) and when, in addition, f satisfies a Nagumo condition

$$|f(t, x, y)| \leqq \varphi(|y|)$$

for $t \in [0, \pi]$, $|x| \leqq M_0$ and $y \in \mathbf{R}$, where $\varphi : \mathbf{R}_+ \to \mathbf{R}_+ \backslash \{0\}$ is continuous and such that

$$\int_0^\infty \frac{s\,\mathrm{d}s}{\varphi(s)} = +\infty.$$

(see e.g. [1, 5]). This is in particular the case if f does not depend upon y.

We now list some conditions upon f which imply that the assumptions of the above corollaries hold.

For simplicity, we consider the case of

$$x'' + f(t, x) = 0, \tag{8}$$

$$x(0) = x(\pi) = 0 \tag{9}$$

with f continuous on $[0, \pi] \times \mathbf{R}$ and $f(t, .)$ analytic on $\mathbf{R}$ for each $t \in [0, \pi]$, so that Remark 2 above can be used. Indeed, each of the following conditions insures that condition (7) holds for the possible solutions of (8), (9) and together with a Leray–Schauder's type argument, they imply indeed the existence of at least one solution, and hence of finitely many by Corollary 2.

Corollary 3. *The problem* (8), (9) *has a finite number of solutions if one of the following conditions hold*:

1. *There exists* $R > 0$ *such that*

$$xf(t, x) < 0$$

for $t \in [0, \pi]$ *and* $|x| \geqq R$.

2. *There exists* $\beta \in L^1(0, \pi)$ *such that* $\beta(t) \leqq 1$ a.e. *on* $[0, \pi]$, $\beta(t) < 1$ *on a subset of positive measure and*

$$\limsup_{|x| \to \infty} x^{-1} f(t, x) \leqq \beta(t)$$

uniformly a.e. *on* $[0, \pi]$.

3. *There exists a positive integer m and L^∞-functions α and β such that* $m^2 \leqq \alpha(t) \leqq \beta(t) \leqq (m+1)^2$ a.e. *on* $[0, \pi]$, $m^2 < \alpha(t)$ *and* $\beta(t) < (m+1)^2$ *on subsets of positive measure and*

$$\alpha(t) \leqq \liminf_{|x|\to\infty} x^{-1} f(t, x), \qquad \limsup_{|x|\to\infty} x^{-1} f(t, x) \leqq \beta(t)$$

uniformly a.e. *on* $[0, \pi]$.

Proof. See [5] for assumption 1, [6] for assumption 2 and [7, 8] for assumption 3.

Remark 3. When

$$\lim_{|x|\to\infty} x^{-1} f(t, x) = +\infty,$$

the set of solutions of (8), (9) can be infinite, as shown by Ehrmann in [3]. The same is true when

$$\lim_{|x|\to\infty} x^{-1} f(t, x) = m^2$$

for some $m \in \mathbf{N}\backslash\{0\}$, as shown by the linear problem

$$x'' + m^2 x = g(t)$$
$$x(0) = x(\pi) = 0$$

with $g : [0, \pi] \to \mathbf{R}$ continuous and such that

$$\int_0^\pi g(t) \sin mt \, dt = 0.$$

Similar results can be proved in an entirely analogous way for the Neumann boundary value problem

$$x'' + f(t, x, x') = 0, \tag{10}$$
$$x'(0) = x'(\pi) = 0 \tag{11}$$

with the same regularity conditions upon f. The reader will easily adapt the proofs to the new situation. We shall denote by $\eta(t, y)$ the solution of the initial value problem

$$x'' + f(t, x, x') = 0,$$
$$x(0) = y, \qquad x'(0) = 0$$

and by Δ_f the (open) set of $y \in \mathbf{R}$ such that $\eta(., y)$ is defined at least over $[0, \pi]$.

Theorem 2. *Assume that there exists a compact subset K of* $\mathbf{R}$ *contained in Δ_f and such that each possible solution x* of (10), (11) *satisfies the condition*

$$x(0) \in K.$$

Then the problem (10), (11) *has at most finitely many solutions.*

Corollary 4. *Assume that $\Delta_f = \mathbf{R}$ and that there exists $R > 0$ such that each possible solution x* of (10), (11) *satisfies the inequality*

$$| x(0) | \leqq R.$$

Then the conclusion of Theorem 2 holds.

Corollary 5. *Assume that there exists $M > 0$ such that each possible solution x* of (10), (11) *satisfies the inequality*

$$\| x \|_\infty + \| x' \|_\infty \leqq M.$$

Then the conclusion of Theorem 2 holds.

Finally, Remark 2 holds for (10), (11) as well and, in Corollary 3, the assumptions have to be modified as follows:

a) in condition 2, replace "$\beta(t) \leqq 1$" and "$\beta(t) < 1$" respectively by "$\beta(t) \leqq 0$" and "$\beta(t) < 0$";

b) in condition 3, replace "positive integer" by "nonnegative integer".

The above results can be applied to the forced pendulum equation with Dirichlet boundary conditions

$$x'' + a \sin x = e(t), \tag{12}$$

$$x(0) = x(\pi) = 0, \tag{13}$$

where $e : [0, \pi] \to \mathbf{R}$ is continuous and $a \in \mathbf{R}$, as condition 2 of Corollary 3 holds. Thus, (12), (13) has finitely many solutions. The same argument does not work for the Neumann conditions

$$x'(0) = x'(\pi) = 0 \tag{14}$$

as

$$\lim_{|x| \to \infty} x^{-1}[a \sin x - e(t)] = 0$$

uniformly in $t \in [0, \pi]$. Indeed, (12), (14) has infinitely many solutions as $x + 2k\pi$, $k \in \mathbf{Z}$ always are solutions of (12), (14) together with x. However, in this case, we can take advantage of the periodicity in x of the nonlinear term to obtain finiteness results on the numbers of solutions modulo 2π.

In addition to the regularity assumption (i) and (ii), let us assume that

(iii) there exists $P > 0$ such that

$$f(t, x + P, y) = f(t, x, y)$$

for all $t \in [0, \pi]$, $x \in \mathbf{R}$ and $y \in \mathbf{R}$.

Theorem 3. *If* (iii) *holds and if $\Delta_f = \mathbf{R}$, then either $\eta(., y)$ is a solution of* (10), (11) *for each $y \in \mathbf{R}$ or* (10), (11) *has at most finitely many solutions modulo P.*

Proof. With the notations above, x is solution of (10), (11) if and only if $x(t) = \eta(t, y)$ with y solution of

$$\varphi(y) := \eta_t'(\pi, y) = 0.$$

By the uniqueness for the Cauchy problem and the P-periodicity of f in x, we necessarily have

$$\varphi(y + P) = \varphi(y)$$

for $y \in \mathbf{R}$ and hence φ is analytic and P-periodic. If φ is identically zero, then $\eta(., y)$ is a solution of (10), (11) for each $y \in \mathbf{R}$; if not, φ has a finite number of zeros in $[0, P]$ and (10), (11) has at most finitely many solutions modulo P.

Remark 4. The case of $f \equiv 0$ shows that the first conclusion of Theorem 3 can be realized.

Remark 5. Other boundary conditions than the Dirichlet and the Neumann ones can be treated as well by a similar approach, for example

$$x(0) = x'\left(\frac{\pi}{2}\right) = 0$$

or

$$x'(0) = x\left(\frac{\pi}{2}\right) = 0$$

for which Theorem 1 and Corollaries 1 and 2 as well as Remark 2 hold with trivial modifications in the proofs. Corollary 3 also holds with "positive integer m" replaced by "positive odd integer m".

III. THE CASE OF PERIODIC SOLUTIONS

The study of boundary value problems for the second order differential equation (1), specially by the functional analytic approach, shows that existence results for the Neumann boundary conditions (3) are in general valid also for the periodic boundary conditions

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0. \tag{15}$$

The reason is that the spectrum of the linearized problem is the same for those two sets of boundary conditions. This similarity is not complete however, as shown for example in bifurcation theory, where the fact that all eigenvalues are simple in the Neumann case and all positive eigenvalues double in the periodic case makes the two situations very different. Hence it is interesting to discuss the possibility of extending the above results to the case of the boundary conditions (15).

If we assume again that the regularity conditions (i) and (ii) of Section II are satisfied, and if we denote by $\zeta(t, y_1, y_2)$ the solution of the initial value problem

$$x'' + f(t, x, x') = 0$$
$$x(0) = y_1, \qquad x'(0) = y_2,$$

then x will be a solution of (1), (15) if and only if $x(t) = \zeta(t, y_1, y_2)$ with (y_1, y_2) solution of the system of equations

$$y_1 - \zeta(2\pi, y_1, y_2) = 0 = y_2 - \zeta_t'(2\pi, y_1, y_2).$$

Again, if $\Gamma_f \subset \mathbf{R}^2$ denotes the set of (y_1, y_2) such that $\zeta(., y_1, y_2)$ is defined at least on $[0, 2\pi]$, the mapping $\varphi : \Gamma_f - \mathbf{R}^2$, $(y_1, y_2) \to (y_1 - \zeta(2\pi, y_1, y_2), y_2 - - \zeta_t'(2\pi, y_1, y_2))$ will be analytic but the results on the zeros of real analytic functions used in Section I a.e not valid for higher dimensions. It is only in the case where equation (1) posses some symmetries with respect to t or x that the solutions of some of the boundary-value problems considered in Section II can be extended to 2π-periodic solutions and the results of this section then give information about the finiteness of the set of those special symmetric periodic solutions. The reader can consult [4] for the link between symmetric periodic solutions and two-point boundary value problems, and [10, 11] for interesting numerical results in this direction for the forced pendulum equation.

We can however proceed like in Section I in the case of the periodic boundary-value problem for a first order scalar differential equation

$$\text{(16)} \qquad \begin{aligned} x' &= f(t, x) \\ x(0) - x(2\pi) &= 0 \end{aligned}$$

when we assume again that

i') $f : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ is continuous

and

ii') for each $t \in [0, 2\pi]$, $f(t, .)$ is analytic on $\mathbf{R}$.

Again, the inital value problem

$$x' = f(t, x),$$
$$x(0) = y$$

has for each $y \in \mathbf{R}$ a unique local solution $\gamma(t, y)$ and $\gamma(t, .)$ is analytic for each t for which the solution exists. Moreover, the set Γ_f of $y \in \mathbf{R}$ such that $\gamma(., y)$ is defined at least over $[0, 2\pi]$ is a (possibly empty) open subset of $\mathbf{R}$. Finally, x is a solution of (16) if and only if $y = x(0) \in \Gamma_f$ and satisfies the equation

$$\psi(y) := y - \gamma(2\pi, y) = 0.$$

It is therefore easy to mimic the proofs in Section II to get the following results:

Theorem 4. *Assume that there exists a compact subset K of $\mathbf{R}$ contained in Γ_f and such that each possible solution x of* (16) *satisfies the condition*

$$x(0) \in K.$$

Then the problem (16) *has at most finitely many solutions.*

Corollary 6. *Assume that $\Gamma_f = \mathbf{R}$ and that there exists $R > 0$ such that each possible solution of* (16) *satisfies the inequality*

$$|x(0)| \leqq R.$$

Then the conclusion of Theorem 4 holds.

Corollary 7. *Assume that there exists $M > 0$ such that each possible solution x of* (16) *satisfies the inequality*

$$\max_{0 \leqq t \leqq 2\pi} |x(t)| \leqq M. \tag{17}$$

Then the conclusion of Theorem 4 holds.

Finally, we can state and prove a sufficient condition on f for which (17) holds.

Corollary 8. *Assume that there exists $M > 0$ such that*

$$f(t, x) \neq 0 \tag{18}$$

for all $t \in [0, 2\pi]$ and $|x| > M$. Then (17) *holds for each possible solution of* (16) *and hence* (16) *has at most finitely many solutions.*

Proof. Let x be a solution of (16) and let $\tau \in [0, 2\pi]$ be such that

$$x(\tau) = \max_{t \in [0, 2\pi]} x(t).$$

If $\tau \in \,]0, 2\pi[$, then

$$0 = x'(\tau) = f(\tau, x(\tau))$$

and hence, by (18),

$$x(\tau) \leqq M.$$

If $\tau = 0$ or 2π, then

$$x(0) = x(2\pi) = \max_{t \in [0, 2\pi]} x(t)$$

so that

$$x'(0) \leqq 0 \leqq x'(2\pi)$$

i.e., as $x(0) = x(2\pi)$,

$$f(0, x(0)) \leqq 0 \leqq f(2\pi, x(0)).$$

By the intermediate value theorem, there exists $\tau_1 \in [0, 2\pi]$ such that

$$f(\tau_1, x(0)) = 0$$

and hence, by (18)

$$x(0) \leqq M.$$

One finds similarly that if $\tau' \in [0, 2\pi]$ is such that

$$x(\tau') = \min_{t \in [0, 2\pi]} x(t),$$

one has

$$x(\tau') \geqq -M$$

and the proof is complete.

The assumptions of Corollary 8 hold in particular for the equation

$$x' = x^n + p_1(t)\, x^{n-1} + \ldots + p_{n-1}(t)\, x + p_n(t), \tag{19}$$

where n is a positive integer and the $p_j : [0, 2\pi] \to \mathbf{R}$ are continuous, so that (19) has at most finitely many 2π-periodic solutions. This is a result proved in [9] in a somewhat more complicated way.

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