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Ondřej Došlý

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RICCATI MATRIX DIFFERENTIAL EQUATION AND CLASSIFICATION OF DISCONJUGATE DIFFERENTIAL SYSTEMS

ONDŘEJ DOŠLÝ

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Abstract. Classification of disconjugate differential systems is established. It is shown that every disconjugate differential system belongs exactly to one of the (n + 1) mutually disjoint classes. Necessary and sufficient condition for determination of this class are given.

Key words. Disconjugate differential systems, principal and nonprincipal solutions, hyperbolic phase matrix, Riccati differential equation.

MS Classification. 34 C 10

1. INTRODUCTION

Let $B(x)$, $C(x)$ be symmetric $n \times n$ matrices of continuous real – valued functions and $B(x)$ be nonnegative definite on some interval I. The aim of the present paper is to study self-adjoint linear differential systems

$$y' = B(x)\,z, \qquad z' = C(x)\,y \tag{1.1}$$

under assumption that these systems are disconjugate on an interval I.

Section 2 involves preliminary statements concerning properties of solutions of investigated differential systems. In Section 3 it is established classification of disconjugate differential systems (1.1) with respect to dimension of the solution space of (1.1) generated by the right and the left principal solution of these systems. Section 4 deals with transformations of certain disconjugate differential systems and in Section 5 the results of the preceding sections are used to study relations between systems (1.1) and associated Riccati matrix differential equation.

The matrix notation is used. E and 0 denote the identity and the zero matrix of any dimension. If we need to emphasize that E is the identity matrix of dimension k, we shall denote this matrix E_k. If A is a symmetric matrix (i.e. $A^T = A$), inequalities $A > 0$ ($\geqq 0$, < 0, $\leqq 0$) mean that the matrix A is positive (nonnegative, negative, nonpositive) definite. Inequality $A > B$ between two symmetric matrices of the same dimension denotes that $A - B > 0$. Inequalities $A \geqq B$, $A < B$ and $A \leqq B$ have similar meaning. If A is a symmetric matrix, $l_1(A)$ and $l_n(A)$ denote the least and the greatest eigenvalue of A.

A pair on n-dimensional vectors $(y(x), z(x))$ is said to be a solution of (1.1) on an interval I if $y(x)$, $z(x) \in C^1(I)$ and (1.1) is identically satisfied on I.

2. PRELIMINARIES

Simultaneously with (1.1) we shall consider the matrix differential system

$$Y' = B(x)\,Z, \qquad Z' = C(x)\,Y, \tag{2.1}$$

where Y, Z are $n \times n$ matrices.

Let $(Y_i(x), Z_i(x))$, $i = 1, 2$, be solutions of (2.1). Then $Y_1^T(x)\,Z_2(x) - Z_1^T(x)\,Y_2(x) = K$, where K is a constant $n \times n$ matrix. If this matrix is nonsingular, the solutions (Y_1, Z_1), (Y_2, Z_2) are said to be *linearly independent* and every solution (Y, Z) of (2.1) can be expressed in the form $(Y, Z) = (Y_1C_1 + Y_2C_2, Z_1C_1 + Z_2C_2)$, where C_1, C_2 are constant $n \times n$ matrices. A solution (Y, Z) of (2.1) is said to be *self-conjoined* if $Y^T(x)\,Z(x) - Z^T(x)\,Y(x) = 0$. Some authors use for solutions having this property concept conjugate solution (Sternberg [9]) or isotropic solution (Coppel[4]) or prepared solution (Hartman [6]). Our terminology due to Reid, e.g. [8].

Two points $a, b \in I$ are said to be conjugate relative to (1.1) or (2.1) if there exists a solution $(y(x), z(x))$ of (1.1) such that $y(a) = 0 = y(b)$ and $y(x)$ is not identically zero between a and b. System (1.1) or (2.1) is said to be identically normal on I whenever the only solution (y, z) of (1.1) for which $y(x) \equiv 0$ on a nondegenerate subinterval of I is the trivial solution $(y, z) = (0, 0)$. System (1.1) is said to be *disconjugate* on an interval I whenever no two distinct points of I are conjugate relative to (1.1).

Let system (1.1) be identically normal and disconjugate on an interval $I = (a, b)$, possibilities $a = -\infty$, $b = \infty$ are not excluded. It is known, cf. [8, p. 325], that there exist self-conjoined solutions (Y_R, Z_R), (Y_L, Z_L) of (2.1) such that $Y_R(x)$, $Y_L(x)$ are nonsingular on I and for some (and hence every) self-conjoined solutions (Y_1, Z_1), (Y_2, Z_2) which are linearly independent on (Y_R, Z_R) and (Y_L, Z_L), respectively, we have $\lim_{x \to b-} Y_1^{-1}(x)\,Y_R(x) = 0$, $\lim_{x \to a+} Y_2^{-1}(x)\,Y_L(x) = 0$. The solutions (Y_R, Z_R), (Y_L, Z_L) are said to be the *right principal solution* and the *left principal solution* of (2.1), respectively. If $(\bar{Y}_R, \bar{Z}_R)$ is another right principal solution of (2.1) then there exists a constant nonsingular $n \times n$ matrix C such that $(\bar{Y}_R, \bar{Z}_R) = (Y_RC, Z_RC)$. The left principal solutions have similar property. It is also known that (Y_R, Z_R), (Y_L, Z_L) are right or a left principal solution if and only if

$$\lim_{x \to b-} l_1\left(\int_c^x Y_R^{-1}(s)\, B(s)\, Y_R^{\mathrm{T}-1}(s)\, \mathrm{d}s\right) = \infty,$$

$$\lim_{x \to a+} l_1\left(\int_x^c Y_L^{-1}(s)\, B(s)\, Y_L^{\mathrm{T}-1}(s)\, \mathrm{d}s\right) = \infty, \qquad c \in I.$$

Every self-conjoined solution (Y_1, Z_1) which is linearly independent on the right principal solution is said to be the *right nonprincipal solution* and we have for this solution

$$\lim_{x \to b-} l_n\left(\int_c^x Y_1^{-1}(s)\, B(s)\, Y_1^{\mathrm{T}-1}(s)\, \mathrm{d}s\right) < \infty, \qquad c \in I.$$

The left nonprincipal solution is defined analogously.

To investigate differential systems (1.1) it seems to be very useful tool the following transformation of these systems.

Theorem A. *Let $H(x)$, $K(x) \in C^1(I)$ be $n \times n$ matrices, $H(x)$ being nonsingular, for which*

$$H^{\mathrm{T}}(x)\, K(x) - K^{\mathrm{T}}(x)\, H(x) = 0, \qquad H'(x) - B(x)\, K(x) = 0. \tag{2.2}$$

Then the transformation

$$Y = H(x)\, U, \qquad Z = K(x)\, U + H^{\mathrm{T}-1}(x)\, V \tag{2.3}$$

transforms (2.1) *into the system*

$$U' = F(x)\, V, \qquad V' = G(x)\, U, \tag{2.4}$$

where

$$F(x) = H^{-1}(x)\, B(x)\, H^{\mathrm{T}-1}(x), \qquad G(x) = -H^{\mathrm{T}}(x)\, K'(x) + H^{\mathrm{T}}(x)\, C(x)\, H(x). \tag{2.5}$$

For more informations concerning this transformation see e.g. [5]. Directly can be verified the following statements:

i) (U, V) is a right or a left principal (nonprincipal) solution of (2.4) if and only if (Y, Z), given by (2.3), is a right or a left principal (nonprincipal) solution of (2.1).

ii) System (2.1) is identically normal on I if and only if (2.4) is identically normal on I.

3. CLASSIFICATION OF DISCONJUGATE SYSTEMS

Consider a system of scalar differential equations

$$y' = b(x)\, z, \qquad z' = c(x)\, v, \tag{3.1}$$

where $b(x)$, $c(x)$ are real functions, $b(x) \geqq 0$, which is disconjugate on $I = (a, b)$. Two cases are possible:

i) the right and the left principal solution of (3.1) are linearly independent.

ii) $(y_L, z_L) = c \cdot (y_R, z_R)$, where $c \neq 0$.

According to the Borůvka's classification, cf. [3], in the case i) (3.1) is said to be *general* on I, in the case ii) – *special.* Disconjugate differential systems can be classified in the following way.

Throughout all paper we shall suppose system (1.1) to be identically normal on I.

Definition 1. *Let system* (1.1) *be disconjugate on I and* (Y_R, Z_R), (Y_L, Z_L) *be its a right and a left principal solution. This system is said to be k-general on I,* $0 \leqq k \leqq n$ *being integer, if rank of the matrix*

$$\begin{bmatrix} Y_R(x) & Y_L(x) \\ Z_R(x) & Z_L(x) \end{bmatrix} \tag{3.2}$$

equals $n + k$ *for every* $x \in I$.

Remark 1. In the scalar case, i.e. $n = 1$, special system of scalar equations is said in the "system" terminology to be 0-general, general system of equations is said to be 1-general.

Theorem 1. *System* (1.1) *is k-general on I if and only if rank of the constant matrix*

$$K = Y_R^T(x)\, Z_L(x) - Z_R^T(x)\, Y_L(x) \tag{3.3}$$

equals k.

Proof. Let (1.1) be k-general on I, i.e. rank of (3.2) equals $n + k$. Using the fact that (Y_R, Z_R) and (Y_L, Z_L) are self-conjoined it can be verified that (3.2) has the same rank as the matrix

$$\begin{bmatrix} Y_R(x) & 0 \\ 0 & Y_R^T(x)\, Z_L(x) - Z_R^T(x)\, Y_L(x) \end{bmatrix}. \tag{3.4}$$

Replacing, if necessary, (Y_R, Z_R), (Y_L, Z_L) by $(Y_R C_1, Z_R C_1)$, $(Y_L C_2, Z_L C_2)$, respectively, C_1, C_2 being suitable constant nonsingular $n \times n$ matrices, we can suppose without loss of generality that $K = \operatorname{diag}\{1, \ldots, 1, 0, \ldots, 0\}$ in (3.3) (of course, the case that K contains no number 1 or no number 0 is possible). As rank of (3.4) equals $n + k$, every minor of $(n + k + 1)$-th order must equal zero and there exists at least one nonzero minor of $(n + k)$-th order. It follows that K has exactly $n - k$ zeros, hence its rank equals k. As all arguments can be reversed, the proof is complete.

Remark 2. Theorem 1 shows that the matrix (3.2) has constant rank on I, hence, every disconjugate system is k-general on I for some k, $0 \leqq k \leqq n$.

Lemma 1. *Let* (1.1) *be disconjugate on I. There exist $n \times n$ matrices $H(x)$, $K(x) \in C^1(I)$, $H(x)$ being nonsingular, satisfying* (2.2), *such that transformation* (2.3) *transforms* (2.1) *into*

$$U' = F(x)\,V, \qquad V' = 0. \tag{3.5}$$

Proof. Disconjugacy of (1.1) implies existence of self-conjoined solution $(Y, Z$ of (2.1) such that $Y(x)$ is nonsingular on I. Letting $H(x) = Y(x)$, $K(x) = Z(x)$, we have the statement of lemma.

Lemma 2. *Let the matrix $F(x)$ in* (3.5) *be of the form*

$$F(x) = \begin{bmatrix} F_1(x) & F_2^{\mathrm{T}}(x) \\ F_2(x) & F_3(x) \end{bmatrix}, \tag{3.6}$$

where F_1, F_2, F_3 are $k \times k$, $(n-k) \times k$, $(n-k) \times (n-k)$ matrices, respectively, $0 \leqq k \leqq n$, for which $\int_c^b F_1(x)\,\mathrm{d}x$, $\int_c^b F_2(x)\,\mathrm{d}x$ exist and are finite (i.e. $\int_c^b f_{ij}(x)\,\mathrm{d}x$ exists finitely for every entry f_{ij} of F_1 or F_2) and $\lim_{x \to b-} l_1(\int_c^x F_3(s)\,\mathrm{d}s) = \infty$, $c \in I$. Then

$$U_R = \begin{bmatrix} \int_x^b F_1(s)\,\mathrm{d}s & 0 \\ \int_x^b F_2(s)\,\mathrm{d}s & E_{n-k} \end{bmatrix}, \qquad V_R = \begin{bmatrix} -E_k & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.7}$$

and

$$U = \begin{bmatrix} E_k & \int_x^b F_2^{\mathrm{T}}(s)\,\mathrm{d}s \\ 0 & \int_c^x F_3(s)\,\mathrm{d}s \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & 0 \\ 0 & E_{n-k} \end{bmatrix},$$

are the right principal and nonprincipal solutions of (3.5), *respectively, for which $U^{\mathrm{T}}(x)\,V_R(x) - V^{\mathrm{T}}(x)\,U_R(x) = -E$.*

Proof. As (3.5) is identically normal on I (since (2.1) is identically normal and transformation (2.3) preserves this property), the matrix $\int_x^b F_1(s)\,\mathrm{d}s$ is nonsingular on I, cf. [8, p. 271], hence $U_R(x)$ is nonsingular. Similarly $U(x)$ is nonsingular near b. Directly we can verify that solutions (U_R, V_R), (U, V) are self-conjoined and $U^{\mathrm{T}} V_R - V^{\mathrm{T}} U_R = -E$. It follows that these solutions are linearly independent.

$$U^{-1} U_R = \begin{bmatrix} E_k & -\int_x^b F_2^{\mathrm{T}}\,\mathrm{d}s(\int_c^x F_3\,\mathrm{d}s)^{-1} \\ 0 & (\int_c^x F_3\,\mathrm{d}s)^{-1} \end{bmatrix} \begin{bmatrix} \int_x^b F_1\,\mathrm{d}s & 0 \\ \int_x^b F_2\,\mathrm{d}s & E_{n-k} \end{bmatrix} =$$

$$= \begin{bmatrix} \int_x^b F_1 \,\mathrm{d}s - \int_x^b F_2^{\mathrm{T}} \,\mathrm{d}s \left(\int_c^x F_3 \,\mathrm{d}s\right)^{-1} \int_x^b F_2 \,\mathrm{d}s & -\int_x^b F_2^{\mathrm{T}} \,\mathrm{d}s \left(\int_c^x F_3 \,\mathrm{d}s\right)^{-1} \\ -\left(\int_c^x F_3 \,\mathrm{d}s\right)^{-1} \int_c^b F_2 \,\mathrm{d}s & \left(\int_c^x F_3 \,\mathrm{d}s\right)^{-1} \end{bmatrix},$$

i.e. $\lim_{x\to b-} U^{-1} U_R = 0$. It completes the proof.

Theorem 2. *Let the matrix* $F(x)$ *in* (3.5) *be of the form* (3.6), *the matrices* F_1, F_2, F_3 *have properties given in Lemma 2 and* $\lim_{x\to a+} l_1\left(\int_x^c F(s)\,\mathrm{d}s\right) = \infty$. *Then* (3.5) *is k-general on I.*

Proof. As $\lim_{x\to a+} l_1\left(\int_x^c F(s)\,\mathrm{d}s\right) = \infty$, $(U_L, V_L) = (E, 0)$ is the left principal solution of (3.5) and according to Lemma 2 the right principal solution (U_R, V_R) is given by (3.7). Obviously $U_L^{\mathrm{T}} V_R - V_L^{\mathrm{T}} U_R = V_R$, hence by Theorem 1 system (3.5) is k-general on I.

Corollary 1. *If system* (2.1) *is n-general on I then there exist the right and the left principal solutions* (Y_R, Z_R), (Y_L, Z_L) *of this system such that the matrix* $Y_R Y_L^{\mathrm{T}}$ *is symmetric and positive definite on I.*

Proof. If (2.1) is n-general on I then there exist matrices $H(x), K(x) \in C^1(I)$ such that transformation (2.3) transforms (2.1) into (3.5), where $\lim_{x\to b-} l_1\left(\int_c^x F(s)\,\mathrm{d}s\right) = \infty$ and $\lim_{x\to a+} l_n\left(\int_x^c F(s)\,\mathrm{d}s\right) < \infty$, i.e. $(U_R, V_R) = (E, 0)$, $(U_L, V_L) = \left(\int_a^x F(s)\,\mathrm{d}s, E\right)$ are principal solutions of (3.5). It implies that $(Y_R, Z_R) = (H, K)$, $(Y_L, Z_L) = \left(H \int_a^x F\,\mathrm{d}s, K \int_a^x F\,\mathrm{d}s + H^{\mathrm{T}-1}\right)$ are principal solutions of (2.1). Directly we can verify that these solutions have all stated properties.

4. TRANSFORMATIONS OF n-GENERAL SYSTEMS

In our investigation of n-general systems the following statement will play important role.

Theorem 3. *Let* (2.1) *be n-general on I. Then there exist* $n\times n$ *matrices* $H(x)$, $K(x) \in C^1(I)$, $H(x)$ *being nonsingular, such that transformation* (2.3) *transforms* (2.1) *into the system*

$$U' = Q(x)\,V, \qquad V' = Q(x)\,U, \tag{4.1}$$

where $Q(x)$ *is a symmetric nonnegative definite* $n\times n$ *matrix.*

Proof. Let (Y_R, Z_R), (Y_L, Z_L) be the right and the left principal solutions for which $Y_R^T Z_L - Z_R^T Y_L = E$ and $Y_R Y_L^T > 0$. It holds

$$\begin{bmatrix} Z_L^T & -Y_L^T \\ -Z_R^T & Y_R^T \end{bmatrix} \begin{bmatrix} Y_R & Y_L \\ Z_R & Z_L \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix},$$

hence

$$\begin{bmatrix} Y_R & Y_L \\ Z_R & Z_L \end{bmatrix} \begin{bmatrix} Z_L^T & -Y_L^T \\ -Z_R^T & Y_R^T \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$$

and thus

$$Y_R Z_L^T - Y_L Z_R^T = E. \tag{4.2}$$

Denote by $D(x)$ the symmetric positive definite $n \times n$ matrix for which $D^2 = 2Y_R Y_L^T$ and let $T(x)$ be the solution of the differential system

$$T' = D^{-1}(x)\,[2B(x)\,Z_R(x)\,Y_L^T(x) + B(x) - D'(x)\,D(x)]\,D^{-1}(x)\,T, \tag{4.3}$$
$$T(c) = E, \qquad c \in I.$$

As

$$\begin{aligned}
&(2BZ_R Y_L^T + B - D'D) + (2BZ_R Y_L^T + B - D'D)^T = \\
&= 2BZ_R Y_L^T + 2B - D'D - DD' + 2Y_L Z_R^T B = \\
&= 2Y_R' Y_L^T + 2B - 2(Y_R Y_L^T)' + 2(Y_R Z_L^T - E)\,B =
\end{aligned}$$

$= 2(Y_R' Y_L^T + Y_R Y_L^{T\prime}) + 2B - 2(Y_R Y_L^T)' - 2B = 0$, where (4.2) has been used, the matrix $T(x)$ is othogonal on I (i.e. $T^T(x) = T^{-1}(x)$). Set

$$\begin{aligned}
H(x) &= D(x)\,T(x), \\
K(x) &= (2Z_R(x)\,Y_L^T(x) + E)\,H^{T-1}(x).
\end{aligned} \tag{4.4}$$

Then

$$\begin{aligned}
&HH^T = DTT^TD = D^2 = 2Y_R Y_L^T\,.\; H' - BK = \\
&= D'T + DT' - B(2Z_R Y_L^T + E)\,D^{-1}T = \\
&= D'T + DD^{-1}(2BZ_R Y_L^T + B - D'D)\,D^{-1}T - 2BZ_R Y_L^T D^{-1}T - BD^{-1}T = \\
&= D'T + 2BZ_R Y_L^T D^{-1}T + BD^{-1}T - D'T - 2BZ_R Y_L^T D^{-1}T - BD^{-1}T = 0, \\
&H^TK - K^TH = H^{-1}(HH^TKH^T - HK^THH^T)\,H^{T-1} = \\
&= H^{-1}[2Y_R Y_L^T(2Z_R Y_L^T + E) - (2Y_L Z_R^T + E)\,2Y_R Y_L^T]\,H^{T-1} = \\
&= H^{-1}[4Y_R Y_L^T Z_R Y_L^T + 2Y_R Y_L^T - (2Y_R Z_L^T - E)\,2Y_R Y_L^T]\,H^{T-1} = \\
&= H^{-1}[4Y_R(Y_L^T Z_R - Z_L^T Y_R)\,Y_L^T + 4Y_R Y_L^T]\,H^{T-1} = 0.
\end{aligned}$$

Let $Q(x) = H^{-1}(x)\,B(x)\,H^{T-1}(x)$. To finish the proof, according to (2.4), it suffices to prove that $H^{-1}BH^{T-1} = -H^TK' + H^TCH$. We have

$$\begin{aligned}
&H^TCH - H^TK' = H^{-1}[-HH^TK'H^T + HH^TCHH^T]\,H^{T-1} = \\
&= H^{-1}[-HH^T(2CY_R Y_L^T + 2Z_R Z_L^T B) + HH^T(2Z_R Y_L^T + E)\,H^{T-1}H^{T\prime} + \\
&\quad + HH^TCHH^T]\,H^{T-1} =
\end{aligned}$$

$$= H^{-1}[-4Y_RY_L^TZ_RZ_L^TB + 2Y_RY_L(2Z_RY_L^T + E)\,H^{T-1}H^{-1}(2Y_LZ_R^T + E)\,B]\,H^{T-1} =$$
$$= H^{-1}[-4Y_RY_L^TZ_RZ_L^TB + (2Y_RZ_L^T - E)\,2Y_RY_L^T(HH^T)^{-1}(2Y_RZ_L^T - E)\,B]\,H^{T-1} =$$
$$= H^{-1}[-4Y_RY_L^TZ_RZ_L + (2Y_RZ_L^T - E)^2]\,BH^{T-1} =$$
$$= H^{-1}[-4Y_RY_L^TZ_RZ_L^T + 4Y_RZ_L^TY_RZ_L^T - 4Y_RZ_L^T + E]\,BH^{T-1} =$$
$$= H^{-1}[-4Y_R(Z_L^TY_R - E)\,Z_L^T + 4Y_RZ_L^TY_RZ_L^T - 4Y_RZ_L^T + E]\,BH^{T-1} =$$
$$= H^{-1}BH^{T-1},$$

which has to be proved. The proof is complete.

Definition 2. *Let* (Y_i, Z_i), $i = 1, 2$, *be self-conjoined solutions of* (2.1) *for which* $Y_1^TZ_2 - Z_1^TY_2 = E$ *and* $Y_1Y_2^T > 0$ *on* I. *Further, let* $D(x)$ *be the symmetric positive definite matrix for which* $D^2 = 2Y_1Y_2^T$, T *be the solution of* $T' = D^{-1}(2BZ_1Y_2^T + B - D'D)\,D^{-1}T$, $T(c) = E$. *The matrix* $Q(x) = H^{-1}(x)\,B(x)\,H^{T-1}(x)$, *where* $H = DT$, *we shall call the hyperbolic phase matrix of* (2.1) *determined by the pair of self-conjoined, linearly independent, solutions* (Y_1, Z_1), (Y_2, Z_2).

Remark 3. According to terminology used in the scalar case, cf. [1], it would be more precise to define the hyperbolic phase matrix of (2.1) as the matrix $\int_c^x Q(s)\,ds$, $c \in I$. However, for matrix differential systems the former definition is more suitable and this definition also agrees with usual matrix notation used e.g. by Reid [7] and Barrett [2] in connection with the generalized Prüfer transformation for systems (2.1).

5. ASSOCIATED RICCATI MATRIX EQUATION

Let (Y, Z) be a solution of (2.1). It is known that in all points where the matrix $Y(x)$ is nonsingular the matrix $W = ZY^{-1}$ is a solution of the Riccati matrix differential equation

$$W' = -WB(x)\,W + C(x) \tag{5.1}$$

and that (2.1) is disconjugate on I if there exists a symmetric solution of (5.1) which is defined on the whole interval I. If (Y_R, Z_R) is a right principal solution of (2.1) then $W_R = Z_RY_R^{-1}$ is said to be the *right distinguished solution* of (5.1). The *left distinguished solution* is defined analogously.

Definition 3. *A hyperbolic phase matrix* $Q(x)$ *of (2.1) is said to be canonical if* $W_R = -E$ *and* $W_L = E$ *are the right and the left distinguished solution of the Riccati matrix equation*

$$W' = -WQ(x)\,W + Q(x). \tag{5.2}$$

Theorem 4. *Every n-general disconjugate differential system (2.1) has at least one canonical hyperbolic phase matrix. Two $n\times n$ matrices $Q_1(x)$, $Q_2(x)$ are canonical hyperbolic phase matrices of the same differential system (2.1) if and only if there exists a constant orthogonal $n\times n$ matrix G_0 such that $Q_2(x) = G_0^{\mathrm{T}} Q_1(x)\, G_0$.*

Proof. Let $Q(x)$ be the hyperbolic phase matrix of (2.1) determined by the pair of solutions (Y_R, Z_R), (Y_L, Z_L) for which $Y_R^{\mathrm{T}} Z_L - Z_R^{\mathrm{T}} Y_L = E$ and $Y_R Y_L^{\mathrm{T}} > 0$, i.e. $Q(x) = H^{-1}(x)\, B(x)\, H^{\mathrm{T}-1}(x)$, where $H(x)$ is given by (4.4) and $T(x)$ by (4.3). As transformation (2.3) transforms principal solutions into principal solutions, $(U_R, V_R) = (H^{-1} Y_R,\ -K^{\mathrm{T}} Y_R + H^{\mathrm{T}} Z_R)$, $(U_L, V_L) = (H^{-1} Y_L,\ -K^{\mathrm{T}} Y_L + H^{\mathrm{T}} Z_L$ are the right and the left principal solution of (4.1). Further, $Y_R Y_L^{\mathrm{T}} = H U_R U_L^{\mathrm{T}} H^{\mathrm{T}}$, hence $U_R U_L^{\mathrm{T}} = H^{-1} Y_R Y_L^{\mathrm{T}} H^{\mathrm{T}-1} = \dfrac{1}{2} E$. Every solution (U, V) of (4.1) can be expressed in the form $(U, V) = (XC_1 + X^{\mathrm{T}-1} C_2,\ XC_1 - X^{\mathrm{T}-1} C_2)$, where C_1, C_2 are constant $n\times n$ matrices and $X(x)$ is the solution of $X' = Q(x)\, X$, $X(c) = E$, $c \in I$. Let $U_R = XA + X^{\mathrm{T}-1} B$, $U_L = XC + X^{\mathrm{T}-1} D$. As the solutions (U_R, V_R), (U_L, V_L) are self-conjoined and $U_R^{\mathrm{T}} V_L - V_R^{\mathrm{T}} U_L = E$, we have

$$
\begin{aligned}
&A^{\mathrm{T}} B - B^{\mathrm{T}} A = 0,\\
(5.3)\qquad &C^{\mathrm{T}} D - D^{\mathrm{T}} C = 0,\\
&A^{\mathrm{T}} D - B^{\mathrm{T}} C = -\frac{1}{2} E.
\end{aligned}
$$

Now, $\dfrac{1}{2} E = U_L^{\mathrm{T}} U_R = (C^{\mathrm{T}} X^{\mathrm{T}} + D^{\mathrm{T}} X^{-1})(XA + X^{\mathrm{T}-1} B) = C^{\mathrm{T}} X^{\mathrm{T}} X A + D^{\mathrm{T}} A + D^{\mathrm{T}} X^{-1} X^{\mathrm{T}-1} B + C^{\mathrm{T}} B$. As the matrix $X(x)$ is nonconstant (since (4.1) is identically normal on I) and (5.3) holds, we have $A = 0$, $D = 0$, $B^{\mathrm{T}} C = \dfrac{1}{2} E$, i.e. $(U_R, V_R) = (X^{\mathrm{T}-1} B,\ -X^{\mathrm{T}-1} B)$, $(U_L, V_L) = (XC, XC)$, where B, C are constant nonsingular $n\times n$ matrices for which $B^{\mathrm{T}} C = \dfrac{1}{2} E$. Hence $W_R = V_R U_R^{-1} = -E$ and $W_L = V_L U_L^{-1} = E$.

Now, let $Q_1(x)$, $Q_2(x)$ be two hyperbolic phase matrices of (2.1) and let these matrices be determined by the pairs of self-conjoined solutions (Y_1, Z_1), $(\bar{Y}_1, \bar{Z}_1)$ and (Y_2, Z_2), $(\bar{Y}_2, \bar{Z}_2)$, respectively, for which

$$
(5.4)\qquad Y_i^{\mathrm{T}} \bar{Z}_i - Z_i^{\mathrm{T}} \bar{Y}_i = E, \qquad i = 1, 2.
$$

Further, let the matrices $H_i(x)$, $K_i(x)$ be given by means of solutions (Y_i, Z_i), $(\bar{Y}_i, \bar{Z}_i)$ in the same way as the matrices $H(x)$, $K(x)$ by (Y_R, Z_R), (Y_L, Z_L) in the proof of Theorem 3. Then $(U_i, V_i) = (H_i^{-1} Y_i,\ -K_i^{\mathrm{T}} Y_i + H_i^{\mathrm{T}} Z_i)$, $(\bar{U}_i, \bar{V}_i) = (H_i^{-1} \bar{Y}_i,\ -K_i^{\mathrm{T}} \bar{Y}_i + H_i^{\mathrm{T}} \bar{Z}_i)$, $i = 1, 2$, are solutions of the differential systems

$$
(5.5)_1\qquad U' = Q_1(x)\, V, \qquad V' = Q_1(x)\, U,
$$

$$(5.5)_2 \qquad U' = Q_2(x)\,V, \qquad V' = Q_2(x)\,U,$$

respectively. Similarly as above we can prove that $(U_i, V_i) = (X_i^{\mathrm{T}-1}A_i, -X_i^{\mathrm{T}-1}A_i)$, $(\bar{U}_i, \bar{V}_i) = (X_iB_i, X_iB_i)$, where $X_i' = Q_i(x)\,X_i$, $X_i(c) = E$, $c \in I$ and A_i, B_i are constant nonsingular $n \times n$ matrices for which $A_i^{\mathrm{T}}B_i = \frac{1}{2}E$, $i = 1, 2$. Since $V_iU_i^{-1} = -E$, $\bar{V}_i\bar{U}_i^{-1} = E$, and $W_R = E$, $W_L = E$ are the right and the left distinguished solutions of (5.2) with $Q = Q_i$, (U_i, V_i) are the right and $(\bar{U}_i, \bar{V}_i)$ the left principal solutions of $(5.5)_i$, respectively. It follows that both (Y_1, Z_1) and (Y_2, Z_2) are the right principal solutions of (2.1), i.e. $(Y_2, Z_2) = (Y_1M, Z_1M)$, where M is a constant nonsingular $n \times n$ matrix. Similarly $(\bar{Y}_2, \bar{Z}_2) = (\bar{Y}_1N, \bar{Z}_1N)$, N being a constant nonsingular $n \times n$ matrix. From (5.4) it follows $N = M^{\mathrm{T}-1}$. Further, $H_2H_2^{\mathrm{T}} = Y_2\bar{Y}_2^{\mathrm{T}} = Y_1MN^{\mathrm{T}}\bar{Y}_1^{\mathrm{T}} = Y_1\bar{Y}_1^{\mathrm{T}} = H_1H_1^{\mathrm{T}}$, hence $H_2(x) = H_1(x)\,G(x)$, where $G(x)$ is an orthogonal $n \times n$ matrix. It holds $K_2 = (2BZ_2\bar{Y}_2^{\mathrm{T}} + E)\,H_2^{\mathrm{T}-1} = (2BZ_1MN^{\mathrm{T}}\bar{Y}_1^{\mathrm{T}} + E)\,.\,H_1^{\mathrm{T}-1}G = (2BZ_1\bar{Y}_1^{\mathrm{T}} + E)\,H_1^{\mathrm{T}-1}G = K_1G$. It follows $H_2' = H_1'G + H_1G' = BK_1G + H_1G'$. From the other hand $H_2' = BK_2 = BK_1G$, hence $H_1G' = 0$, i.e. $G(x) = G_0$ is a constant orthogonal $n \times n$ matrix. Thus $Q_2(x) = H_2^{-1}(x)\,B(x)\,H_2^{\mathrm{T}-1}(x) = G_0^{\mathrm{T}}H_1^{-1}(x)\,B(x)\,H_1^{\mathrm{T}-1}(x)\,G_0 = G_0^{\mathrm{T}}Q_1(x)\,G_0$. The proof is complete.

Theorem 5. *System (2.1) is k-general on I if and only if rank of the matrix $W_R - W_L$ equals k, where W_R and W_L are the right and the left distinguished solution of the associated Riccati matrix differential equation (5.1).*

Proof. Let (2.1) be k-general on I and W_R, W_L be the right and the left distinguished solutions of (5.1). There exist matrices $H(x)$, $K(x) \in C^1(I)$, $H(x)$ being nonsingular on I, such that transformation (2.3) transforms (2.1) into (3.5), where $\lim_{x \to b-} l_1(\int_c^x F(s)\,\mathrm{d}s, = \infty$. Let $\bar{W}_R$, $\bar{W}_L$ be the right and the left distinguished solutions of the Riccati matrix equation

$$W' = -WF(x)\,W.$$

We have $W_R - W_L = Z_RY_R^{-1} - Z_LY_L^{-1} = (KU_R + H^{\mathrm{T}-1}V_R)\,U_R^{-1}H^{-1} - (KU_L + H^{\mathrm{T}-1}V_L)\,U_L^{-1}H^{-1} = H^{\mathrm{T}-1}V_RU_R^{-1}H^{-1} - H^{\mathrm{T}-1}V_LU_L^{-1} = H^{\mathrm{T}-1}(\bar{W}_R - \bar{W}_L)\,H^{-1}$, thus $\operatorname{rank}(W_R - W_L) = \operatorname{rank}(\bar{W}_R - \bar{W}_L)$. As $\lim_{x \to b-} l_1(\int_c^x F(s)\,\mathrm{d}s) = \infty$ and (3.5) is k-general on I, $(U_R, V_R) = (E, 0)$ and

$$U_L = \begin{bmatrix} \mathfrak{F}_1 & 0 \\ \mathfrak{F}_2 & E_{n-k} \end{bmatrix}, \qquad V_L = \begin{bmatrix} E_k & 0 \\ 0 & 0 \end{bmatrix},$$

where $\mathfrak{F}_1$, $\mathfrak{F}_2$ are $k \times k$ and $(n-k) \times k$ matrices, respectively, $\mathfrak{F}_1$ being nonsingular on I, for which $\mathfrak{F}_1' = F_1$, $\mathfrak{F}_2' = F_2$ and F_1, F_2 are given by (3.6). Now, $\bar{W}_R = V_RU_R^{-1} = 0$ and hence $\operatorname{rank}(\bar{W}_R - \bar{W}_L) = \operatorname{rank}\bar{W}_L = \operatorname{rank}V_LU_L^{-1} = \operatorname{rank}V_L = k$. As all arguments can be reversed, the proof is complete.

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O. Došlý
Department of Mathematics
J. E. Purkyně University
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia