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THE LATTICES OF TOPOLOGIES ON A PARTIALLY ORDERED SET

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Abstract. The paper deals with some types of a compatibility of a topology and an order. The aim is to state conditions on a partially ordered set $(P, \leqq)$ under which the system of all topologies on P compatible in a certain sense is a lattice.

Key words. Partially ordered set, topological space, compatibility of topology and order.

MS Classification. 06 B 30, 06 F 30.

1. INTRODUCTION

If an ordering relation $\leqq$ and a topology $\mathcal{O}$ on a set P are given, various kinds of connections can be considered. In this paper we deal with four types of compatibility (see 2.1), so called i-compatibility for $i \in \{1, 2, 3, 4\}$, introduced in the papers [1], [4], [5]. The system $C_i(P)$ of all i-compatible topologies on a partially ordered set $(P, \leqq)$ can be ordered in a natural way. The aim of this paper is to describe all partially ordered sets $(P, \leqq)$ for which $C_i(P)$ $(i \in 2, 3, 4\})$ are lattices and solve the problem of the existence of the least element in $C_i(P)$ for $i \in \{2, 3, 4\}$. For $i = 1$ these questions were investigated in the paper [3].

Let $(P, \leqq)$ be a partially ordered set. For $x, y \in P$ we shall write $x \prec y$ in the case that $x < y$ and no $z \in P$ exists satisfying $x < z < y$. The denotation $x \parallel y$ will mean that x, y are incomparable, while $x \nparallel y$ will express that x, y are comparable, i.e. $x \leqq y$ or $x \geqq y$. For $x \in P$ set

$$\uparrow x = \{y \in P \mid y \geqq x\},$$
$$\downarrow x = \{y \in P \mid y \leqq x\},$$
$$\updownarrow x = \{y \in P \mid v \geqq x \text{ or } y \leqq x\},$$
$$N(x) = \{y \in P \mid y \parallel x\}.$$

For $X \subseteq P, X \neq \emptyset$ let $\uparrow X = \bigcup_{x \in X} \uparrow x, \downarrow X = \bigcup_{x \in X} \downarrow x, \updownarrow X = \bigcup_{x \in X} \updownarrow x$, $\operatorname{conv} X = \uparrow X \cap \downarrow X =$

$= \{x \in P \mid y \leqq x \leqq z \text{ for some } y, z \in X\}$, while $\uparrow\emptyset$, $\downarrow\emptyset$, $\updownarrow\emptyset$ and $\operatorname{conv}\emptyset$ define as $\emptyset$.

The system of all subsets of a set P will be denoted by $\exp P$. By a topology on P a system $\mathcal{O} \subseteq \exp P$ containing $\emptyset$, P and closed under finite intersections and arbitrary unions will be meant. The elements of $\mathcal{O}$ will be called open subsets of P.

The interval topology $\mathcal{I}$ on a partially ordered set $(P, \leqq)$ is the topology generated by the system $\{P - \uparrow x \mid x \in P\} \cup \{P - \downarrow x \mid x \in P\}$.

2. COMPATIBILITY OF A TOPOLOGY WITH AN ORDERING

In what follows, by $(P, \leqq)$ (or briefly P) an arbitrary fixed partially ordered set will be meant.

Let $\mathcal{O}$ be a topology on P. Consider the following conditions for $a, b \in P$:
(C1) There exists $A \in \mathcal{O}$ such that $a \in A$, $b \notin \operatorname{conv} A$;
(C2) There exist $A, B \in \mathcal{O}$ such that $a \in A$, $b \in B$ and $x \ngeq y$ for every $x \in A$, $y \in B$.

2.1. Definition. *The topology $\mathcal{O}$ is called:*
1-compatible, if $\mathcal{O}$ is T_1-topology and (C1) *holds for every $a, b \in P$, $a \neq b$, $a \parallel b$;*
2-compatible, if (C1) *holds for every $a, b \in P$, $a \neq b$;*
3-compatible, if $\mathcal{O}$ is T_1-topology and (C2) *holds for every $a, b \in P$, $a < b$;*
4-compatible, if (C2) *holds for every $a, b \in P$, $a \ngeq b$.*

2.2. Note. 1- and 3-compatibility were introduced in [4], 2-compatibility is a slight modification of the convex compatibility dealt with in [1] and 4-compatibility issues from [5]. It is easy to see that $\mathcal{O}$ is 4-compatible if and only if the relation $\leqq$ on P is closed with respect to $\mathcal{O}$ in the sense of the definition given in [2].

Denote by $C_i(P)$ ($i \in \{1, 2, 3, 4\}$) the system of all i-compatible topologies on P. Clearly $C_4(P)$ is contained in $C_2(P)$ and $C_3(P)$, and both $C_2(P)$ and $C_3(P)$ are subsets of $C_1(P)$. It is easy to see that in general $C_2(P)$ and $C_3(P)$ are incomparable sets. Notice that $C_i(P)$ ($i \in \{1, 2, 3, 4\}$) are increasing subsets of the set $T_1(P)$ of all T_1-topologies on P, i.e. $\mathcal{O}_1 \subseteq \mathcal{O}_2$, $\mathcal{O}_1 \in C_i(P)$, $\mathcal{O}_2 \in T_1(P)$ implies $\mathcal{O}_2 \in C_i(P)$. This, together with the fact that the set $T_1(P)$ is closed under intersections yields that $C_i(P)$ is a lattice if and only if $\mathcal{O}_1 \cap \mathcal{O}_2 \in C_i(P)$ whenever $\mathcal{O}_1, \mathcal{O}_2 \in C_i(P)$.

2.3. Definition (cf. [3]). *Define a mapping $v : \exp P \to \exp P$ as follows:* $v(X) = X \cup \{x \in P \mid X - \uparrow M - \downarrow N$ *is infinite for all finite* $M \subseteq \uparrow x - \{x\}$, $N \subseteq \downarrow x - \{x\}\}$.

It is evident that $v(\emptyset) = \emptyset$ and $v(X) \supseteq X$ for every $X \subseteq P$. It can be verified that $v(X \cup Y) = v(X) \cup v(Y)$ for all $X, Y \subseteq P$ (for the proof see [3]).

2.4. Definition. *Let $i \in \{1, 2, 3, 4\}$, $x \in P$, $X \subseteq P$. We say that x is i-separated from X, if for every $\mathcal{O} \in C_i(P)$ there is $A \in \mathcal{O}$ such that $x \in A$, $A \cap X = \emptyset$.*

Now we will show that for $i \in \{1, 2, 3\}$, $x \in P$, $X \in P$, $x \notin \nu(X)$ if and only if x is i-separated from X. For $i = 1$ the statement follows immediately from 3.18 of [3]. To show this for $i = 2, 3$, it is sufficient to prove that if x is i-separated from X, then $x \notin \nu(X)$ (the converse implication, also for $i = 4$, follows from that for $i = 1$, since $C_1(P) \supseteq C_i(P)$).

2.5. Lemma. *Let $X \subseteq P$ be infinite, $x \in P$, $X \subseteq N(x)$. Then x is not 2-separated from X.*

Proof. Since X is infinite, it contains an infinite antichain or an infinite chain. Let $R \subseteq X$ be an infinite antichain. Put $\mathcal{O} = \{A \in P \mid x \notin A \text{ or } R - A \text{ is finite}\}$. Evidently $\mathcal{O} \in C_2(P)$ and x has no neighbourhood disjoint from X.

Let $R \subseteq X$ be an infinite chain. Without loss of generality we can assume that $R = \{r_1, r_2, \ldots\}$ is a descending chain. We put again $\mathcal{O} = \{A \subseteq P \mid x \notin A$ or $R - A$ is finite$\}$. It holds that $\mathcal{O} \in T_1(P)$ and x has no neighbourhood disjoint from X. We show 2-compatibility of the topology $\mathcal{O}$. Let $a, b \in P$, $a \neq b$. If $a \neq x$, then we put $A = \{a\} \in \mathcal{O}$. Let $a = x$. If there is $r_i \in R$ such that $r_i \ngeq b$, then we put $A = \{a, r_i, r_{i+1}, \ldots\}$, else $A = \{a\} \cup R$. In every case $a \in A$, $A \in \mathcal{O}$ and $b \notin \operatorname{conv} A$.

2.6. Lemma. *Let $x \in P$, $X \subseteq P$. If x is 2-separated from X, then $x \notin \nu(X)$.*

Proof. Let $x \in \nu(X)$. If $X \cap N(x)$ is infinite, then x is not 2-separated from X by 2.5. Let $X \cap N(x)$ be finite. Since $x \notin \nu(X \cap N(x))$ and $\nu(X) = \nu(X \cap \uparrow x) \cup \cup \nu(X \cap \downarrow x) \cup \nu(X \cap N(x))$, it must be $x \in \nu(X \cap \uparrow x)$ or $x \in \nu(X \cap \downarrow x)$. Without loss of generality we can assume $x \in \nu(X \cap \uparrow x) = \nu(X_1)$. If $x \in X$, then evidently x is not 2-separated from X. Let $x \notin X$. Then for all finite $M \subseteq \uparrow x - \{x\}$, $N \subseteq \subseteq \downarrow x - \{x\}$ the set $X_1 - \uparrow M - \downarrow N = X_1 - \uparrow M$ is infinite. The system $\{X_1 - - \uparrow M \mid M \subseteq \uparrow x - \{x\}, M \text{ is finite}\}$ obviously has the finite intersection property and hence generates a filter $\mathcal{F}$ on P. We put $\mathcal{O} = \{A \subseteq P \mid x \notin A$ or $A \in \mathcal{F}\}$. Evidently $\mathcal{O}$ is a T_1-topology and each neighbourhood of x has a non-empty intersection with X. We show that $\mathcal{O} \in C_2(P)$. Let $a, b \in P$, $a \neq b$. If $a \neq x$, then we put $A = \{a\}$. If $a = x$ and $b \ngtr a$, we can take $A = X_1 \cup \{x\}$. If $a = x$ and $b > a$, set $A = (X_1 \cup \{x\}) - \uparrow b$. In every case $a \in A$, $A \in \mathcal{O}$ and $b \notin \operatorname{conv} A$. We have obtained that $\mathcal{O} \in C_2(P)$ and again x is not 2-separated from X.

2.7. Lemma. *Let $x \in P$, $X \subseteq P$. If x is 3-separated from X, then $x \notin \nu(X)$.*

Proof. First observe that if $x \in \nu(X) - X$, then there exists $\mathcal{O} \in C_2(P)$ satisfying:

(i) $\{y\} \in \mathcal{O}$ for $y \neq x$,

(ii) $A \cap X \neq \emptyset$ whenever $A \in \mathcal{O}$, $x \in A$.

Namely if $X \cap N(x)$ is infinite, we can take the topology constructed in 2.5, in the opposite case the topology given in 2.6.

Now let $x \in \nu(X) - X$, $\mathcal{O}$ be a topology as above. To prove that x is not 3-separated from X, it is sufficient to show that $\mathcal{O} \in C_3(P)$. Let $a, b \in P$, $a < b$.

If $x \notin \{a, b\}$, we put $A = \{a\}$, $B = \{b\}$. If $x = a$ (the case $x = b$ is symmetrical), then there is $A \in \mathcal{O}$ such that $a \in A$, $b \notin \operatorname{conv} A$. Put $B = \{b\}$. Evidently $u \ngeq v$ for $u \in A$, $v \in B$. We have obtained $\mathcal{O} \in C_3(P)$.

For 4-compatibility a different result is obtained:

2.8. Theorem. *Let $x \in A$, $X \subseteq P$. It holds: x is 4-separated from X if and only if there exist finite sets $M_1 \subseteq \uparrow x - \{x\}$, $M_2 \subseteq \downarrow x - \{x\}$, $M_3 \subseteq N(x)$ such that $X - \uparrow M_1 - \downarrow M_2 - \updownarrow M_3 = \emptyset$.*

Proof. I. Let such M_1, M_2, M_3 exist and let $\mathcal{O} \in C_4(P)$. For $y \in M_1$ we get $A_y \in \mathcal{O}$ such that $x \in A_y$, $A_y \cap \uparrow y = \emptyset$. Analogously, for $y \in M_2$ and $y \in M_3$ we have an open neighbourhood A_y of x disjoint from $\downarrow y$ and $\updownarrow y$, respectively. Let $A = \cap \{A_y \mid y \in M_1 \cup M_2 \cup M_3\}$. Then $A \in \mathcal{O}$, $x \in A$ and A is disjoint from $\uparrow M_1 \cup \downarrow M_2 \cup \updownarrow M_3 \supseteq X$. Hence x is 4-separated from X.

II. Let $X - \uparrow M_1 - \downarrow M_2 - \updownarrow M_3 \neq \emptyset$ for every finite $M_1 \subseteq \uparrow x - \{x\}$, $M_2 \subseteq \downarrow x - \{x\}$, $M_3 \subseteq N(x)$. Then the system $\{X - \uparrow M_1 - \downarrow M_2 - \updownarrow M_3 \mid M_1, M_2, M_3$ finite, $M_1 \subseteq \uparrow x - \{x\}$, $M_2 \subseteq \downarrow x - \{x\}$, $M_3 \subseteq N(x)\}$ generates a filter $\mathcal{F}$ on P. Put $\mathcal{O} = \{A \supseteq P \mid x \notin A$ or $A \in \mathcal{F}\}$.

$\mathcal{O}$ is a T_1-topology and each neighbourhood of x has a nonempty intersection with X. We show $\mathcal{O} \in C_4(P)$. Let $a, b \in P$, $a \ngeq b$. If $x \notin \{a, b\}$, then we can put $A = \{a\}$, $B = \{b\}$. Let $x = a$ (the case $x = b$ is symmetrical). If $a \parallel b$, then put $A = (X \cup \{a\}) - \updownarrow b$, $B = \{b\}$, else (i.e. if $a < b$) put $A = (X \cup \{a\}) - \uparrow b$, $B = \{b\}$. In every case $a \in A$, $b \in B$, $A \in \mathcal{O}$, $B \in \mathcal{O}$ and $u \ngeq v$ for $u \in A$, $v \in B$.

3. THE LATTICES $C_i(P)$

Consider the topologies $\mathcal{N}_i = \cap \{\mathcal{O} : \mathcal{O} \in C_i(P)\} \in T_1(P)$ for $i \in \{1, 2, 3, 4\}$. It is obvious, by Definition 2.4, that $\mathcal{N}_i$ contains just those $A \subseteq P$, for which every $x \in A$ is i-separated from $P - A$. In view of the foregoing results it is $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \{A \subseteq P : v(P - a) = P - A\}$. Consequently further we shall use the denotation $\mathcal{N}$ instead of $\mathcal{N}_i$ for $i \in \{1, 2, 3\}$. Evidently $C_i(P)$ ($i \in \{1, 2, 3\}$) contains the least element if and only if $\mathcal{N} \in C_i(P)$.

In [3] the equivalence of the following conditions was proved:

(i) $C_1(P)$ is a lattice,

(ii) $C_1(P)$ contains the least element,

(iii) $v(v(X)) = v(X)$ for every $X \subseteq P$,

(iv) $v(v(\uparrow y)) = v(\uparrow y)$ and $v(v(\downarrow y)) = v(\downarrow y)$ for every $y \in P$. We shall prove similar results for 2-, 3- and 4-compatibility.

3.1. Lemma. *Let $x, y \in P$, $x \neq y$. If $x \in v(\uparrow y)$ and $x \in v(\downarrow y)$, then $C_2(P)$ is not a lattice.*

Proof. By 2.6 the relation $x \in v(\uparrow y)$ implies the existence of a topology $\mathcal{O}_1 \in C_2(P)$ satisfying $A \cap \uparrow y \neq \emptyset$ for each $A \in \mathcal{O}_1$ that contains x.

Analogously because of $x \in v(\downarrow y)$ there exists a topology $\mathcal{O}_2 \in C_2(P)$ such that $A \cap \downarrow y \neq \emptyset$ whenever $x \in A \in \mathcal{O}_2$. Let $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$. Then $x \in A \in \mathcal{O}$ implies $A \cap \uparrow y \neq \emptyset$ and $A \cap \downarrow y \neq \emptyset$, hence $y \in \operatorname{conv} A$. We have proved that $\mathcal{O}$ is not 2-compatible. Hence $C_2(P)$ is not a lattice.

3.2. Lemma. *If $v(\uparrow y) \cap v(\downarrow y) = \{y\}$ for every $y \in P$, then $C_2(P) = C_1(P)$.*

Proof. Let $\mathcal{O} \in C_1(P)$, $a, b \in P$, $a \neq b$. Then $a \notin v(\uparrow b)$ or $a \notin v(\downarrow b)$. Suppose e.g. that the first possibility holds. Then a is 1-separated from $\uparrow b$. Hence there exists $A \in \mathcal{O}$ such that $a \in A$, $A \cap \uparrow b = \emptyset$. We have proved that $\mathcal{O} \in C_2(P)$.

3.3. Theorem. *The following conditions are equivalent*:

(i) $C_2(P)$ is a lattice,

(ii) $C_2(P) = \{\mathcal{O} \in T_1(P) : \mathcal{O} \supseteq \mathcal{N}\}$,

(iii) $v(v(X)) = v(X)$ for every $X \subseteq P$ and $v(\uparrow y) \cap v(\downarrow y) = \{y\}$ for every $y \in P$.

Proof. If $C_2(P)$ is a lattice, then $v(\uparrow y) \cap v(\downarrow y) = \{y\}$ for every $y \in P$ by 3.1. Considering 3.2 and the result for 1-compatibility it follows the first part of (iii). We have proved that (i) implies (iii). The first part of (iii) gives $\mathcal{N} \in C_1(P)$ and using the second part of (iii) we have $\mathcal{N} \in C_2(P)$ by 3.2. Hence (iii) implies (ii). The validity of the implication (ii) $\Rightarrow$ (i) is obvious.

3.4. Lemma. Let $x, y \in P$, $x \neq y$, $x \nparallel y$. If $x \in v(N(y))$, then $C_3(P)$ is not a lattice.

Proof. If $x \in v(N(y))$, then by 2.7 x is not 3-separated from $N(y)$. Hence there exists $\mathcal{O}_1 \in C_3(P)$ such that $A \cap N(y) \neq \emptyset$ whenever $x \in A \in \mathcal{O}_1$. Since $\mathcal{O}_1$ is a T_1-topology and $x \notin N(y)$, $A \cap N(y)$ must be infinite whenever $x \in A \in \mathcal{O}_1$. Let $\mathcal{O}_2 = \{A \subseteq P \mid y \notin A \text{ or } N(y) - A \text{ is finite}\}$. We show that $\mathcal{O}_2 \in C_3(P)$. Clearly $\mathcal{O}_2$ is a T_1-topology. Let $a, b \in P$, $a < b$. If $y \notin \{a, b\}$, put $A = \{a\}$, $B = \{b\}$. If $y = a$ (the case $y = b$ is symmetrical), put $A = \{a\} \cup N(a)$, $B = \{b\}$. In both cases $A, B \in \mathcal{O}_2$ and $u \ngeq v$ for $u \in A$, $v \in B$. Let $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$. If $A, B \in \mathcal{O}$, $x \in A$, $y \in B$, then $A \cap N(y)$ is infinite and $N(y) - B$ is finite. That is why $A \cap B \neq \emptyset$. This shows $\mathcal{O} \notin C_3(P)$ and $C_3(P)$ is not a lattice.

3.5. Theorem. *The following conditions are equivalent*:

(i) *$C_3(P)$ is a lattice,*

(ii) $C_3(P) = \{\mathcal{O} \in T_1(P) : \mathcal{O} \supseteq \mathcal{N}\}$,

(iii) *$x \notin v(N(y))$ for every $x, y \in P$, $x \neq y$, $x \nparallel v$.*

Proof. (i) implies (iii) by 3.4, the implication (ii) $\Rightarrow$ (i) is evident. So it is sufficient to show that (iii) implies (ii). Hence let us suppose that (iii) holds. Take any $a, b \in P$, $a < b$.

I. If $a \prec b$, put $A = \downarrow b - \{b\}$, $B = \uparrow a - \{a\}$. Evidently $a \in A$, $b \in B$. For $x \in A$ it holds $x \notin v(\uparrow b)$ (because of $x < b$) and $x \notin v(N(b))$ by (iii), which follows $x \notin v(\uparrow b \cup N(b)) = v(P - A)$. Hence $A \in \mathcal{N}$ and analogously $B \in \mathcal{N}$. Furthermore $x \ngeq y$ for $x \in A$, $y \in B$.

II. If $a < s < b$ for some $s \in P$, put $A = \downarrow s - \{s\}$, $B = \uparrow s - \{s\}$. Again $a \in A \in \in \mathcal{N}$, $b \in B \in \mathcal{N}$ and $x \not\geq y$ for $x \in A$, $y \in B$.

We have shown that $\mathcal{N} \in C_3(P)$, hence (ii) holds.

3.6. Lemma. *Let $\mathcal{I}$ be the interval topology on P, $\mathcal{O} \in C_4(P)$. Then $\mathcal{I} \subseteq \mathcal{O}$.*

Proof. It is sufficient to show that $P - \uparrow x \in \mathcal{O}$ for every $x \in P$ (the proof for $P - \downarrow x$ would be symmetrical). Let $y \in P - \uparrow x$, we will find a neighbourhood A of y in $\mathcal{O}$ such that $A \subseteq P - \uparrow x$. Because of $\mathcal{O} \in C_4(P)$ and $y \not\geq x$, there exists $A \in \mathcal{O}$ such that $y \in A$ and $u \not\geq x$ for every $u \in A$. Hence $A \subseteq P - \uparrow x$.

3.7. Lemma. *Let $\mathcal{I}$* be the interval topology on *P*. If $C_4(P)$ is a lattice, then $\mathcal{I} \in \in C_4(P)$.

Proof. Let $C_4(P)$ be a lattice. Let $a, b \in P$, $a \not\geq b$. Set

$$\mathcal{I}_a = \{A \subseteq P \mid a \notin A \text{ or } A \in \mathcal{D}(a)\},$$

$$\mathcal{I}_b = \{A \subseteq P \mid b \notin A \text{ or } A \in \mathcal{D}(b)\},$$

where $\mathcal{D}(a)$ and $\mathcal{D}(b)$ denotes the system of all neighbourhoods of a and b in $\mathcal{I}$, respectively. It is easy to see that $\mathcal{I}_a \in C_4(P)$, $\mathcal{I}_b \in C_4(P)$. Since $C_4(P)$ is a lattice, $\mathcal{O} = \mathcal{I}_a \cap \mathcal{I}_b \in C_4(P)$. Then there are $A_1, B_1 \in \mathcal{O}$ such that $a \in A_1$, $b \in B_1$ and $x \not\geq y$ whenever $x \in A_1$, $y \in B_1$. Clearly $A_1 \in \mathcal{D}(a)$, $B_1 \in \mathcal{D}(b)$ and there are $A \in \mathcal{I}$, $B \in \mathcal{I}$ such that $a \in A \subseteq A_1$, $b \in B \subseteq B_1$. Then $x \not\geq y$ for $x \in A$, $y \in B$. The proof is finished.

3.8. Theorem. *The following conditions are equivalent:*

(i) *$C_4(P)$ is a lattice,*

(ii) $C_4(P) = \{\mathcal{O} \in T_1(P) : \mathcal{O} \supseteq \mathcal{I}\}$,

(iii) *for every $x, y \in P$, $x \not\leq y$ there exist finite sets $M_1 \subseteq \uparrow x - \{x\}$, $M_2 \subseteq \subseteq \downarrow x - \{x\}$, $M_3 \subseteq N(x)$, $N_1 \subseteq \uparrow y - \{y\}$, $N_2 \subseteq \downarrow y - \{y\}$, $N_3 \subseteq N(y)$ such that $z \not\leq t$ whenever $z \in P - \uparrow M_1 - \downarrow M_2 - \updownarrow M_3$ and $t \in P - \uparrow N_1 - \downarrow N_2 - \updownarrow N_3$.*

Proof. (i) implies (ii) by 3.6 and 3.7, the implication (ii) $\Rightarrow$ (i) is evident. We are going to show that (ii) and (iii) are equivalent. Let (ii) hold. Take any $x, y \in P$, $x \not\leq y$. Since $\mathcal{I} \in C_4(P)$, there are $A, B \in \mathcal{I}$ such that $x \in A$, $y \in B$ and $u \not\leq v$ for every $u \in A$, $v \in B$. Since $\mathcal{I} \subseteq \mathcal{O}$ for every $\mathcal{O} \in C_4(P)$, x is 4-separated from $P - A$. Then 2.8 yields the existence of finite sets $M_1 \subseteq \uparrow x - \{x\}$, $M_2 \subseteq \downarrow x - \{x\}$, $M_3 \subseteq N(x)$ satisfying $P - A - \uparrow M_1 - \downarrow M_2 - \updownarrow M_3 = \emptyset$, i.e. $P - \uparrow M_1 - \downarrow M_2 - \updownarrow M_3 \subseteq A$. Analogously we can find finite sets N_1, N_2, N_3 with $P - \uparrow N_1 - \downarrow N_2 - \updownarrow N_3 \subseteq B$. Clearly $z \not\leq t$ for $z \in P - \uparrow M_1 - \downarrow M_2 - \updownarrow M_3$, $t \in P - \uparrow N_1 - \downarrow N_2 - \updownarrow N_3$. Hence (iii) holds.

Now let (iii) be satisfied. We are going to show that $\mathcal{I} \in C_4(P)$. Let $x, y \in P$, $x \not\leq y$. Put $A = P - \uparrow M_1 - \downarrow M_2 - \updownarrow M_3$, $B = P - \uparrow N_1 - \downarrow N_2 - \updownarrow N_3$. It is clear that $x \in A$, $y \in B$, $A \in \mathcal{I}$, $B \in \mathcal{I}$ and $u \not\leq v$ for $u \in A$, $v \in B$.

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