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Jiří Svoboda
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QUASI-UNIFORMISATION OF CLOSURE SPACES

JIŘÍ SVOBODA

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Abstract. It is shown that the set of all quasi-uniformities (in the sense of Isbell) inducing a weakly regular closure structure forms a complete lattice. A variant of precompactness and completness is given.

Key words. Closure structure, quasi-uniformity, coarse and fine quasiuniformities.

MS Classification. 54 E 15.

NOTATION

Let X be a set, $\exp X = \{A/A \subset X\}$. The complement of $A \in \exp X$ is denoted by A^c.

For $\mathscr{A}, \mathscr{B} \subset \exp X$, $C \subset X$, $x \in X$ we define $\mathscr{A} \prec \mathscr{B} \Leftrightarrow (\forall A \in \mathscr{A})$ $(\exists B \in \mathscr{B})$ $(A \subset B)$ and put $\mathscr{A} \wedge \mathscr{B} = \{A \cap B/A \in \mathscr{A}, B \in \mathscr{B}\}$,
$\mathrm{St}(C, \mathscr{A}) = \bigcup\{A \in \mathscr{A}/C \cap A \neq \emptyset\}$,
$\mathrm{St}(x, \mathscr{A}) = \mathrm{St}(\{x\}, \mathscr{A})$.
If $\mathbf{A} \subset \exp\exp X$, we write
$\mathrm{St}(x, \mathbf{A}) = \{\mathrm{St}(x, \mathscr{A})/\mathscr{A} \in \mathbf{A}.\}$
$\mathscr{A} \in \mathrm{Cov}\, X$ means $\emptyset \neq \mathscr{A} \subset \exp X$ and $\bigcup\mathscr{A} = X$.
Finally $\mathscr{F} \in \mathrm{Filt}\, X$ means that $\mathscr{F}$ is a proper filter on X.

PRELIMINARY REMARKS

Let us recall some definitions and known facts.

P1. We write $\mathbf{U} \in \mathrm{Qn}X$ and call $\mathbf{U}$ a quasi-uniformity on X and $(X, \mathbf{U})$ a quasi-uniform space iff $\emptyset \neq \mathbf{U} \subset \mathrm{Cov}\, X$ and

$$\mathscr{U}_1, \mathscr{U}_2 \in \mathbf{U} \Rightarrow \mathscr{U}_1 \wedge \mathscr{U}_2 \in \mathbf{U};$$
$$\mathscr{U} \in \mathbf{U}, \qquad \mathscr{U} \prec \mathscr{U}' \subset \exp X \Rightarrow \mathscr{U}' \in \mathbf{U}.$$

Let $\mathbf{U} \in \mathrm{Qn}X$, $\mathbf{V}$ is a u-base for $\mathbf{U}$ iff

$$\mathbf{U} = \{\mathscr{U} \subset \exp X / (\exists\, \mathscr{V} \in \mathbf{V})\, (\mathscr{V} \prec \mathscr{U})\};$$

$\mathbf{W}$ is a u-subbase for $\mathbf{U}$ iff

$$\{\mathscr{W}_1 \wedge \ldots \wedge \mathscr{W}_n / \mathscr{W}_1, \ldots, \mathscr{W}_n \in \mathbf{W},\ n = 1, 2, \ldots\}$$

is a u-base for $\mathbf{U}$. We also say that $\mathbf{W}$ generates $\mathbf{U}$. It follows easily that $(\mathrm{Qn}X, \subset)$ is a complete lattice and if $\{\mathbf{U}_\alpha;\ \alpha \in A\}$ is a non-void family in $\mathrm{Qn}X$, then $\cup \{\mathbf{U}_\alpha;\ \alpha \in A\}$ is a u-sub-base for $\sup_\alpha \mathbf{U}_\alpha$ in $(\mathrm{Qn}X, \subset)$. [2]

P2. A closure space will be given in the form $(X, \mathfrak{N})$ where $\mathfrak{N}(x)$ denotes the neighbourhood-filter at $x \in X$. By $A^-(A^i)$ we denote the closure (interior) of $A \subset X$ in $(X, \mathfrak{N})$. $\mathfrak{N}$ itself will be termed as a closure structure (although "n-hood structure" would be more adequate).

A closure space $(X, \mathfrak{N})$ is compact iff one of the two equivalent conditions is fulfilled:

(a) $\mathscr{F} \in \mathrm{Filt}\, X \Rightarrow \mathscr{F}$ clusters in $(X, \mathfrak{N})$;

(b) $\mathscr{A} \subset \exp X$, $\{A^i / A \in \mathscr{A}\} \in \mathrm{Cov}\, X \Rightarrow$

$$(\exists\, n)\, (\exists\, A_1, \ldots, A_n \in \mathscr{A})\, (\{A_1, \ldots, A_n\} \in \mathrm{Cov}\, X). \ [1]$$

RESULTS

1. Proposition and definition. *Let* $(X, \mathfrak{N})$ *be a closure space. Then*

(1) $(\forall\, x, y \in X)\, (y \in \{x\}^- \Rightarrow x \in \{y\}^-)$;

(2) $(\forall\, x \in X)\, (\{x\}^- = \cap \{N / N \in \mathfrak{N}(x)\})$

are equivalent conditions and $(X, \mathfrak{N})$ *is an* S_1*-closure space* (*weakly regular*) *iff* (1) *holds.*

Proof. (1) $\Rightarrow$ (2): Let $y \in \{x\}^-$, so that $x \in \{y\}^-$. If $N \in \mathfrak{N}(x)$ is arbitrary, we get $N \cap \{y\} \neq \emptyset$, so that $y \in N$. It follows $\{x\}^- \subset \bigcap \{N / N \in \mathfrak{N}(x)\}$. If conversely $y \notin \{x\}^-$, then $x \notin \{y\}^-$, so that for some $N_0 \in \mathfrak{N}(x)$ we have $N_0 \cap \{y\} = \emptyset$, $y \notin N_0$. Hence equality holds.

(2) $\Rightarrow$ (1): Let $y \in \{x\}^-$. By (2) $N \cap \{y\} \neq 0$ for all $N \in \mathfrak{N}(x)$, so that $x \in \{y\}^-$.

2. Proposition and definition. *Let* $(X, \mathbf{U})$ *be a quasiuniform space. For* $x \in X$ *put* $\mathfrak{N}(x) = \mathrm{St}\,(x, \mathbf{U})$. *Then* $(X, \mathfrak{N})$ *is an* S_1*-closure space. We write* $\mathfrak{N} = \mathrm{St}\,(-, \mathbf{U})$ *and call* $\mathrm{St}\,(-, \mathbf{U})$ *a closure structure induced by* $\mathbf{U}$.

Proof. Clearly $X \in \mathfrak{N}(x)$, $N \in \mathfrak{N}(x) \Rightarrow x \in N$ and $N_1, N_2 \in \mathfrak{N}(x) \Rightarrow N_1 \cap N_2 \in \mathfrak{N}(x)$. (Since $\mathrm{St}\,(x, \mathscr{U}_1 \wedge \mathscr{U}_2) = \mathrm{St}\,(x, \mathscr{U}_1) \cap \mathrm{St}\,(x, \mathscr{U}_2)$).

Let $\mathscr{U} \in \mathbf{U}$ and $\mathrm{St}\,(x, \mathscr{U}) \subset A \subset X$. Put $\mathscr{U}' = \mathscr{U} \cup \{A\}$, so that $\mathscr{U} \prec \mathscr{U}'$, $\mathscr{U}' \in \mathbf{U}$ and $\mathrm{St}\,(x, \mathscr{U}') = A$.

Since $y \in \mathrm{St}\,(x, \mathscr{U} \Leftrightarrow x \in \mathrm{St}\,(y, \mathscr{U})$ we get $\{x\}^- = \bigcap \mathrm{St}\,(x, \mathscr{U})$, so that $(X, \mathfrak{N})$ is an S_1-closure space.

3. Definition. *Let* $(X, \mathfrak{N})$ *be a closure space and* $\mathbf{U} \in \mathrm{Qn}X$. $\mathbf{U}$ *is continuous on* $(X, \mathfrak{N})$ *iff* $\mathrm{id}_X\colon (X, \mathfrak{N}) \to (X, \mathrm{St}(-, \mathbf{U}))$ is continuous, or – equivalently $\forall x \in X : \mathrm{St}(x, \mathbf{U}) \subset \mathfrak{N}(x)$.

4. Proposition. *Let* $\mathbf{U} \in \mathrm{Qn}X$ *be continuous on a closure space* $(X, \mathfrak{N})$. *Then* $\mathbf{U}$ *induces* $\mathfrak{N}$ *iff the following condition is satisfied: If* $\{x(\mathcal{U});\ \mathcal{U} \in \mathbf{U}\}$ *is an arbitrary net in* X *(defined on the down-ward directed set* $(\mathbf{U}, <)$*) and* $x \in \bigcap \{\mathrm{St}(x(\mathcal{U}), \mathcal{U}) \mid \mathcal{U} \in \mathbf{U}\}$, *then* $x(\mathcal{U}) \to x$ *in* $(X, \mathfrak{N})$.

Proof. $\Rightarrow$: Let $x \in \bigcap \{\mathrm{St}(x(\mathcal{U}), \mathcal{U}) / \mathcal{U} \in \mathbf{U}\}$. Let $N \in \mathfrak{N}(x)$. Since $\mathfrak{N}(x) = \mathrm{St}(x, \mathbf{U})$, $N = \mathrm{St}(x, \mathcal{U}_0)$ for some $\mathcal{U}_0 \in \mathbf{U}$ and we have $\mathcal{U} \in \mathbf{U}, \mathcal{U} \prec \mathcal{U}_0 \Rightarrow x \in \mathrm{St}(x(\mathcal{U}), \mathcal{U}) \Rightarrow x(\mathcal{U}) \in \mathrm{St}(x, \mathcal{U}) \subset \mathrm{St}(x, \mathcal{U}_0) = N$, so that $x(\mathcal{U}) \to x$ in $(X, \mathfrak{N})$. Assume conversely that $\mathrm{St}(x, \mathcal{U}) \neq \mathfrak{N}(x)$ for some $x \in X$. There is $N \in \mathfrak{N}(x)$ such that $\forall \mathcal{U} \in \mathbf{U}\colon \mathrm{St}(x, \mathcal{U}) \not\subset N$ and for each $\mathcal{U} \in \mathbf{U}$ we can select $x(\mathcal{U})$ with $x(\mathcal{U}) \in \mathrm{St}(x, \mathcal{U})$ and $x(\mathcal{U}) \notin N$. It follows $x \in \bigcap \{\mathrm{St}(x(\mathcal{U}), \mathcal{U}) \mid \mathcal{U} \in U\}$ and $x(\mathcal{U}) \not\to x$.

5. Proposition and definition. *Let* $(X, \mathfrak{N})$ *be an* S_1*-closure space. Put*
$\mathbf{W} = \{\{\{x\}^c, N\} / N \in \mathfrak{N}(x),\ x \in X\}$.
If $\mathbf{U}_0 \in \mathrm{Qn}X$ *is generated by* $\mathbf{W}$, *then* $\mathbf{U}_0$ *induces* $\mathfrak{N}$ *and is called the coarse quasi-uniformity on* $(X, \mathfrak{N})$.

Proof. a) Let $\mathcal{W} = \{\{x\}^c, N\} \in \mathbf{W}$ and $y \in X$ be arbitrary. If $y = x$, then $\mathrm{St}(y, \mathcal{W}) = N$. If $y \in N$ and $y \neq x$, then $\mathrm{St}(y, \mathcal{W}) = X$. If finally $y \in N^c$, then $\mathrm{St}(y, \mathcal{W}) = \{x\}^c$ and since $y \notin \{x\}^-$ – on account of S_1 – $y \in \{x\}^{ci}$. It follows $\mathrm{St}(y, \mathcal{W}) \in \mathfrak{N}(y)$ for all $y \in X$.

b) Let $\mathcal{U} \in \mathbf{U}_0$, so that $\mathcal{W}_1 \wedge \ldots \wedge \mathcal{W}_n \prec \mathcal{U}$, for some n and $\mathcal{W}_1, \ldots, \mathcal{W}_n \in \mathbf{W}$.. It follows $\mathrm{St}(y, \mathcal{W}_1) \wedge \ldots \wedge \mathrm{St}(y, \mathcal{W}_n) \subset \mathrm{St}(y, \mathcal{U})$, so that $\mathrm{St}(y, \mathcal{U}) \in \mathfrak{N}(y)$ for each $y \in X$ on account of a). Hence $\mathbf{U}_0$ is continuous on $(X, \mathfrak{N})$.

c) If $x \in X$ and $N \in \mathfrak{N}(x)$, then $\mathcal{U} = \{\{x\}^c, N\} \in \mathbf{U}_0$ and $\mathrm{St}(x, \mathcal{U}) = N$, so that $\mathbf{U}_0$ induces $\mathfrak{N}$.

6. Proposition and definition. *Let* $(X, \mathfrak{N})$ *be an* S_1*-closure space and for* $\mathcal{U} \subset \exp X$ *define.* $\mathcal{U} \in \mathbf{U}_1 \Leftrightarrow (\forall x \in X)\ (\mathrm{St}(x, \mathcal{U}) \in \mathfrak{N}(x))$. *Then* $\mathbf{U}_1$ *is a quasi-uniformity on* X. *that induces* $\mathfrak{N}$. $\mathbf{U}_1$ *is called the fine quasi-uniformity on* $(X, \mathfrak{N})$.

Proof. Clearly $\{X\} \in \mathbf{U}_1$, $\emptyset \neq \mathbf{U}_1 \subset \mathrm{Cov}\,X$, and it follows easily that $\mathbf{U}_1$ is a quasi-uniformity on X. By its construction it is continuous on $(X, \mathfrak{N})$, and since it contains the coarse quasi-uniformity, it induces $\mathfrak{N}$ as well.

7. Theorem. *Let* $(X, \mathfrak{N})$ *be an* S_1*-closure space. Then the* set $\mathrm{Qn}(X, \mathfrak{N})$ *of all quasi-uniformities on* X *inducing* $\mathfrak{N}$, *and ordered by inclusion is a complete lattice. The minimal (maximal) element of this lattice is the coarse (fine) quasi-uniformity on* $(X, \mathfrak{N})$.

Proof. a) Let $\mathbf{U}_0$ and $\mathbf{U}_1$ be the coarse and the fine quasi-uniformities on $(X, \mathfrak{N})$.

Assume $\mathbf{U} \in \mathrm{Qn}(X, \mathfrak{N})$. If $N \in \mathfrak{N}(x)$, then $\mathrm{St}(x, \mathbf{U}) = \mathfrak{N}(x)$, so that $N = \mathrm{St}(x, \mathcal{U})$ foı some $\mathcal{U} \in \mathbf{U}$. It follows $\mathcal{U} \prec \{\{x\}^c, N\}$, and—with the notation from 5—we get $\mathbf{W} \subset \mathbf{U}$, so that $\mathbf{U}_0 \subset \mathbf{U}$. The relation $\mathbf{U} \subset \mathbf{U}_1$ is obvious.

b) Let $\{\mathbf{U}_\alpha; \alpha \in \Lambda\}$ be a non-void family in $\mathrm{Qn}(X, \mathfrak{N})$. Put $\mathbf{U} = \sup\{\mathbf{U}_\alpha; \alpha \in \Lambda\}$, where the supremum is taken in $(\mathrm{Qn}X, \subset)$ (see P1). Let $\mathcal{U} \in \mathbf{U}$, $x \in X$. For some n and $\alpha_1, \ldots, \alpha_n \in \Lambda$ and $\mathcal{U}_k \in \mathbf{U}_\alpha$ $(k = 1, \ldots, n)$ we have $\mathcal{U}_1 \wedge \ldots \wedge \mathcal{U}_n \prec \mathcal{U}$. It follows $\mathrm{St}(x, U_1) \cap \ldots \cap \mathrm{St}(x, \mathcal{U}_n) \subset \mathrm{St}(x, \mathcal{U})$, and since $\mathrm{St}(x, \mathcal{U}_k) \in \mathfrak{N}(x)$ for $k = 1, \ldots, n$, we get $\mathrm{St}(x, \mathcal{U}) \in \mathfrak{N}(x)$, so that $\mathbf{U}$ is continuous on $(X, \mathfrak{N})$. Fix $\alpha \in \Lambda$. Since $\mathbf{U}_\alpha \subset \mathbf{U}$, we have for $x \in X$: $\mathfrak{N}(x) = \mathrm{St}(x, \mathbf{U}_\alpha) \subset \mathrm{St}(x, \mathbf{U})$, so that $\mathbf{U} \in \mathrm{Qn}(X, \mathfrak{N})$.

8. Remark. Let $\mathfrak{N}$ be the closure structure induced by $\mathbf{U} \in \mathrm{Qn}X$. Recall that $\mathbf{U}$ is a nearness on X iff $\{\{U^i/U \in \mathcal{U}\} \mid \mathcal{U} \in \mathbf{U}\}$ is a u-base for $\mathbf{U}$. [3] It follows easily that in this case $(X, \mathfrak{N})$ is a topological S_1-space and the following theoıem can be similarly proved:

9. Theorem. *Let $(X, \mathfrak{N})$ be a topological* S_1-space. *Then the set* $\mathrm{Nr}(X, \mathfrak{N})$ *of all nearnesses on X inducing $\mathfrak{N}$ and ordered by inclusion is a complete lattice. The minimal element in* $\mathrm{Nr}(X, \mathfrak{N})$ *is generated by all covers of the form $\{\{x\}^{-c}, G\}$ where G is $\mathfrak{N}$-open and $x \in G$. The maximal element in* $\mathrm{Nı}(X, \mathfrak{N})$ *has the set of all $\mathfrak{N}$-open covers of X as u-base.*

10. Remark. It is known that the notions of precompactness and completness of uniform spaces can be extended on quasi-uniform spaces in many (non-equivalent) manners [4]. We give here a certain generalisation that preserves the required relation to compactness.

11. Definitions. *Let $(X, \mathbf{U})$ be a quasi-uniform space.*

(a) $\mathcal{F} \in \mathrm{Filt}\, X$ *is c-*$\mathbf{U}$*-filter iff*

$$(\forall\, \mathcal{U} \in \mathbf{U})\, (\exists\, x \in X)\, (\mathrm{St}(x, \mathcal{U}) \in \mathcal{F});$$

(b) $(X, \mathbf{U})$ *is precompact iff*

$$(\forall\, \mathcal{U} \in \mathbf{U})\, (\exists\, n)\, (\exists\, x_1, \ldots, x_n \in X)\, (X = \bigcup \{\mathrm{St}(x_k, \mathcal{U})/k = 1, \ldots, n\}$$

(c) $(X, \mathbf{U})$ *is complete iff each c-*$\mathbf{U}$*-filter clusters in*

$$(X, \mathrm{St}(-, \mathbf{U})).$$

Note. If $(X, \mathbf{U})$ is a uniform space, then just introduced notions coincide with the usual ones, as can be easily proved.

12. Theorem. *Let $(X, \mathbf{U})$ be a quasi-uniform space. Then* $(X, \mathrm{St}(-, \mathbf{U}))$ *is compact iff $(X, \mathbf{U})$ is precompact and complete.*

Proof. Put $\mathfrak{N} = \mathrm{St}(-, \mathrm{U})$ and assume first that $(X, \mathfrak{N})$ is compact. Since each $\mathscr{F} \in \mathrm{Filt}\ X$ clusters in $(X, \mathfrak{N})$, (X, U) is complete. Let $\mathscr{U} \in \mathrm{U}$. Since $X = \bigcup \{\mathrm{St}(x, \mathscr{U})^i / x \in X\}$, $X = \bigcup \{\mathrm{St}(x_k, \mathscr{U}) / k = 1, \dots, n\}$ for some n and $x_1, \dots, x_n \in X$, so that (X, U) is precompact.

Assume conversely that (X, U) is precompact and complete without $(X, \mathfrak{N})$ being compact. There is $\mathscr{A} \subset \exp X$ with $\bigcup \{A^i / A \in \mathscr{A}\} = X$ and $\mathscr{A}_1 \subset \mathscr{A}$, $\mathscr{A}_1$ – finite $\Rightarrow \bigcup \mathscr{A}_1 \neq X$. Put $\mathscr{B} = \{A^c / A \in \mathscr{A}\}$. It follows that $\mathscr{B}$ is centered, so that $\mathscr{B} \subset \mathscr{F}$ for some ultra-filter $\mathscr{F}$ on X, and it follows easily that $\mathscr{F}$ is $\mathrm{c} - \mathrm{U}$. By completness $x \in \bigcap \{F^- / F \in \mathscr{F}\}$ for some $x \in X$. But $x \in A^i$ for some $A \in \mathscr{A}$ and $A^c \in \mathscr{B} \subset \mathscr{F}$, so that $x \in A^{c-}$ and we get a contradiction $A^{c-c} \cap A^{c-} \neq \emptyset$.

13. Definition. *A quasi-uniform space* (X, U) *is fine iff* U *is the fine quasi-uniformity for* $\mathrm{St}(-, \mathrm{U})$.

14. Theorem. *Let* (X, U), (Y, V) *be quasi-uniform spaces,* (X, U) *fine. If* $f\colon (X, \mathrm{St}(-, \mathrm{U})) \to (Y, \mathrm{St}(-, \mathrm{V}))$ *is continuous, then* $f\colon (X, \mathrm{U}) \to (Y, \mathrm{V})$ *is uniformly continuous.*

Proof. Let $\mathscr{V} \in \mathrm{V}$. Since

$$f^{-1}[\mathrm{St}(f(x), \mathscr{V})] \subset \mathrm{St}(x, f^{-1}(\mathscr{V}))$$

and f is continuous, it follows

$$\mathrm{St}(x, f^{-1}(\mathscr{V})) \in \mathrm{St}(x, \mathrm{U}) \qquad \text{for all } x \in X,$$

so that $f^{-1}(\mathscr{V}) \in \mathscr{U}$ by 6.

REFERENCES

[1] E. Čech, *Topological Spaces*, Academia Prague, 1966.

[2] M. Katětov, *On continuity structures and spaces of mappings*, CMUC 6, 2, 1965, P. 257–278.

[3] H. Bentley and H. Herrlich, *Completion as reflection*, CMUC 19, 3, 1978, p. 541–568.

[4] H. Bentley and H. Herrlich, *Completness for nearness spaces*, Structure of top categories. Math. Arbeitspapiere No. 18, Teil A, Bremen 1979.

J. Svoboda
Department of Mathematics
Faculty of Science, J. E. Purkyně University,
Janáčkovo nám. 2a
662 95 Brno
Czechoslovakia