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LORENTZIAN GEOMETRY AS DETERMINED BY THE VOLUMES OF SMALL TRUNCATED LIGHT CONES

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Abstract. Lorentzian manifolds, i.e. pseudo-Riemannian manifolds of signature $(+ - \ldots -)$, are considered. It is shown that from the volume of small truncated light cones there may be recognized whether the manifold is flat or Ricci-flat respectively.

Key words. Lorentzian manifold, light cone, volume.

MS Classification. 53 B 20, 58 C 35.

INTRODUCTION

A. Gray, L. Vanhecke [5, 6, 7] and others [2, 9, 10] studied the following problem for properly Riemannian manifolds: To what extent does the volume of small geodesic balls determine the geometry? (The papers [8, 1, 11, 12, 13, 16, 17] are closely related to this problem. Note also that there is a rich literature on the volumes of tubes, which we will not quote here. For the historical sources cf. [7, 4]). The topic of the present paper is an analogous problem for Lorentzian manifolds, i.e. for smooth manifolds equipped with a pseudo-Riemannian metric of signature $(+ - \ldots -)$: To what extent does the volume of small truncated light cones determine the geometry? For the case of four dimensions, i.e. for the spacetimes of general relativity theory, this question has been raised by F. and B. Gackstatter [4].

The "volume conjecture" due to A. Gray and L. Vanhecke reads:

An n-dimensional properly Riemannian manifold is flat if and only if the volume of its geodesic balls of small radius $R > 0$ equals $\Gamma\left(\frac{n}{2} + 1\right)^{-1} \pi^{n/2} R^n$. The Lorentzian analog of this conjecture reads:

An $(n + 1)$-dimensional Lorentzian manifold is flat if and only if the volume of its truncated light cones of small altitude $T > 0$ equals $(n + 1)^{-1} \Gamma\left(\frac{n}{2} + 1\right)^{-1} \pi^{n/2} T^{n+1}$.

For the case $n+1=4$ this has been supposed by F. Gackstatter [personal communication].

While Gray's and Vanhecke's conjecture remains still open, we decide here Gackstatter's conjecture, generalized to $n+1 \geqq 3$ dimensions, in the affirmative. Thus, flatness is a property of a Lorentzian manifold which is reflected by the volume of small truncated light cones. We prove that Ricci-flatness is another such property. As a by-product we derive a seemingly new Pizzetti-type expansion formula for the mean value of a function over a truncated light cone in flat R^{n+1}.

Let us fix some conventions and notations. The manifolds and all geometric objects on them are assumed to be of differentiability class C^∞. The (pseudo-)-Riemannian metric and its inverse read

$$g = g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \qquad (g^{\alpha\beta}) := (g_{\alpha\beta})^{-1},$$

the curvature tensor, Ricci tensor, and scalar curvature respectively

$$\text{Riem} = R_{\alpha\beta\mu\nu}(dx^\alpha \wedge dx^\beta)(dx^\mu \wedge dx^\nu), \qquad \text{Ric} = R_{\alpha\beta}\, dx^\alpha\, dx^\beta, \qquad R.$$

The sign conventions for the curvature quantities are the same as in [7, 19]. We abbreviate

$$(\text{Riem})^2 := R_{\alpha\lambda\varrho\beta} R_\mu{}^{\lambda\varrho}{}_\nu\, dx^\alpha\, dx^\beta\, dx^\mu\, dx^\nu.$$

A symmetric differential form of degree p

$$A_p = A_{\alpha_1\alpha_2 \cdots \alpha_p} dx^{\alpha_1}\, dx^{\alpha_2} \ldots dx^{\alpha_p}$$

is a special notation for a symmetric covariant tensor field of degree p. Apart from the usual tensorial operations there are specific operations for symmetric forms:

– Symmetric product of a p-form A_p and a q-form B_q

$$A_p B_q := A_{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_q} dx^{\alpha_1} \ldots dx^{\alpha_p}\, dx^{\beta_1} \ldots dx^{\beta_q}.$$

– Trace = tr with respect to the metric g

$$\text{tr}\, A_p := g^{\alpha\beta} A_{\alpha\beta\alpha_3 \cdots \alpha_p} dx^{\alpha_3} \ldots dx^{\alpha_p} \qquad \text{for } p \geqq 3,$$

$$\text{tr}\, A_0 := 0, \qquad \text{tr}\, A_1 := 0, \qquad \text{tr}\, A_2 := g^{\alpha\beta} A_{\alpha\beta}.$$

– Trace-free part ${}^-A_p$ of A_p with respect to g; cf. [19] for concrete formulas.

– Value of A_p on a vector field $a = a^\alpha \dfrac{\partial}{\partial x^\alpha}$

$$A_p(a, \ldots, a) := A_{\alpha_1\alpha_2 \cdots \alpha_p} a^{\alpha_1} a^{\alpha_2} \ldots a^{\alpha_p}.$$

– Symmetric differential d built by means of the Levi–Civita derivatives ∇_α

$$dA_p := \nabla_\alpha A_{\alpha_1 \cdots \alpha_p} dx^\alpha\, dx^{\alpha_1} \ldots dx^{\alpha_p}.$$

The Laplace operator of (M, g) reads

$$\Delta := g^{\alpha\beta} \nabla_\alpha \nabla_\beta.$$

A PIZZETTI-TYPE FORMULA FOR TRUNCATED LIGHT CONES IN R^{n+1}

Let R^n denote the flat space with points $\mathfrak{r} = (x^1, x^2, \dots, x^n)$ and equipped with the Euclidean metric

$$d\mathfrak{r}^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2,$$

the Laplace operator

$$\Delta = \frac{\partial^2}{(\partial\mathfrak{r})^2} = \frac{\partial^2}{(\partial x^1)^2} + \dots + \frac{\partial^2}{(\partial x^n)^2},$$

and the Lebesgue measure $d^n\mathfrak{r} = dx^1\, dx^2 \dots dx^n$. Let further R^{n+1} denote the flat spacetime with points $x = (x^\alpha) = (x_0, x^1, \dots, x^n) = (t, \mathfrak{r})$ and equipped with the Minkowski metric

$$\eta = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta = dt^2 - d\mathfrak{r}^2,$$

the D'Alembert operator

$$\Box = \eta^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha\, \partial x^\beta} = \frac{\partial^2}{\partial t^2} - \Delta,$$

and the Lebesgue measure $d^{n+1}x = dx^\circ\, dx^1 \dots dx^n = dt\, d^n\mathfrak{r}$. Our aim is to calculate the mean value of a function $F(x) = F(t, \mathfrak{r})$ over the truncated light cones in R^{n+1}. The mean value $M_D F$ of a function F over the domain D is defined as the quotient of the integral of F over D and the measure of D.

Definition 1. *Let* $a = a^\alpha \frac{\partial}{\partial x^\alpha}$ *be a timelike vector. The point set*

$$C(a) := \{x \in R^{n+1} \mid \eta(x, x) \geqq 0, 0 \leqq \eta(a, x) \leqq \eta(a, a)\}$$

is called the truncated light cone with vertex 0, *axis* a *and altitude* $|a| := \eta(a, a)^{1/2}$.

Theorem 1. *There holds the asymptotic power series expansion with respect to* $|a|$

$$(1) \qquad M_{C(a)} F \sim \sum_{k,l=0}^{\infty} \alpha_{kl} (-|a|^2)^k (\Box^k a^l F)(0)$$

with the numerical coefficients α_{kl} *given by*

$$4^k k!\, l! (n + 2k + l + 1) \Gamma\left(\frac{n}{2} + k + l - \left[\frac{l}{2}\right] + \frac{1}{2}\right) \Gamma\left(\frac{n}{2} + k + \left[\frac{l}{2}\right] + 1\right) \alpha_{kl} :=$$

$$:= (n+1) \Gamma\left(\frac{n}{2} + 1\right) \Gamma\left(\frac{n}{2} + k + l + \frac{1}{2}\right).$$

Proof. There exists a Lorentz transformation which results in

$$(a^\alpha) = (T, 0, \dots, 0), \qquad T = |a|,$$
$$C(a) = \{(t, \mathfrak{r}) \in R^{n+1} \mid |\mathfrak{r}| \leqq t \leqq T\}.$$

We will work in the corresponding coordinate system. Let us write (like, e.g., in quantum field theory) integrations as operators which act on the succedent functional expressions. Then Fubini's theorem applied to $C(a)$ reads

$$\int_{C(a)} F(x)\,\mathrm{d}^{n+1}x = \int_0^T \mathrm{d}t \int_0^t \mathrm{d}r \int_{S(r)} F(t, \mathfrak{r})\,\mathrm{d}S,$$

where

$$S(r) := \{\mathfrak{r} \in R^n \mid |\mathfrak{r}| = r\}$$

denotes the sphere with centre 0, radius r, and measure $\mathrm{d}S$. Changing from integrals to mean values we obtain

$$T^{n+1} M_{C(a)} F = n(n+1) \int_0^T \mathrm{d}t \int_0^t \mathrm{d}r\, r^{n-1} M_{S(r)} F.$$

Herein we insert the well-known Pizzetti expansion formula [3]

$$M_{S(r)} F \sim \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \left[k!\,\Gamma\left(\frac{n}{2}+k\right)\right]^{-1} (\Delta^k F)(t, 0)$$

and simultaneously the Taylor expansion with respect to t

$$F(t, \mathfrak{r}) \sim \sum_{l=0}^{\infty} t^l (l!)^{-1} (\partial^l F)(0, \mathfrak{r}) \qquad \text{with } \partial := \frac{\partial}{\partial t}.$$

The result is

$$M_{C(a)} F \sim \sum_{k,l=0}^{\infty} \beta_{kl} T^{2k+l} (\Delta^k \partial^l F)(0) \tag{2}$$

with the numerical coefficients

$$\beta_{kl} := (n+1)\,\Gamma\left(\frac{n}{2}+1\right)\left[4^k k!\, l!\,(n+2k+l+1)\,\Gamma\left(\frac{n}{2}+k+1\right)\right]^{-1}.$$

In order to transform (2) into the Lorentz-invariant formula (1) a rearrangement of terms is to be done:

$$\Delta^k = (\partial^2 - \Box)^k = \sum_{r=0}^{k} \binom{k}{r} (-\Box)^{k-r} \partial^{2r},$$

$$a = T\,\partial, \qquad T^{2k+1}\,\partial^l = |a|^{2k} a^l,$$

$$\alpha_{kl} = \sum_{r=0}^{[l/2]} \binom{k+r}{r} \beta_{k+r,\,l-2r}.$$

The last formula is, by some elementary transformations, equivalent to

$$\binom{2m}{l} = \sum_{r=0}^{[l/2]} 2^{l-2r} \binom{m}{r} \binom{m-r}{l-2r}, \qquad \text{where } m := \frac{n}{2} + k + l.$$

This "combinatorial identity" is, finally, established by means of the method of a generating function:

$$\sum_{l=0}^{\infty}\binom{2m}{l}z^l = (z+1)^{2m} = (z^2+2z+1)^m = \sum_{r=0}^{\infty}\binom{m}{r}z^{2r}(2z+1)^{m-r} = \ldots$$
$$\ldots = \sum_{l=0}^{\infty}\sum_{r=0}^{[l/2]}2^{l-2r}\binom{m}{r}\binom{m-r}{l-2r}z^l.$$

The theorem is proved.

We give the Pizzetti-type foımula (1) an appearance which is more suitable for our purposes by changing from the language of differential operators to the language of symmetric differential forms.

Proposition 1. *There holds the asymptotic power series expansion with respect to a:*

(3) $$M_{C(a)}F \sim \sum_{p=0}^{\infty} A_p(a, \ldots, a)$$

with the symmetric p-forms

(4) $$A_p := \sum_{2k+l=p} \beta_{kl}(-\eta)^k \operatorname{tr}^k F_p, \qquad F_p := (\mathrm{d}^p F)(0).$$

The proof follows from

$$|a|^{2k}(\Box^k a^l F)(0) = |a|^{2k}(\operatorname{tr}^k F_p)(a, \ldots, a) =$$
$$= (\eta^k \operatorname{tr}^k F_p)(a, \ldots, a) \qquad \text{with } p := 2k + l.$$

Conjecture. For each $p \geqq 0$ the linear transformation of symmetric p-forms (4) has an inverse

$$F_p = \sum_{2k+l=p} \gamma_{kl}\eta^k \operatorname{tr}^k A_p$$

with the coefficients γ_{kl} being rational functions of n.

Proposition 2. *The preceding conjecture holds true for $p \leqq 4$.*

The proof is done by direct calculation. We omit the concrete expressions for the γ_{kl} for $2k + l \leqq 4$; they do not matter here.

Proposition 3. *If a symmetric p-form A_p vanishes on all timelike vectors then it vanishes identically. That means, $A_p(a, \ldots, a) = 0$ for every timelike a implies $A_p = 0$.*

Proof. Let a_0 be a fixed timelike vector and v an arbitrary vector. For sufficiently small $|\varepsilon|$ the vector $a = a_0 + \varepsilon v$ remains timelike and $A_p(a, \ldots, a)$ becomes a polynomial of degree p in ε which vanishes identically (in ε) by assumption. Particularly, the coefficient of ε^p gives $A_p(v, \ldots, v) = 0$, hence $A_p = 0$.

Proposition 4. *If two functions $F_{\mathrm{I}}, F_{\mathrm{II}}$ on R^{n+1} satisfy*

$$M_{C(a)}F_{\mathrm{I}} = M_{C(a)}F_{\mathrm{II}}$$

for each timelike vector a, then

$$(\mathrm{d}^p F_{\mathrm{I}})(0) = (\mathrm{d}^p F_{\mathrm{II}})(0) \qquad \textit{for } p \leqq 4.$$

Proof. Because of the linearity it is sufficient to show that $M_{C(a)}F = 0$ for every timelike a implies $(d^pF)(0) = 0$ for $p \leqq 4$. This assertion follows, in fact, from the propositions 1, 2, 3.

THE MAIN RESULT AND ITS PROOF

To any Riemannian manifold there are associated some "natural" two-point functions [18, 14, 19].

Definition 2. *Let (M, g) be a Riemannian manifold of arbitrary signature. The distance function $\sigma = \sigma(x, y)$ is the solution of the problem*

$$\begin{gathered} g^{\alpha\beta} \nabla_\alpha \sigma \nabla_\beta \sigma = 2\sigma, \\ (\nabla_\alpha \sigma)(y, y) = 0, \qquad (\nabla_\alpha \nabla_\beta \sigma)(y, y) = g_{\alpha\beta}(y). \end{gathered} \tag{5}$$

The function $u = u(x, y)$ is defined by

$$2u := \Delta\sigma - \dim M. \tag{6}$$

The normal volume function $\varrho = \varrho(x, y)$ is the solution of the problem

$$g^{\alpha\beta} \nabla_\alpha \sigma \nabla_\beta \varrho = 2u\varrho, \qquad \varrho(y, y) = 1. \tag{7}$$

Here and in the following differential operators ∇, Δ, d, ... refer to the first argument of the two-point functions.

Both the two-point functions σ and ϱ are defined in some neighbourhood of the diagonal of $M \times M$ and are symmetric in their arguments, i.e. $\sigma(x, y) = \sigma(y, x)$ and $\varrho(x, y) = \varrho(y, x)$. In normal coordinates (x^α) of $x \in M$ with respect to the origin $y \in M$ there holds

$$\sigma(x, y) = \frac{1}{2} g_{\alpha\beta}(y)\, x^\alpha x^\beta,$$

$$\varrho(x, y) = |\det g_{\alpha\beta}(x)|^{1/2}\ |\det g_{\mu\nu}(y)|^{-1/2}.$$

For properly Riemannian manifolds σ and the geodesic distance s between two sufficiently neighboured points are related by $2\sigma(x, y) = s(x, y)^2$. For pseudo-Riemannian manifolds the geodesic distance s should be defined by $2\sigma\,|(x, y)| = s(x, y)^2$.

The limit for $x \to y$, if existing, of a two-point quantity with the arguments x, y is called its coincidence limit. The equality of the coincidence limits is an equivalence relation between two-point quantities and shall be denoted by $\doteq$. One-point quantities and constants may be looked upon as special two-point quantities.

Proposition 5. *There holds*

$$\begin{gathered} d\varrho \doteq 0, \qquad -3\, d^2\varrho \doteq \mathrm{Ric}, \qquad -2\, d^3\varrho \doteq d\,\mathrm{Ric}, \\ -15\, d^4\varrho \doteq 2(\mathrm{Riem})^2 - 5(\mathrm{Ric})^2 + 9\, d^2\,\mathrm{Ric}. \end{gathered}$$

Proof. We apply $\mathrm{d}^p = \mathrm{dd} \dots \mathrm{d}$ and after that the limit operation $x \to y$ to the differential equation (7). The recursion formula

$$p\,\mathrm{d}^p\varrho \doteq 2\mathrm{d}^p\mu + 2\sum_{q=2}^{p-2}\binom{p}{q}\mathrm{d}^{p-q}\mu\,\mathrm{d}^q\varrho \qquad \text{for } p \geqq 4$$

comes out. The formula holds for $p < 4$ too if an "empty sum" is taken as 0. Next, the coincidence limits of the $\mathrm{d}^q\mu$ $(q = 1, 2, \dots)$, as they follow from the celebrated Ledger formula [18, 19], are inserted. The concrete evaluation for $p = 1, 2, 3, 4$ gives the above result.

Note that proposition 5 is equivalent to the expressions for the first four terms in the Taylor expansion of ϱ with respect to normal coordinates, which have been presented in [5, 6, 7].

Proposition 6. *A Lorentzian manifold (M, g) is*
- *Einstein iff* $^-\mathrm{d}^2\varrho \doteq 0$,
- *Ricci-flat iff* $\mathrm{d}^2\varrho \doteq 0$,
- *of constant curvature iff* $^-\mathrm{d}^2\varrho \doteq 0$ *and* $^-\mathrm{d}^4\varrho \doteq 0$,
- *(locally) flat iff* $\mathrm{d}^2\varrho \doteq 0$ *and* $^-\mathrm{d}^4\varrho \doteq 0$.

Here iff := if and only if.

Proof. Proposition 5 is used. While the first two assertions are obvious, the last two assertions folow from [19], proposition 1.2, and are essentially based on arguments given by A. Lichnerowicz and A. G. Walker [15].

Definition 3. *Let (M, g) be a Lorentzian manifold, $y \in M$, and $a = a^\alpha \dfrac{\partial}{\partial y^\alpha}$ a timelike vector at y. For sufficiently small $|a|^2 := g_{\alpha\beta}(y)\, a^\alpha a^\beta > 0$ the compact set*

$$C(y, a) := \{x \in M \mid \sigma(x, y) \geqq 0,\ 0 \leqq (a\sigma)(x, y) \leqq |a|^2\}$$

is defined; it is called the truncated light cone with vertex y, axis a, and altitude $|a|$. (Note that a acts as a differential operator on the second argument of σ.) The volume of $C(y, a)$ with respect to the invariant Riemannian measure of (M, g) is denoted by $Vol\, C(y, a)$.

Proposition 7. *Let the altitude $T := |a|$ be so small that $C(y, a)$ is defined. There exists a system of normal coordinates $(x^\alpha) = (x^0, x^1, \dots, x^n)$ of $x \in C(y, a)$ with respect to the origin y such that $C(y, a)$ is characterized by the inequalities*

$$0 \leqq x^0 \leqq T \qquad \textit{and} \qquad (x^1)^2 + (x^2)^2 + \dots + (x^n)^2 \leqq (x^0)^2.$$

Proof. By assumption $\sigma(x, y)$ and normal coordinates (x^α) are defined for $x \in$ $\in C(y, a)$. There holds

$$a\sigma = -a^\alpha \frac{\partial \sigma}{\partial x^\alpha} = -g_{\alpha\beta}(y)\, a^\alpha x^\beta.$$

Further, there exists a linear transformation of the normal coordinates which results in

$$(g_{\alpha\beta}(y)) = (\eta_{\alpha\beta}) = \mathrm{diag}(1, -1, \ldots, -1),$$
$$a^\alpha = -T\delta_0^\alpha, \qquad a\sigma = Tx^0.$$

The corresponding coordinate system is the desired one.

Proposition 8. *The relative deviation of the curved manifold expression* Vol $C(y, a)$ *from the flat manifold expression*

$$\mathrm{Vol}\, C(a) = (n+1)^{-1}\Gamma\left(\frac{n}{2}+1\right)^{-1}\pi^{n/2}\,|\,a\,|^{n+1}$$

admits a power series expansion in the axis vector $a = a^\alpha \dfrac{\partial}{\partial y^\alpha}$, *namely*

$$576(n+1)^{-1}\binom{n+5}{4}[\mathrm{Vol}\, C(y, a)/\mathrm{Vol}\, C(a) - 1] =$$
$$= 12(n+4)(n+5)[(n+3)F_2 - g\,\mathrm{tr}\, F_2](a, a) +$$
$$+ 4(n+3)(n+5)[(n+5)F_3 - 3g\,\mathrm{tr}\, F_3](a, a, a) +$$
$$+ (n+3)[(n+5)(n+7)F_4 - 6(n+5)g\,\mathrm{tr}\, F_4 - 3g^2\,\mathrm{tr}^2 F_4](a, a, a, a) + \ldots,$$

the points indicating higher order terms. Here the symmetric p-forms

$$F_p := [\mathrm{d}^p\varrho]_{x=y} \qquad (p = 2, 3, 4, \ldots)$$

are built from the normal volume function ϱ *and are taken at the vertex* y.

Proof. Working with suitably chosen normal coordinates $x = (x^\alpha) = (x^0, x^1, \ldots, x^n)$ with respect to the origin y we have

$$\mathrm{Vol}\, C(y, a) = \int_{C(y,a)} \varrho(x, y)\, \mathrm{d}x^0, \mathrm{d}x^1 \ldots \mathrm{d}x^n =$$
$$= \int_{C(a)} \varrho\, \mathrm{d}^{n+1}x = [\mathrm{Vol}\, C(a)]\, M_{C(a)}\varrho.$$

(For the moment we notationally identify the point x with its normal coordinates.) As is well known, the Taylor expansion of a function $\varrho = \varrho(x, y)$ in the normal coordinates of x with respect to the origin y has the covariant derivatives taken at $x = y$ as its coefficients:

$$\varrho = 1 + \frac{1}{2}F_2(x, x) + \frac{1}{6}F_3(x, x, x) + \frac{1}{24}F_4(x, x, x, x) + \ldots$$

The operator $M_{C(a)}$ transforms a power series in x into a power series in a. The calculation of the first terms is based on proposition 1 and theorem 1.

In the following, $o(T^r)$ denotes a remainder term of the form

$T^r \cdot$ (regular function of y and a).

Theorem 2. *A Lorentzian manifold (M, g) of dimension $n + 1 \geqq 3$ is flat if and only if*

$$\operatorname{Vol} C(y, a) = \operatorname{Vol} C(a)\,(1 + o(T^5))$$

for sufficiently small altitudes $T := |a|$ of the truncated light cones $C(y, a)$. Likewise (M, g) is Ricci-flat if and only if

$$\operatorname{Vol} C(y, a) = \operatorname{Vol} C(a)\,(1 + o(T^3)).$$

Proof. The quadratic term in the expansion of proposition 8 is missing if and only if $d^2\varrho \doteq 0$. Likewise, the quadratic and quaitic terms are missing if and only if $d^2\varrho \doteq 0$ and $d^4\varrho \doteq 0$. Considering this, proposition 6 gives the assertion.

F. and B. Gackstatter [4] derived a partial result, namely for $n + 1 = 4$

$$\operatorname{Vol} C(y, a) = \operatorname{Vol} C(a)\left[1 + \frac{1}{45}(6R_{00}(y) - R(y))\,T^2 + \ldots\right]$$

Herefrom the assertions which are based on the Ricci tensor alone can be obtained for the spacetimes of general relativity theory.

DISCUSSION

Let us indicate the dependence on the metric g:

$B(y, r, g) :=$ geodesic ball in (M, g) with centre y and radius r for a properly Riemannian manifold (M, g) and

$C(y, a, g) :=$ truncated light cone in (M, g) with axis a and vertex y for a Lorentzian manifold (M, g). The intuitive notion of recognizing geometric properties from the volume of $B(y, r, g)$ or $C(y, a, g)$ respectively is made precise as follows.

Definition. *Two properly Riemannian manifolds (M, g), (M_0, g_0) of the same dimension n are called to possess the same volume of small geodesic balls or, for short, isovolumal, if M is covered by neighbourhoods U and diffeomorphisms $\varphi: U \to \varphi(U) \subseteq M_0$ which satisfy*

$$\operatorname{Vol} B(y, r, g) = \operatorname{Vol} B(y, r, \varphi^* g_0)$$

for $y \in M$, $r > 0$ such that $B(y, r, g) \subset U$.

(Here $\varphi^* g_0$ is the pull-back of the metric g_0 from M_0 to M.) Two Lorentzian manifolds (M, g), (M_0, g_0) of the same dimension $n + 1$ are called to possess the same volume of small truncated light cones or, for short, isovolumal, if M is covered by neighbourhoods U and diffeomorphisms $\varphi : U \to \varphi(U) \subseteq M_0$ which satisfy

$$\operatorname{Vol} C(y, a, g) = \operatorname{Vol} C(y, a, \varphi^* g_0)$$

for $y \in M$, $a \in T_y M$ such that $C(y, a, g) \subset U$.

Clearly, this "isovolumality" is an equivalence relation between Riemannian manifolds of the same dimension and the same signature. The definition is of local character and there is ambiguity in the choice of the covering by U, φ. It is plausible to compare manifolds (M, g) with simple "model manifolds" (M_0, g_0); that means, to impose on (M_0, g_0) geometric conditions easy to describe. One would start with flat model manifolds and by this rephrase the "volume conjecture": If a manifold is isovolumal to a flat manifold then it is flat itself.

That is true for Lorentzian manifolds; namely we have the

Theorem. *If a Lorentzian manifold (M, g) of dimension $n + 1 \geqq 3$ is isovolumal to a flat or Ricci-flat manifold (M_0, g_0) then it is flat or Ricci-flat itself respectively.*

Proof. A local diffeomorphism $\varphi : U \to M_0$ may be realized by describing the points $x \in M$ and $\varphi(x) \in M_0$ by the same coordinates $(x^0, \ldots, x^n) \in R^{n+1}$. The isovolumality implies then that the symmetric tensors $F_p := [\mathrm{d}^p \varrho]_{x=y}$ $(p = 2, 3, 4, \ldots)$ have equal components for (M, g) and for (M_0, g_0) in this coordinate system. Particularly, if $F_2 = 0$ and/or $F_4 = 0$ for (M_0, g_0) then the same condition holds for (M, g).

It is an interesting problem to produce analogous theorems for other model manifolds (M_0, g_0). For instance, if $n + 1 = 4$, could the Petrov type of the space-time (M, g) be recognized from Vol $C(y, a, g)$?

It is not too surprising that Vol $C(y, a)$ seemingly comprises more information than Vol $B(y, r)$: the volume of the truncated cones depends on 2(dim M) variables, while the volume of the balls depends on only (dim M) + 1 variables. We do not know how to define "maximally intrinsic" solid point sets which depend on fewer variables.

Another remarkable feature of Lorentzian manifolds is the energy-like behaviour of the traceless symmetric four-form $^-(\mathrm{Riem})^2$, cf. [15, 18, 19]. That means, this four-form is definite in some sense; it is akin to the Bel – Robinson tensor of general relativity theory.

More "volume problems" than those discussed here are studied/could be studied for properly Riemannian manifolds as well as for Lorentzian manifolds; let us mention the volume of tubes and the volume of geodesic disks [11, 12, 13].

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