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ON THE VARIATIONAL PRINCIPLES FOR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Using a modified theory of potential operators as given in]6], we find explicit formulas for the construction of the potential for the given partial differential equation.

Key words. Gâteaux differential, Gâteaux derivative, potential operators.

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1. PRELIMINARIES

Let X be a real Banach space, X^* its dual. Let $F : X \to Y$ be a map between two Banach spaces. When F has a *Gâteaux differential* at a point $x \in X$ in direction $h \in X$ we will denote it by $DF(x, h)$. When the map $h \to DF(x, h)$ from X to Y is linear and continuous, we say that F has a *Gâteaux derivative* at the point x and we denote this map by $DF(x)$ and write $DF(x).h$ for $DF(x, h)$. When f is a functional (i.e. $Y = R$), we write $\langle Df(x), h\rangle = Df(x).h$.

Definition. *Let $F : X \to X^*$ and let X_0 be a closed subspace of X. We say that F is a potential operator with respect to X_0, if there exists a functional $f : X \to R$ such that:*

1. $Df(x)$ exists for all $x \in X$.

2. $\langle Df(x), h\rangle = \langle F(x), h\rangle$

for all $x \in X$ and all $h \in X_0$. Such functional is called a potential for F with respect to X_0.

The following theorem ([6]) is an extension of the classical theorem of Vajnberg [5].

Theorem. *Let $F : X \to X^*$ have a Gâteaux-derivative $DF(x)$ at all $x \in X$. Let the functional $\langle DF(x).h, k\rangle$ be continuous in X for all $h, k \in X$. Let X_0 be a closed subspace of X. Then F is a potential operator with respect to X_0, if and only if*

*) This article was presented in Poster session of Equadiff 6.

$$\langle DF(x).\, h, k\rangle = \langle DF(x).\, k, h\rangle$$

for all $h, k \in X_0$ and all $x \in X$.

2. FORMULATION OF THE PROBLEM AND RESULTS

Let n be a positive integer, ε a smooth function of real variables $x, y, y_{j_1}, \ldots, y_{j_1 \ldots j_r}$ where $1 \leqq i \leqq n$, $1 \leqq j_1 \leqq \ldots \leqq j_r \leqq n$. Consider a partial differential equation

$$\text{(1)} \qquad \varepsilon(x^i, u(x^i), \quad D_{j_1}u, \ldots, D_{j_1} \ldots Dj_r u) = 0.$$

Necessary and sufficient conditions for equation (1) to be variational are given the following relations ([2]):

$$\text{(2)} \quad \frac{\partial \varepsilon}{\partial y_{j_1 \ldots j_l}} = (-1)^l \frac{\partial \varepsilon}{\partial y_{j_1 \ldots j_l}} + \sum_{m=l+1}^{r} (-1)^m \binom{m}{l} d_{j_{l+1}} \ldots d_{j_m} \frac{\partial \varepsilon}{\partial y_{j_1 \ldots j_m}} \qquad 0 \leqq l \leqq r,$$

where $d_i u$ denotes the *formal* (= *total*) *derivative* of a function u with respect to the coordinate x^i.

Remark. As it is proved in [1] equations (2) can be satisfied only for r even. Let us put $r = 2k$.

Let Ω be a domain in R^n, with the Lipschitz's boundary $\partial\Omega$, $\bar{\Omega} = \Omega \cup \partial\Omega$. Let $X = C^{2k}(\bar{\Omega})$, $X_0 \subset X$ resp. $X_1 \subset X$ be subspaces of functions whose partial derivatives up to the $(k-1)$st order resp. $(2k-1)$st order vanish on $\partial\Omega$. For the equation (1) we define an operator $A : X \to X^*$ by the following relation

$$\langle Au, v\rangle = \int_\Omega v(x)\, \varepsilon(x^i, u, \ldots, D_{j_1} \ldots D_{j_{2k}}u)\, \mathrm{d}x.$$

Remark. In the next we suppose ε to be a sufficiently smooth function.

Theorem. *The operator A is potential with respect to X_0 if and only if conditions* (2) *are satisfied.*

Proof. It is necessary to prove

$$\langle DA(u, v), w\rangle = \langle DA(u, w), v\rangle$$

for all $u \in X$ and all $w, v \in X_0$. We have

$$\langle DA(u, v), w\rangle = \int_\Omega w \sum_{r=1}^{2k} \frac{\partial \varepsilon}{\partial y_{j_1 \ldots j_r}} D_{j_1} \ldots D_{j_r} v\, \mathrm{d}x.$$

Now, we integrate by parts and first we apply partial integration on the last terms. We can find the following expression:

$$\langle DA(u, v), w\rangle = -\int_\Omega d_{j_{2k}} \left(w \frac{\partial \varepsilon}{\partial y_{j_1 \ldots j_{2k}}} \right) D_{j_1} \ldots D_{j_{2k-1}} v\, \mathrm{d}x +$$

$$+\int_{\partial\Omega} w \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_{2k}}} D_{j_1} \dots D_{j_{2k-1}} v \gamma_{j_{2k}} \, dS + \dots$$

$$\dots = \int_{\Omega} d_{j_{2k-1}} d_{j_{2k}} \left(w \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_{2k}}} \right) D_{j_1} \dots D_{j_{2k-2}} v \, dx -$$

$$- \int_{\partial\Omega} d_{j_{2k}} w \left(\frac{\partial \varepsilon}{\partial y_{j_1 \dots j_{2k}}} \right) D_{j_1} \dots D_{j_{2k-2}} \gamma_{j_{2k-1}} \, dS +$$

$$+ \int_{\partial\Omega} w \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_{2k}}} D_{j_1, \dots, j_{2k-1}} v \gamma_{j_{2k}} \, dS -$$

$$- \int_{\Omega} d_{j_{2k-1}} \left(w \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_{2k-1}}} \right) D_{j_1} \dots D_{j_{2k-2}} v \, dx +$$

$$+ \int_{\partial\Omega} w \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_{2k-1}}} D_{j_1} \dots D_{j_{2k-2}} v \gamma_{j_{2k-1}} \, dS + \dots$$

$$\dots = \sum_{l=0}^{2k} \int_{\Omega} P^{(0)}_{(j_1, \dots, j_l)} v D_{j_1} \dots D_{j_l} w \, dx +$$

$$+ \sum_{r=1}^{2k} \sum_{r \le l, m \le 2k}^{2k} \int_{\partial\Omega} P^{(j_1, \dots, j_{r+1})}_{(j_m, \dots, j_l)} D_{j_1} \dots D_{j_r} v D_{j_m} \dots D_{j_l} w \gamma_{j_{r+1}} \, dS,$$

where γ_{jk} is j_k component of the outer unit normal vector and $P^{(\dots)}_{(\dots)} = P^{(\dots)}_{(\dots)}(x^i, u, D_{j_1}u, D_{j_1}D_{j_2}u, \dots)$ are defined by the following reccurence relations:

$$P^{(j_1, \dots, j_{2k})}_{(0)} = \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_{2k}}},$$

$$P^{(j_1, \dots, j_{2k-2})}_{(0)} = \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_{2k-2}}} - d_{j_{2k-l+1}} \quad P^{(j_1, \dots, j_{2k-l+1})}_{(0)},$$

$$1 \leqq l \leqq 2k - 1,$$

(3) $$P^{(j_1, \dots, j_{2k-r-1})}_{(j_{2k-l}, \dots, j_{2k-r})} = -P^{(j_1, \dots, j_{2k-r})}_{(j_{2k-l}, \dots, j_{2k-r+1})} - d_{j_{2k-l+1}} P^{(j_1, \dots, j_{2k-r})}_{(j_{2k-l+1}, \dots, j_{2k-r+1})},$$

$$0 \leqq l \leqq 2k - r, \qquad 0 \leqq r \leqq 2k.$$

Since $v \in X_0$, $w \in X_0$, all boundary terms vanish. From (3) we can obtain

$$P^{(0)}_{(j_1, \dots, j_r)} = (-1)^l \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_r}} + \sum_{m=l+1}^{2k} (-1)^m \binom{m}{l} d_{j_{l+1}} \dots d_{j_m} \frac{\partial \varepsilon}{\partial y_{j_1 \dots j_m}},$$

From here it follows the symmetry condition is valid if and only if the conditions (2) are satisfied.

Lemma. *Functions* $P^{(j_1, \dots, j_r)}_{(j_m, \dots, j_l)}$, $1 \leqq r \leqq 2k - 1$, $r \leqq l$, $m \leqq 2k$, *do not depend on derivatives of the order greater than* $2k$.

Proof is an immediate consequence of reccurence relations (3).

Our problem consists in constructing a potential for A with respect to X_0. In the paper [6], the potential is constructed by means of a projection operator. We shall try to construct the potential without a definition of a projection operator. Our main result can be formulated as follows:

Theorem. *Let conditions* (2) *be satisfied and let functions* $P^{(j_1,\dots,j_r)}_{(j_m,\dots,j_r)}$ $k \leqq r \leqq 2k-1$, $r \leqq l$, $m \leqq 2k-1$ *do not depend on the* $2k$*-th derivatives.*

The functional f defined by the relation

$$f(u) = \int_0^1 \mathrm{d}s \int_\Omega u\varepsilon(x^i, su, sD_{j_1}u, \dots, sD_{j_1} \dots D_{j_{2k}}u)\,\mathrm{d}x -$$
$$- \int_0^1 s\,\mathrm{d}s \int_0^1 \mathrm{d}s' \int_\Omega \sum_{k=r}^{2k-1} \sum_{l,m=r+1}^{2k-1} P^{(j_1,\dots,j_{r+1})}_{(j_1,\dots,j_m)}(x^i, su, \dots$$
$$\dots sD_{j_1} \dots D_{j_{k-1}}u, ss'D_{j_1} \dots D_{j_k}u, \dots, ss'D_{j_1} \dots D_{j_{2k-1}}u)\, D_{j_1} \dots D_{j_r}uD_{j_m} \dots D_{j_r}u\gamma_{j_{r+1}}\,\mathrm{d}S.$$

is the potential for A with respect to X_0, *i.e.*

$$\langle Df(u), v\rangle = \langle Au, v\rangle$$

for all $u \in X$ *and* $v \in X_0$.

Proof. We shall write the functional f in the form

$$f(u) = f_1(u) - f_2(u), \tag{4}$$

where

$$f_1(u) = \int_0^1 \mathrm{d}s \int_\Omega u\varepsilon(x^i, su, \dots, sD_{j_1} \dots D_{j_{2k}}u)\,\mathrm{d}x,$$
$$f_2(u) = \int_0^1 s\,\mathrm{d}s \int_0^1 \mathrm{d}s' \int_{\partial\Omega} \sum_{r=k}^{2k-1} \sum_{l,m=r+1}^{2k-1} P^{(j_1,\dots,j_{r+1})}_{(j_1,\dots,j_m)} \times$$
$$\times (x^i, su, \dots, ss'D_{j_1} \dots D_{j_{2k-1}}u)\, D_{j_1} \dots D_{j_r}uD_{j_m} \dots D_{j_l}u\gamma_{j_{r+1}}\,\mathrm{d}S.$$

Now, we calculate the Gâteaux derivative of f_1. We get

$$\langle Df_1(u), v\rangle = \int_0^1 \mathrm{d}s \int_\Omega v\varepsilon(x^i, su, \dots, sD_{j_1} \dots D_{j_{2k}}u)\,\mathrm{d}x$$
$$\int_0^1 s\,\mathrm{d}s \int_\Omega u \sum_{f=0}^{2k} \frac{\partial\varepsilon}{\partial y_{j_1\dots j_l}} D_{j_1} \dots D_{j_l}v\,\mathrm{d}x.$$

As we apply a partial integration for the second term and use the symmetry conditions (2) we can write

$$\langle Df_1(u), v\rangle = \int_0^1 \mathrm{d}s \int_\Omega v\varepsilon(x^i, su, \dots, sD_{j_1} \dots D_{j_{2k}}u)\,\mathrm{d}x +$$
$$+ \int_0^1 s\,\mathrm{d}s \int_\Omega v \sum_{l=0}^{2k} \frac{\partial\varepsilon}{\partial y_{i_1\dots j_l}} D_{j_1} \dots D_{j_l}u + \text{boundary terms} =$$
$$= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\Big[\int_\Omega v\varepsilon(x^i, su, \dots, sD_{j_1} \dots D_{j_{2k}}u)\,\mathrm{d}x\Big]\,\mathrm{d}s +$$
$$+ \textit{boundary terms} = \langle Au, v\rangle + \textit{boundary terms}.$$

For boundary terms (BT) we have

$$BT = \int_0^1 s\,\mathrm{d}s \int_{\partial\Omega} \sum_{r=0}^{2k-1} \sum_{l,m=r+1}^{2k-1} P^{(j_1,\dots,j_{r+1})}_{(j_m,\dots,j_l)}(x^i, su, \dots,$$

$$\dots, sD_{j_1}, \dots, D_{j_{2k-1}}u)\, D_{j_1} \dots D_{j_r}vD_{j_1} \dots D_{j_m}u\gamma_{j_{r+1}}\,\mathrm{d}S.$$

For $x \in X_1$, it is $BT = 0$ and

$$\langle Df_1(u), v\rangle = \langle Au, v\rangle, \qquad u \in X, v \in X_1.$$

Now, we can define an operator $\bar{A} : X \to X^*$ as follows:

$$\langle \bar{A}u, v\rangle = BT$$

for all $v, u \in X$.

Then

(5) $$\langle Df_1(u), v\rangle = \langle Au, v\rangle - \langle \bar{A}u, v\rangle$$

for all $u, v \in X$.

The operator A is potential with respect to X_0, $Df_1(u)$ is potential with respect to every subspace and then $\bar{A}$ is potential with respect to X_0.

We shall prove that the potential for $\bar{A}$ is the functional $f_2(u)$.

Let us compute $\langle Df_2(u), v\rangle$

It is

$$\langle Df_2(u), v\rangle = \int_0^1 s\,\mathrm{d}s \int_0^1 \mathrm{d}s' \int_{\partial\Omega} \sum_{r=k}^{2k-1} \sum_{l,m=r+1}^{2k-1}$$

$$\left\{\left[\sum_{p=1}^{k-1} \frac{\partial P^{(\dots)}_{(\dots)}}{\partial y_{j_1\dots j_p}} sD_{j_1\dots j_p}v + \sum_{p=k}^{2k-1} ss' \frac{\partial P^{(\dots)}_{(\dots)}}{\partial y_{j_1\dots j_p}} D_{j_1} \dots D_{j_p}v\right] D_{j_1} \dots\right.$$

$$\dots D_{j_r}uD_{j_m} \dots D_{j_l}u\gamma_{j_{r+1}} + P^{(\dots)}_{(\dots)}D_{j_1} \dots D_{j_r}uD_{j_m} \dots D_{j_l}v\gamma_{j_{r+1}} +$$

$$\left. + P^{(\dots)}_{(\dots)}D_{j_1} \dots D_{j_r}vD_{j_m} \dots D_{j_l}u\gamma_{j_{r+1}}\right\} \mathrm{d}S.$$

For $v \in X_0$ we obtain

$$\langle Df_2(u), v\rangle = \int_0^1 s\,\mathrm{d}s \int_0^1 \mathrm{d}s' \int_{\partial\Omega} \sum_{r=k}^{2k-1} \sum_{l,m=r+1}^{2k-1}$$

$$\left[\sum_{p=k}^{2k} ss' \frac{\partial P^{(\dots)}_{(\dots)}}{\partial y_{j_1\dots j_p}} D_{j_1} \dots D_{j_p}vD_{j_1} \dots D_{j_r}uD_{j_m} \dots D_{j_l}u\gamma_{j_{r+1}} +\right.$$

$$\left. + P^{(\dots)}_{(\dots)}D_{j_1} \dots D_{j_r}vD_{j_m} \dots D_{j_l}u\gamma_{j_{r+1}}\right] \mathrm{d}S =$$

$$= \int_0^1 s\,\mathrm{d}s \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s'}\left[\sum_{r=k}^{2k-1} \sum_{l,m=r+1}^{2k-1} \int_{\partial\Omega} s'P^{(j_1,\dots,j_{r+1})}_{(j_m,\dots,j_l)}\right.$$

$$(x^i, su, \dots, sD_{j_1} \dots D_{j_{k-1}}u, ss'D_{j_1} \dots D_{j_k}u, \dots, ss'D_{j_1} \dots$$

$$\left.\dots D_{j_{2k-1}}u)\, D_{j_1} \dots D_{j_r}vD_{j_m} \dots D_{j_l}u\gamma_{j_{r+1}}\,\mathrm{d}S\right] \mathrm{d}s'.$$

From here it follows:

$$\langle Df_2(u), v\rangle = \langle \bar{A}u, v\rangle \tag{6}$$

for all $u \in X$ and $v \in X_0$.

Combining (4), (5), (6) we get our final result

$$\langle Df(u), v\rangle = \langle Df_1(u), v\rangle - \langle Df_2(u), v\rangle = \langle Au, v\rangle + \langle \bar{A}u, v\rangle - \langle Df_2u, v\rangle$$

and for all $u \in X$ and $v \in X_0$:

$$\langle Df(u), v\rangle = \langle Au, v\rangle.$$

Examples. We shall illustrate the preceding theory on some examples. In all cases conditions (2) are satisfied and we shall only construct the potential.

1. $$-\frac{\partial}{\partial x}\left(|\operatorname{grad} u|^2 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(|\operatorname{grad} u|^2 \frac{\partial u}{\partial y}\right) = 0 \qquad (x, y) \in \Omega.$$

The potential is of the form

$$f(u) = f_1(u) - f_2(u),$$

where

$$f_1(u) = \int_0^1 s^3 \, ds \int_\Omega u \left\{-\frac{\partial}{\partial x}\left(|\operatorname{grad} u|^2 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(|\operatorname{grad} u|^2 \frac{\partial u}{\partial y}\right)\right\} dx\, dy.$$

$$f_2(u) = \int_0^1 s \, ds \int_0^1 ds' \int_{\partial\Omega} \left[us^2 s'^2 |\operatorname{grad} u|^2 \frac{\partial u}{\partial x} \gamma_1 + \right.$$
$$+ 2s^2 s'^2 \left(\frac{\partial u}{\partial x}\right)^3 u\gamma_1 + 2us^2 s'^2 \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 \gamma_1 +$$
$$+ us^2 s'^2 |\operatorname{grad} u|^2 \frac{\partial u}{\partial y}\gamma_2 + 2s^2 s'^2 u \left(\frac{\partial u}{\partial y}\right)^3 \gamma_2 +$$
$$\left. + 2s^2 s'^2 \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)^2 \gamma_2\right] dS = \frac{1}{4}\int_{\partial\Omega} u \, |\operatorname{grad} u|^2 \frac{\partial u}{\partial \nu}\, dS.$$

Then for $f(u)$ we have

$$f(u) = \frac{1}{4}\int_\Omega \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)^2 dx\, dy.$$

In the following examples the potential is constructed in a similar way.

2. $$-\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{p-2} \frac{\partial u}{\partial y}\right) = 0 \qquad (x, y) \in \Omega,\ p > 2.$$

The potential takes the following form

$$f(u) = \frac{1}{p}\int_\Omega \left\{\left|\frac{\partial u}{\partial x}\right|^p + \left|\frac{\partial u}{\partial y}\right|^p\right\} dx\, dy.$$

3. $$u'u^{(iv)} + u''u''' + \frac{1}{2}u''^2 = 0, \qquad x \in \langle 0, 1\rangle.$$

The potential is

$$f(u) = \frac{1}{3}\left\{2\int_0^1 u''^2 u' \,\mathrm{d}x + \frac{1}{2}\int_0^1 u'' u \,\mathrm{d}x\right\}.$$

4. $$-\frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{1+|\operatorname{grad} u|^2}}\frac{\partial u}{\partial x}\right\} - \frac{\partial}{\partial y}\left\{\frac{1}{\sqrt{1+|\operatorname{grad} u|^2}}\frac{\partial u}{\partial y}\right\} = 0 \qquad (x, y) \in \Omega.$$

The potential is of the form

$$f(u) = \int_\Omega \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

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