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Zuzana Došlá; Ondřej Došlý

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ON TRANSFORMATIONS OF SINGULAR QUADRATIC FUNCTIONALS CORRESPONDING TO EQUATION $(py')' + qy = 0$

ZUZANA DOŠLÁ and ONDŘEJ DOŠLÝ

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Abstract. There are studied extremal properties of the singular quadratic functionals $J(y) = \int_a^b (p(t) y'^2 - q(t) y^2) \, dt$. Using transformations of these functionals it is derived the so called singularity condition and it is also shown that the main results of [2] and [3] are valid only for regular functionals.

Key words. Singular and regular quadratic functionals, associated Euler equation, conjugate points, phase function.

MS Classification: 34 A 30, 34 C 10, 49 A 10.

1. INTRODUCTION

The rise of this paper was motivated by [2] and [3] where the unified approach to study of extremal properties of singular quadratic functionals

$$J(y) = \int_0^b [p(t) y'^2(t) - q(t) y^2(t)] \, dt$$

was introduced. Here $p(t), q(t) \in C^0(0, b]$, $p(t) > 0$ and $y(t)$ are A-admissible functions on $[0, b]$ (see § 2 below). This approach to study of quadratic functionals is based on the principal result of the Borůvka's transformations theory [1] consisting in the fact that each linear differential equation of the second order on its whole definition interval can be globally transformed into the equation $y'' + y = 0$ on a suitable interval.

The idea to use Borůvka's theory in order to investigate quadratic functionals was used for the first time by Krbila [4, 5] for regular functionals, i.e. for the case when $p(t), q(t) \in C^0[0, b]$ and extremal properties are studied for functions $y(t)$ having the property: $y(t) \in C^1[0, b]$, $y(0) = 0 = y(b)$.

The theory of singular quadratic functionals was initiated by Leighton and Morse [6] and developed in [7, 8]. In this case the functions $p(t)$, $q(t)$ are supposed continuous only on the half-open interval $(0, b]$ and also the class of admissible functions is larger than in the regular case. The classical sufficient conditions of the variational theory were found lacking for the singular problem and a condition, termed the "singularity condition" was discovered which together with the classical conditions yields necessary and sufficient condition for singular functional to be nonnegative.

In the present paper we shall derive Leighton's singularity condition by other method than in [6] and [8], namely by transformation of investigated functionals, and we shall show that results of [2] and [3] hold only for regular functionals.

2. STATEMENT OF THE PROBLEM

Consider the functional

$$J(y)\,|_e^b = \int_e^b [p(t)\, y'^2 - q(t)\, y^2]\, \mathrm{d}t, \qquad 0 < e < b,$$

where $p, q \in C^0(0, b]$, $p(t) > 0$ on $(0, b]$. Following Morse and Leighton [6] we call the function $y(t)$ A-admissible on $[0, b]$ if:

(i) $y(t) \in C^0[0, b]$, $y(0) = 0 = y(b)$

(ii) $y(t)$ is absolutely continuous and $y'^2(t)$ is Lebesque integrable on each closed subinterval of $(0, b]$.

We shall seek conditions under which

$$\liminf_{e \to 0+} \int_e^b [p(t)\, y'^2 - q(t)\, y^2]\, \mathrm{d}t \geqq 0 \tag{1}$$

for each A-admissible function $y(t)$. Note that the Euler equation of (1) is of the form

$$(p(t)\, y')' + q(t)\, y = 0 \tag{p, q}$$

and $t = 0$ may be the singular point of this equation.

A solution $y_0(t)$ of (p, q) is said to be principal at $t = 0$ if $\lim_{t \to 0} \dfrac{y_0(t)}{v(t)} = 0$ for every solution $v(t)$ of (p, q) which is linearly independent on $y_0(t)$. The principal solution $y_b(t)$ at $t = b$ is defined analogously. If there exists a principal solution $y_0(t)$ of (p, q) at $t = 0$ the right conjugate points of $t = 0$ are defined as positive zeros of $y_0(t)$. If (p, q) possesses no principal solution at $t = 0$, i.e. every solution of (p, q) has infinitely many zeros on $(0, b]$, we say that $t = 0$ is its own conjugate point. Finally, we say that (p, q) is disconjugate on $[0, b]$ if there exists no conjugate point of $t = 0$ on $[0, b)$.

3. TRANSFORMATION OF FUNCTIONALS

Consider a pair of functionals and associated Euler equations

$$\int_a^b (p(t)\, y'^2 - q(t)\, y^2)\, \mathrm{d}t, \qquad (p(t)\, y')' + q(t)\, y = 0, \qquad t \in (a, b)$$

and

$$\int_A^B (p_1(T)\, \dot{u}^2 - q(T)\, u^2)\, \mathrm{d}T, \qquad (p_1(T)\, \dot{u})^{\cdot} + q_1(T)\, u = 0, \qquad T \in (A, B)$$

The transformation

$$y(t) = h(t)\, u(T), \qquad T = x(t), \tag{2}$$

where $h(t), x(t) \in C^2(0, b)$, $h(t) \neq 0$, $x(a) = A$, $x(b) = B$, transforms equation (p, q) on (a, b) into $(\mathrm{p}_1, \mathrm{q}_1)$ on (A, B), i.e.

$$\begin{aligned} p_1(T) &= p(t)\, h^2(t)\, x'(t) \\ q_1(T) &= \frac{1}{x'(t)}\, [h(t)\, (p(t)\, h'(t))' + q(t)\, h^2(t)], \qquad t = x^{-1}(T). \end{aligned} \tag{3}$$

The following lemma describes the transformation (2) applied for corresponding functionals.

Lemma 1. *Let the transformation* (2) *be given. Then*

$$\int_a^b (p(t)\, y'^2 - q(t)\, y^2)\, \mathrm{d}t = \int_A^B (p_1(T)\, \dot{u}^2 - q_1(T)\, u^2)\, \mathrm{d}T + \\ + p(t)\, \frac{h'(t)}{h(t)}\, y^2(t) \Big|_a^b , \tag{4}$$

where $p_1(T)$, $q_1(T)$ are given by (3).

Proof. Routine computation gives $\int_a^b (p(t)\, y'^2(t) - q(t)\, y^2(t))\, \mathrm{d}t = \int_a^b \{p(t) [h'^2(t)\, u^2(x(t)) + 2h'(t)\, h(t)\, u(x(t))\, u'(x(t))\, x'(t) + h^2(t)\, u'^2(x(t))\, x'^2(t)] - q(t)\, h^2(t)\, u^2(x(t))\}\, \mathrm{d}t$. Integrating the relation $[p(t)\, h(t)\, h'(t)\, u^2(x(t))]' = p(t)\, h'^2(t)\, u^2(x(t)) + (p(t)\, h'(t))'\, h(t)\, u^2(x(t)) + 2p(t)\, h(t)\, h'(t)\, u(x(t))\, u'(x(t))\, x'(t)$, we have $\int_a^b [p(t)\, h'^2(t)\, u^2(x(t)) + 2p(t)\, h(t)\, h'(t)\, u(x(t))\, u'(x(t))\, x'(t)]\, \mathrm{d}t = p(t)\, h(t)\, h'(t)\, u^2(x(t))\, |_a^b - \int_a^b h(t) (p(t)\, h'(t))'\, u^2(x(t))\, \mathrm{d}t$, thus $\int_a^b [p(t)\, y'^2(t) - q(t)\, y^2(t)]\, \mathrm{d}t = p(t) \frac{h'(t)}{h(t)}\, y^2(t)\, |_a^b + \int_a^b [p(t)\, h^2(t)\, u'^2(x(t)) - \frac{1}{x'(t)}\, (h(t)\, (p(t)\, h'(t))' + q(t)\, h^2(t))\, u^2(x(t))]\, x'(t)\, \mathrm{d}t$. Substituting $T = x(t)$ in the last integral we have the conclusion.

Corollary 1. *Let $\alpha(t)$ be a first phase function of* (p, q) (see [1]). *Then*

$$(5)\qquad \int_a^b [p(t)\,y'^2(t) - q(t)\,y^2(t)]\,\mathrm{d}t = \int_{\alpha(a)}^{\alpha(b)} [\dot{u}^2(T) - u^2(T)]\,\mathrm{d}T - \frac{\alpha''(t)}{2\alpha'(t)}\,y^2(t)\Big|_a^b .$$

Proof. The statement follows immediately from Lemma 1 choosing $x(t) = \alpha(t)$, $h(t) = (|\,\alpha'(t)\,|)^{-1/2}$.

Now, using Lemma 1 we shall derive principal result of [6] and [8].

Theorem 1. *In order that* (1) *holds it is necessary and sufficient:*

i) (p, q) *is disconjugate on* $[0, b)$.

ii) *singularity condition is satisfied, i.e. for each A-admissible function $y(t)$ such that* $\liminf_{e\to 0_+} J(y)\,|_e^b < \infty$ *it holds*

$$\liminf_{t\to 0_+}\left(-y^2(t)\,p(t)\,\frac{y_b'(t)}{y_b(t)}\right) \geqq 0,$$

where $y_b(t)$ is a principal solution of (p, q) *at $t = b$.*

Proof. Let (p, q) be disconjugate on $[0, b)$ and $y_b(t)$ be a principal solution at $t = b$. Then $y_b(t) \neq 0$ on $(0, b)$, $y_b(b) = 0$, $y_b'(b) \neq 0$ and the transformation (2) with $h(t) = y_b(t)$, $x(t) = t$ gives $p_1(t) = p(t)\,y_b^2(t)$, $q_1(t) = y_b(t)\,[(p(t)\,y_b'(t))' + q(t)\,y_b(t)] = 0$. According to Lemma 1 it holds

$$(6)\qquad \int_e^b [p(t)\,y'^2(t) - q(t)\,y^2(t)]\,\mathrm{d}t = \int_e^b p_1(t)\,u'^2(t)\,\mathrm{d}t + p(t)\,\frac{y_b'(t)}{y_b(t)}\,y^2(t)\Big|_e^b, \qquad 0 < e < b.$$

Using the l'Hospital rule we get $\lim_{t\to b_-} y^2(t)/y_b(t) = \lim_{t\to b_-} 2y(t)\,y'(t)/y_b(t) = 0$ for each A-admissible function $y(t)$ and thus $\lim_{t\to b_-} p(t)\,y_b(t)\,y^2(t)/y_b(t) = 0$. Then it follows from (6) $\liminf_{e\to 0+}\int_e^b [p(t)\,y'^2(t) - q(t)\,y^2(t)]\,\mathrm{d}t \geqq \liminf_{e\to 0+}\int_e^b p_1(t)\,u'^2(t)\,\mathrm{d}t + \liminf_{e\to 0+}[-p(t)\,y_b'(t)\,y^2(t)/y_b(t)]$, from which the sufficiency of the singularity condition follows.

The proof of the necessity of the singularity condition is similar to that of Leighton [6]. Suppose that there exists a A-admissible function $y(t)$ for which $\liminf_{e\to 0+} J(y)\,|_e^b < \infty$ and $\liminf_{t\to 0+}[-p(t)\,y_b'(t)\,y^2(t)/y_b(t)] = -k^2 < 0$. Let $e \in (0, b)$ and define

$$y_e(t) = \begin{cases} y(t) & \text{for } t \in (0, e), \\ y_b(t) & \text{for } t \in [e, b), \end{cases}$$

where $y_b(t)$ is the principal solution of (p, q) at $t = b$ for which $y_b(e) = y(e)$. Then $\liminf_{x\to 0+} \int_x^b (py_e'^2 - qy_e^2)\,dt = \liminf_{x\to 0+} \int_x^e (py'^2 - qy^2)\,dt + \int_e^b (py_b'^2 - qy_b^2)\,dt =$
$= \liminf_{x\to 0+} \int_x^e (py'^2 - qy^2)\,dt + y_b(t)\,y_b'(t)\,p(t)\,|_e^b - \int_e^b [(py_b')' + qy_b]\,y_b\,dt =$
$= \liminf_{x\to 0+} \int_x^e (py'^2 - qy)\,dt - p(e)\,y_b'(e)\,y^2(e)/y_b(e)$. As $\liminf_{x\to 0+} J(y)\,|_x^b$ is finite and $\liminf_{x\to 0+}\left(-p(x)\,y_b'(x)\,\frac{y^2(x)}{y_b(x)}\right) = -k^2$, there exists $e \in (0, b]$ sufficiently close to $t = 0$ such that $J(y)\,|_0^e < \frac{k^2}{3}$ and $\frac{-y_b'(e)}{y_b(e)}\,p(e)\,y^2(e) < -\frac{2}{3}k^2$, hence $J(y_e)\,|_0^b <$
$< -\frac{1}{3}k^2 < 0$.

4. REMARK TO RESULTS OF [2] AND [3]

The main theorem of [2] is:

Theorem A. *Let $y(t)$ be any A-admissible function on $[0, b]$ and $q(t) \in C^0(0, b]$. Then*

$$\liminf_{e\to 0+} \int_e^b (y'^2(t) - q(t)\,y^2(t))\,dt \geqq 0 \tag{7}$$

if and only if the associated Euler equation $y'' + q(t)\,y = 0$ is disconjugate on $[0, b)$.

We shall show that this statement is valid, in general, only if $q(t) \in C^0[0, b]$.

Counter-example. Consider the functional

$$J = \int_e^{1/2} \left(y'^2 - \frac{1}{4t^2}\,y^2\right) dt, \qquad 0 < e < \frac{1}{2}. \tag{8}$$

The associated Euler equation

$$y'' + \frac{1}{4t^2}\,y = 0 \tag{9}$$

possesses the linearly independent solutions $\sqrt{t}$ and $\sqrt{t}\ln t$, so (9) is disconjugate on $(0, \infty)$. Nevertheless an easy computation shows that along the curve $y = \sqrt{2t}(\sqrt{2t} - 1)$ which is surely A-admissible on $[0, 1/2]$

$$\lim_{e\to 0+} J(y)\Big|_e^{1/2} = -\frac{1}{2}.$$

Note that the singularity condition is not satisfied. The principal solution of (9) at $t = 1/2$ is $y_b(t) = \sqrt{t}\ln 2t$ and by routine computation we get

$$\lim_{t\to 0+} [-y^2(t)\, y_b'(t)/y_b(t)] = -\lim_{t\to 0+} 2t(\sqrt{2t} - 1)^2 \left(\frac{1}{2t} + \frac{1}{t\ln 2t}\right) = -1.$$

The proof of Theorem A is based on the transformation of functionals described in Corollary 1, where $a = 0$ and $\alpha(t)$ is a phase function $y'' + q(t)\, y = 0$ for which $\lim_{t\to 0+} \alpha(t) = 0$. If the Euler equation $y'' + q(t)\, y = 0$ is disconjugate on $[0, b)$ then $\int_0^{\alpha(b)} [\dot{u}^2(T) - u^2(T)]\, \mathrm{d}T \geqq 0$ for every A-admissible function on $[0, \alpha(b)]$ but the second term on the right-side hand of (5) equals 0 (as it is stated in [2]) only if $q(t)$ is continuous also for $t = 0$. Indeed, when $t = 0$ is the singular point it can happen that this term is negative, particularly in our counter-example $\alpha(t) = \operatorname{arctg} \dfrac{1}{\ln t}$ and $\lim_{t\to 0+} \dfrac{\alpha''(t)}{2\alpha'(t)}\, y^2(t) = -1.$

Now, we shall state the correct version of Theorem A in terminology of phase functions.

Theorem 2. *In order that* (7) *holds it is necessary and sufficient:*

i) the equation

$$y'' + q(t)\, y = 0 \tag{10}$$

is disconjugate on $[0, b)$

ii) it holds

$$\limsup_{t\to 0+} \left(-\frac{\alpha''(t)}{2\alpha'(t)} + \alpha'(t)\operatorname{cotg}\alpha(t)\right) y^2(t) \leqq 0 \tag{11}$$

for some phase function $\alpha(t)$ *of* (10) *for which* $\alpha(b) = k\pi$ *and for each A-admissible functions* $y(t)$ *on* $[0, b]$ *such that*

$$\liminf_{\varepsilon\to 0+} \int_\varepsilon^b (y'^2 - q(t)\, y^2)\, \mathrm{d}t < \infty.$$

Proof. Let $\alpha(t)$ be an arbitrary phase function of (10) such that $\alpha(b) = k\pi$. Then

$$y_b(t) = (|\alpha'(t)|)^{-1/2} \sin\alpha(t)$$

is the principal solution of (10) at $t = b$ and

$$\limsup_{t\to 0+} \left(\frac{y_b'(t)}{y_b(t)}\, y^2(t)\right) = \limsup_{t\to 0+} \left(-\frac{\alpha''(t)}{2\alpha'(t)} + \alpha'(t)\operatorname{cotg}\alpha(t)\right) y^2(t).$$

The conclusion now follows from Theorem 1.

Remark. Let k be a fixed integer. There exist infinitely many phase functions $\alpha(t)$ of (10) for which $\alpha(b) = k\pi$ (see [1]) whereby all these functions are given by the relation

$$\operatorname{tg} \alpha(t) = \frac{y_b(t)}{v(t)}, \tag{12}$$

where $y_b(t)$ is a principal solution of (10) at $t = b$ and $v(t)$ is any solution of (10) linearly independent on $y_b(t)$. We shall show that (11) does not depend on the choise of $\alpha(t)$. Differentiating (12) we get $\alpha'(t) = w(y_b^2(t) + v^2(t))^{-1}$, $\alpha'' = -2w(y_b(t)\, y_b'(t) + v(t)\, v'(t))\,(y_b^2(t) + v^2(t))^{-2}$, where $w = y_b'(t)\, v(t) - y_b(t)\, v'(t)$ is the wronskian of solutions $y_b(t)$, $v(t)$. Hence

$$-\frac{\alpha''(t)}{2\alpha'(t)} + \alpha'(t) \operatorname{cotg} \alpha(t) =$$

$$= \frac{y_b(t)\, y_b'(t) + v(t)\, v'(t)}{y_b^2(t) + v^2(t)} + \frac{v(t)\,(y_b'(t)\, v(t) - y_b'(t)\, v(t))}{y_b(t)\,(y_b^2(t) + v^2(t))} = \frac{y_b'(t)}{y_b(t)},$$

which was to be proved.

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Zuzana Došlá, Ondřej Došlý,
Department of Mathematics,
J. E. Purkyně University
Janáčkovo nám. 2a
662 95 Brno, Czechoslovakia